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ON MODIFIED NOOR ITERATIONS FOR NONEXPANSIVE MAPPINGS

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Abstract. In this paper, we suggest and analyze some new iterative methods for finding the fixed point of nonexpansive mapping in Banach spaces, which are called modified Noor iterations. We show that the approximate solution converges to a fixed point of the nonexpansive mapping, which is a solution of a variational inequality, under some mild conditions. Results obtained in this paper may be viewed as a significant refinement of the previously known results in this area.

1. Introduction

Variational inequalities, which were introduced and studied in early sixties, have played a critical and significant part in the study of several unrelated problems arising in finance, economics, network analysis, elasticity, optimization, water resources, medical images and structural analysis. Variational inequalities have witnessed an dynamic growth in theoretical advances, algorithmic development and new applications across all disciplines of pure and applied sciences and proved to productive. As a result of interaction among various branches of mathematical and engineering sciences, we now have a variety of techniques to suggest and analyze several numerical techniques for solving variational inequalities and related optimization problems. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis, see [2, 3, 8-25] and the references therein for more details. Related to the variational inequalities, is the problem of finding the fixed points of the nonexpansive mappings, which is the subject of current interest in functional analysis. In 2000, Noor [11] suggested and analyze three-step iterative method for finding the approximate solution of the nonexpansive mapping using the technique of updating the solution. It is well known [13, 18] that the three-step iterative methods perform better numerically than the two-step(Ishikawa) and one-step(Mann) iterations. Three-step iterations are also called Noor iteration and this has initiated a quite a new direction of research in functional analysis. Noor [17] and Noor and Huang [19] have considered some three-step iterative methods for the nonexpansive mappings in conjunction with variational

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inequalities. In recent years, viscosity methods introduced by Moudafi [10] are being considered for finding the approximate solution of the nonexpansive mappings, see, for example, [14, 21] and the references therein for more details. Motivated and inspired by the research going on in this direction, we use the ideas and technique of Moudafi [10] and Noor [11-13] to suggest and analyze two new iterative methods for finding the approximate solutions of the nonexpansive mappings in the Banach spaces. We also consider the convergence criteria of these new iterations under some mild conditions and show that the fixed point of the nonexpansive mapping solves a certain variational inequality.

Let X be a real Banach space with dual X^* and C a nonempty closed convex subset of X. Let $J : X \to 2^{X^*}$ denote the normalized duality mapping defined by $J(x) := \{x^* \in X^* : \langle x, x^* \rangle = \|x\|^2, \|x^*\| = \|x\|, x \in X\}$.

Recall that a mapping $f : C \to C$ is called contractive if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leqslant \alpha \|x - y\|, x, y \in C$. We use Π_C to denote the collection of all contractive mappings on C. Let now $T : C \to C$ be a nonexpansive mapping; namely,

$$\|Tx - Ty\| \leqslant \|x - y\|,$$

for all $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Throughout the paper we assume that $F(T) \neq \emptyset$.

Construction of fixed points of nonexpansive mapping is an important subject in the fixed point theory and its applications in a number of applied areas, for example image recovery and signal processing (please see, e.g., [1-3]). Halpern [4] considered the following iterative scheme:

For a given $x_0 \in C$ and $u \in C$, find the approximate solution x_{n+1} by the

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T x_n. \tag{1}$$

He pointed out that both of the conditions (C1): $\lim_{n\to\infty} \alpha_n = 0$ and (C2): $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary in the sense that if the iteration scheme (1) converges to a fixed point of T, then these conditions must be satisfied. After that, many authors considered several conditions of the iterative method (1) concerning the choice of the parameters $\{\alpha_n\}$ (see [5-8]). In particular, Xu [9] considered the following iteration scheme

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) T x_n. \tag{2}$$

which is generalization of (1) and is known as the viscosity approximation method, the origin of which goes to Moudafi [10]. Xu [9] proved the strong convergence of the sequence $\{x_n\}$ by using the conditions (C1), (C2) and the following condition:

(C3): either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n\to\infty} (\alpha_{n+1}/\alpha_n) = 1$.

Essentially using the idea and technique of Noor [11-13], Su and Qin [15] considered the following modified Noor iteration for nonexpansive mappings.

$$\begin{cases} w_n = \delta_n x_n + (1 - \delta_n) T x_n, \\ z_n = \gamma_n x_n + (1 - \gamma_n) T w_n, \\ y_n = \beta_n x_n + (1 - \beta_n) T z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n) y_n, \end{cases} \tag{3}$$

It is clear from (3) that modified Noor iteration include the two-step and one-step iterations as special cases of Noor iterations.

Su and Qin [15] obtained the following result.

THEOREM SQ. *Let C be a closed convex subset of a uniformly smooth Banach space X and let $T : C \to C$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Given a point $u \in C$, the initial guess $x_0 \in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $[0,1]$, the following conditions are satisfied*

(*i*) $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(*ii*) $\beta_n + (1+\beta_n)(1-\gamma_n)(2-\delta_n) \in (0,a)$ *for some* $a \in (0,1)$;

(*iii*) $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$, $\sum_{n=0}^{\infty} |\beta_{n+1} - \beta_n| < \infty$, $\sum_{n=0}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$ *and* $\sum_{n=0}^{\infty} |\delta_{n+1} - \delta_n| < \infty$.

Then $\{x_n\}$ defined by (3) converges strongly to a fixed point of T.

Motivated and inspired by the ongoing research in this direction, we consider and construct two multi-step iterations algorithms for approximating fixed pints of nonexpansive mappings. The main purpose of this paper is twofold. First we extend Su and Qin's result [15] to a general situation with less restrictions on parameters for finding the fixed point of the nonexpansive mapping, which solve a certain variational inequality. Secondly, we propose a new modified Halpern iteration which enriches and complements the iterative methods of nonexpansive mappings, which is called the Noor-Halpern iteration. We prove that the the proposed iteration schemes converge strongly to a fixed point of T which solves some variational inequalities. Results proved in this paper may be viewed as an improvement and refinement of the previous known results.

2. Preliminaries

Let X be a real Banach space with its dual X^*. Let $S = \{x \in X : \|x\| = 1\}$ denote the unit sphere of X. The norm on X is said to be G$\hat{a}$teaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{4}$$

exists for each $x, y \in S$ and in this case X is said to be smooth. X is said to have a uniformly Frechet differentiable norm if the limit (4) is attained uniformly for $x, y \in S$ and in this case X is said to be uniformly smooth. It is well-known that if X is uniformly smooth then the duality map is norm-to-norm uniformly continuous on bounded subsets of X

The first lemma is very well-known (subdifferential) inequality.

LEMMA 2.1. *Let X be a real Banach space and J the normalized duality map on X. Then for any given $x, y \in X$, the following inequality holds:*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall j(x+y) \in J(x+y).$$

LEMMA 2.2. ([16]) *Let* $\{x_n\}$ *and* $\{z_n\}$ *be bounded sequences in a Banach space* X *and let* $\{\alpha_n\}$ *be a sequence in* $[0,1]$ *which satisfies the following condition*

$$0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1.$$

Suppose

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)z_n, \quad n \geqslant 0,$$

and

$$\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \leqslant 0.$$

Then $\lim_{n\to\infty}\|z_n - x_n\| = 0$.

LEMMA 2.3. ([9]) *Assume* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leqslant (1-\gamma_n)a_n + \delta_n, \quad n \geqslant 0,$$

where $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence in* R *such that*

(*i*) $\sum_{n=0}^{\infty}\gamma_n = \infty$;

(*ii*) $\limsup_{n\to\infty}\delta_n/\gamma_n \leqslant 0$ *or* $\sum_{n=0}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.4. ([9]) *Let* C *be a nonempty closed convex subset of a real uniformly Banach space* X. *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$, *and* $f \in \Pi_C$. *Then* $\{x_t\}$ *defined by*

$$x_t = tf(x_t) + (1-t)Tx_t, \quad t \in [0,1] \tag{5}$$

converges strongly to a point p *in* $F(T)$ *which is the unique solution of the variational inequality*

$$\langle (I-f)p, j(x-p)\rangle \geqslant 0, \quad x \in F(T). \tag{6}$$

LEMMA 2.5. *Let* C *be a nonempty closed convex subset of a real uniformly smooth Banach space* X. *Let* $T : C \to C$ *be a nonexpansive mapping with* $F(T) \neq \emptyset$, *and* $f \in \Pi_C$. *Given bounded sequence* $\{x_n\} \subset C$ *satisfying* $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$. *Then*

$$\limsup_{n\to\infty}\langle f(z) - z, j(x_n - z)\rangle \leqslant 0, \quad z \in F(T).$$

Proof. Let x_t be the unique fixed point of the contraction mapping z_t given by

$$z_t x = tf(x) + (1-t)Tx.$$

Then

$$x_t - x_n = t(f(x_t) - x_n) + (1-t)(Tx_t - x_n).$$

We apply Lemma 2.1 to get

$$\begin{aligned}\|x_t - x_n\|^2 &\leqslant (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n)\rangle \\ &\leqslant (1-t)^2(\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2 \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle + 2t\|x_t - x_n\|^2 \\ &\leqslant (1-t)^2\|x_t - x_n\|^2 + a_n(t) + 2t\|x_t - x_n\|^2 \\ &\quad + 2t\langle f(x_t) - x_t, j(x_t - x_n)\rangle,\end{aligned} \tag{7}$$

where

$$\begin{aligned} a_n(t) &= \|Tx_n - x_n\|(2\|x_t - x_n\| + \|Tx_n - x_n\|) \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{8}$$

The last inequality (7) implies

$$\langle x_t - f(x_t), j(x_t - x_n)\rangle \leqslant \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}a_n(t).$$

It follows that

$$\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \leqslant \frac{t}{2}M_1^2, \tag{9}$$

where $M_1 > 0$ is a constant such that $M_1 \geqslant \|x_t - x_n\|$ for all $t \in (0,1)$ and $n \geqslant 0$.

Letting $t \to 0$ in (9) yields

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \leqslant 0.$$

Moreover, we have that

$$\begin{aligned} &\langle z - f(z), j(z - x_n)\rangle \\ &= \langle z - f(z), j(z - x_n)\rangle - \langle z - f(z), j(x_t - x_n)\rangle \\ &\quad + \langle z - f(z), j(x_t - x_n)\rangle - \langle x_t - f(z), j(x_t - x_n)\rangle \\ &\quad + \langle x_t - f(z), j(x_t - x_n)\rangle - \langle x_t - f(x_t), j(x_t - x_n)\rangle \\ &\quad + \langle x_t - f(x_t), j(x_t - x_n)\rangle \\ &= \langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle \\ &\quad + \langle z - x_t, j(x_t - x_n)\rangle + \langle f(x_t) - f(z), j(x_t - x_n)\rangle \\ &\quad + \langle x_t - f(x_t), j(x_t - x_n)\rangle. \end{aligned}$$

Then, we obtain

$$\begin{aligned} &\limsup_{n\to\infty}\langle z - f(z), j(z - x_n)\rangle \\ &\leqslant \sup_{n\in N}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle \\ &\quad + \|z - x_t\|\limsup_{n\to\infty}\|x_t - x_n\| + \|f(x_t) - f(z)\|\limsup_{n\to\infty}\|x_t - x_n\| \\ &\quad + \limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle \\ &\leqslant \sup_{n\in N}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle \\ &\quad + (1+\alpha)\|z - x_t\|\limsup_{n\to\infty}\|x_t - x_n\| \\ &\quad + \limsup_{n\to\infty}\langle x_t - f(x_t), j(x_t - x_n)\rangle. \end{aligned}$$

From Lemma 2.4, we know that $x_t \to z \in F(T)$ as $t \to 0$ and j is norm-to-weak* uniformly continuous on bounded subset of C, we obtain

$$\limsup_{t\to 0}\sup_{n\in N}\langle z - f(z), j(z - x_n) - j(x_t - x_n)\rangle = 0.$$

Therefore we have

$$\begin{aligned}\limsup_{n\to\infty}\langle z-f(z),j(z-x_n)\rangle &= \limsup_{t\to 0}\limsup_{n\to\infty}\langle z-f(z),j(z-x_n)\rangle\\ &\leqslant \limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t-f(x_t),j(x_t-x_n)\rangle\\ &\leqslant 0.\end{aligned}$$

This completes the proof.

3. Main Results

3.1. Modified Noor iteration

Our first result is the continuous study of Theorem SQ. First we note that the restrictions on parameters given below are different from that of Theorem SQ. We now consider and study the following iteration which includes (3) as a special case. For $f \in \prod_C$ and $x_0 \in C$, find the approximate solution x_n by the iterative schemes.

$$\begin{cases} w_n = \delta_n x_n + (1-\delta_n)Tx_n,\\ z_n = \gamma_n x_n + (1-\gamma_n)Tw_n,\\ y_n = \beta_n x_n + (1-\beta_n)Tz_n,\\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)y_n, \end{cases} \tag{10}$$

which is called the modified Noor iteration. Clearly iteration (10) includes (3) as a special case. In a similar way, one can show that Noor iteration (10) include two-step (Ishikawa) and one-step(Mann) iteration as special cases. This shows that modified Noor iteration is more general and unified one.

Now we consider the convergence criteria of the modified Noor iteration (1) and is the main motivation of our next result.

THEOREM 3.1. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T: C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, suppose the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$;

(ii) $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$;

(iii) $\lim_{n\to\infty}(\gamma_{n+1}-\gamma_n) = 0$ *and* $\lim_{n\to\infty}(\delta_{n+1}-\delta_n) = 0$;

(iv) $\beta_n + (1-\beta_n)(1-\gamma_n)(2-\delta_n) \in [0,a)$ *for some* $a \in (0,1)$.

Then, for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (10) strongly converges to a fixed point $p \in F(T)$ which is the unique solution of the variational inequality (6).

Proof. Under the conditions of Theorem 3.1, we know from Lemma 2.4 that $\{x_t\}$ defined by (5) converges strongly to $p \in F(T)$ which is the unique solution of the variational inequality (6) in $F(T)$.

First, we observe that $\{x_n\}$ is bounded. Indeed, if we take a fixed point p of T, then we have

$$\begin{aligned}\|w_n - p\| &= \delta_n\|x_n - p\| + (1-\delta_n)\|Tx_n - p\| \\ &\leqslant \|x_n - p\|.\end{aligned} \tag{11}$$

It follows from (10) and (11) that

$$\begin{aligned}\|z_n - p\| &\leqslant \gamma_n\|x_n - p\| + (1-\gamma_n)\|Tw_n - p\| \\ &\leqslant \gamma_n\|x_n - p\| + (1-\gamma_n)\|w_n - p\| \\ &\leqslant \|x_n - p\|,\end{aligned}$$

and

$$\begin{aligned}\|y_n - p\| &\leqslant \beta_n\|x_n - p\| + (1-\beta_n)\|Tz_n - p\| \\ &\leqslant \beta_n\|x_n - p\| + (1-\beta_n)\|z_n - p\| \\ &\leqslant \|x_n - p\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant \alpha_n\|f(x_n) - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leqslant \alpha_n\|f(x_n) - f(p)\| + \alpha_n\|f(p) - p\| \\ &\quad + (1-\alpha_n)\|x_n - p\| \\ &\leqslant [1-(1-\alpha)\alpha_n]\|x_n - p\| + \alpha_n\|f(p) - p\| \\ &\leqslant \max\{\|f(p) - p\|/(1-\alpha), \|x_n - p\|\}.\end{aligned}$$

An induction yields

$$\|x_n - p\| \leqslant \max\{\|f(p) - p\|/(1-\alpha), \|x_0 - p\|\}, \quad n \geqslant 0.$$

Hence, $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$. Set $\sigma_n = (1-\alpha_n)\beta_n, \quad n \geqslant 0$. It follows from (i) and (ii) that

$$0 < \liminf_{n\to\infty} \sigma_n \leqslant \limsup_{n\to\infty} \sigma_n < 1. \tag{12}$$

Define

$$x_{n+1} = \sigma_n x_n + (1-\sigma_n)u_n. \tag{13}$$

Observe that

$$\begin{aligned}u_{n+1} - u_n &= \frac{x_{n+2} - \sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} - \frac{x_{n+1} - \sigma_n x_n}{1-\sigma_n} \\ &= \frac{\alpha_{n+1}f(x_{n+1}) + (1-\alpha_{n+1})y_{n+1} - \sigma_{n+1}x_{n+1}}{1-\sigma_{n+1}} \\ &\quad - \frac{\alpha_n f(x_n) + (1-\alpha_n)y_n - \sigma_n x_n}{1-\sigma_n} \\ &= \left(\frac{\alpha_{n+1}f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}\right) - \frac{(1-\alpha_n)(1-\beta_n)Tz_n}{1-\sigma_n} \\ &\quad + \frac{(1-\alpha_{n+1})(1-\beta_{n+1})Tz_{n+1}}{1-\sigma_{n+1}}\end{aligned}$$

$$= (\frac{\alpha_{n+1} f(x_{n+1})}{1-\sigma_{n+1}} - \frac{\alpha_n f(x_n)}{1-\sigma_n}) + Tz_{n+1} - Tz_n \\ - \frac{\alpha_{n+1} Tz_{n+1}}{1-\sigma_{n+1}} + \frac{\alpha_n Tz_n}{1-\sigma_n}. \tag{14}$$

It follows from (14) that

$$\begin{aligned} \|u_{n+1} - u_n\| - \|x_{n+1} - x_n\| \leqslant & \frac{\alpha_{n+1}}{1-\sigma_{n+1}} (\|f(x_{n+1})\| + \|Tz_{n+1}\|) \\ & + \frac{\alpha_n}{1-\sigma_n} (\|f(x_n)\| + \|Tz_n\|) \\ & + \|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|. \end{aligned} \tag{15}$$

From (10), we have

$$\begin{aligned} w_{n+1} - w_n = & \ \delta_{n+1} x_{n+1} - \delta_n x_n \\ & + (1-\delta_{n+1}) Tx_{n+1} - (1-\delta_n) Tx_n \\ = & \ (\delta_{n+1} - \delta_n) x_{n+1} + \delta_n (x_{n+1} - x_n) \\ & + (1-\delta_n)(Tx_{n+1} - Tx_n) + (\delta_n - \delta_{n+1}) Tx_{n+1}. \end{aligned}$$

It follows that

$$\begin{aligned} \|w_{n+1} - w_n\| \leqslant & \ |\delta_{n+1} - \delta_n| (\|x_{n+1}\| + \|Tx_{n+1}\|) \\ & + \delta_n \|x_{n+1} - x_n\| + (1-\delta_n) \|Tx_{n+1} - Tx_n\| \\ \leqslant & \ |\delta_{n+1} - \delta_n| (\|x_{n+1}\| + \|Tx_{n+1}\|) \\ & + \|x_{n+1} - x_n\|. \end{aligned} \tag{16}$$

Again from (10), we obtain

$$\begin{aligned} z_{n+1} - z_n = & \ \gamma_{n+1} x_{n+1} - \gamma_n x_n + (1-\gamma_{n+1}) Tw_{n+1} - (1-\gamma_n) Tw_n \\ = & \ (\gamma_{n+1} - \gamma_n) x_{n+1} + \gamma_n (x_{n+1} - x_n) \\ & + (1-\gamma_n)(Tw_{n+1} - Tw_n) + (\gamma_n - \gamma_{n+1}) Tw_{n+1}. \end{aligned} \tag{17}$$

It follows from (16) and (17) that

$$\begin{aligned} \|z_{n+1} - z_n\| \leqslant & \ |\gamma_{n+1} - \gamma_n| (\|x_{n+1}\| + \|Tw_{n+1}\|) \\ & + \gamma_n \|x_{n+1} - x_n\| + (1-\gamma_n) \|Tw_{n+1} - Tw_n\| \\ \leqslant & \ |\gamma_{n+1} - \gamma_n| (\|x_{n+1}\| + \|Tw_{n+1}\|) \\ & + \gamma_n \|x_{n+1} - x_n\| + (1-\gamma_n) \|w_{n+1} - w_n\| \\ \leqslant & \ |\gamma_{n+1} - \gamma_n| (\|x_{n+1}\| + \|Tw_{n+1}\|) + \gamma_n \|x_{n+1} - x_n\| \\ & + (1-\gamma_n) |\delta_{n+1} - \delta_n| (\|x_{n+1}\| + \|Tx_{n+1}\|) \\ & + (1-\gamma_n) \|x_{n+1} - x_n\| \\ \leqslant & \ |\gamma_{n+1} - \gamma_n| (\|x_{n+1}\| + \|Tw_{n+1}\|) \\ & + |\delta_{n+1} - \delta_n| (\|x_{n+1}\| + \|Tx_{n+1}\|) + \|x_{n+1} - x_n\|. \end{aligned} \tag{18}$$

Substituting (18) into (15) that

$$\begin{aligned}\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\| \leqslant{}& \frac{\alpha_{n+1}}{1-\sigma_{n+1}}(\|f(x_{n+1})\|+\|Tz_{n+1}\|)\\ &+\frac{\alpha_n}{1-\sigma_n}(\|f(x_n)\|+\|Tz_n\|)\\ &+|\gamma_{n+1}-\gamma_n|(\|x_{n+1}\|+\|Tw_{n+1}\|)\\ &+|\delta_{n+1}-\delta_n|(\|x_{n+1}\|+\|Tx_{n+1}\|).\end{aligned} \tag{19}$$

Since $\{x_n\}$, $\{f(x_n)\}$, $\{Tx_n\}$, $\{Tz_n\}$ and $\{Tw_n\}$ are bounded, by (i),(iii) and (19) we obtain that

$$\limsup_{n\to\infty}(\|u_{n+1}-u_n\|-\|x_{n+1}-x_n\|)\leqslant 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{20}$$

It follows from (12), (13) and (20) that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{21}$$

From (10), we have

$$\begin{aligned}\|x_n-Tx_n\| &\leqslant \|x_{n+1}-x_n\|+\|x_{n+1}-Tx_n\|\\ &\leqslant \|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-Tx_n\|+(1-\alpha_n)\|y_n-Tx_n\|\\ &\leqslant |x_{n+1}-x_n\|+\alpha_n\|f(x_n)-Tx_n\|+(1-\alpha_n)\beta_n\|x_n-Tx_n\|\\ &\quad+(1-\alpha_n)(1-\beta_n)\|Tz_n-Tx_n\|\\ &\leqslant \|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-Tx_n\|+(1-\alpha_n)\beta_n\|x_n-Tx_n\|\\ &\quad+(1-\alpha_n)(1-\beta_n)\|z_n-x_n\|.\end{aligned} \tag{22}$$

Note that

$$\begin{aligned}\|z_n-x_n\| &= (1-\gamma_n)\|Tw_n-x_n\|\\ &\leqslant (1-\gamma_n)\|Tw_n-Tx_n\|+(1-\gamma_n)\|x_n-Tx_n\|\\ &\leqslant (1-\gamma_n)\|w_n-x_n\|+(1-\gamma_n)\|x_n-Tx_n\|\\ &\leqslant (1-\gamma_n)(1-\delta_n)\|x_n-Tx_n\|+(1-\gamma_n)\|x_n-Tx_n\|\\ &= (1-\gamma_n)(2-\delta_n)\|x_n-Tx_n\|.\end{aligned} \tag{23}$$

Substituting (23) into (22) that

$$\begin{aligned}\|x_n-Tx_n\| \leqslant{}& \|x_{n+1}-x_n\|+\alpha_n\|f(x_n)-Tx_n\|\\ &+(1-\alpha_n)[\beta_n+(1-\beta_n)(1-\gamma_n)(2-\delta_n)]\|x_n-Tx_n\|.\end{aligned} \tag{24}$$

From (21), (i), (iv) and (24), we obtain

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.$$

It follows from Lemma 2.5 that

$$\limsup_{n\to\infty}\langle f(p)-p, j(x_n-p)\rangle \leqslant 0.$$

Finally we show that $x_n \to p$ as $n \to \infty$.

Write

$$x_{n+1}-p=\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p),$$

and apply Lemma 2.1 to get

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|\alpha_n(f(x_n)-p)+(1-\alpha_n)(y_n-p)\|^2\\
&\leqslant (1-\alpha_n)^2\|y_n-p\|^2+2\alpha_n\langle f(x_n)-p, j(x_{n+1}-p)\rangle\\
&\leqslant (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\langle f(x_n)-f(p), j(x_{n+1}-p)\rangle\\
&\quad +2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\leqslant (1-\alpha_n)^2\|x_n-p\|^2+2\alpha_n\|f(x_n)-f(p)\|\|x_{n+1}-p\|\\
&\quad +2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\leqslant (1-\alpha_n)^2\|x_n-p\|^2+2\alpha\alpha_n\|x_n-p\|\|x_{n+1}-p\|\\
&\quad +2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\leqslant (1-\alpha_n)^2\|x_n-p\|^2+\alpha\alpha_n(\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\
&\quad +2\alpha_n\langle f(p)-p, j(x_{n+1}-p)\rangle .
\end{aligned}$$

It then follows that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leqslant \frac{1-(2-\alpha)\alpha_n+\alpha_n^2}{1-\alpha\alpha_n}\|x_n-p\|^2\\
&\quad +\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&= (1-\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n})\|x_n-p\|^2+\frac{2\alpha_n}{1-\alpha\alpha_n}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad +\frac{\alpha_n^2}{1-\alpha\alpha_n}\|x_n-p\|^2,
\end{aligned}$$

that is

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= (1-s_n)\|x_n-p\|^2+s_n[\frac{1}{1-\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad +\frac{\alpha_n}{2(1-\alpha)}\|x_n-p\|^2]\\
&\leqslant (1-s_n)\|x_n-p\|^2+s_n[\frac{1}{1-\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle\\
&\quad +\frac{\alpha_n}{2(1-\alpha)}M_2]\\
&= (1-s_n)\|x_n-p\|^2+t_n,
\end{aligned} \tag{25}$$

where $s_n=\frac{2(1-\alpha)\alpha_n}{1-\alpha\alpha_n}$, $t_n=s_n[\frac{1}{1-\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle+\frac{\alpha_n}{2(1-\alpha)}M_2]$ and $M_2>0$ is a constant such that $\|x_n-p\|^2\leqslant M_2,\ n\geqslant 0$.

It is easily seen that $\sum_{n=0}^{\infty} s_n = \infty$, and

$$\limsup_{n\to\infty} t_n/s_n = \limsup_{n\to\infty}[\frac{1}{1-\alpha}\langle f(p)-p, j(x_{n+1}-p)\rangle + \frac{\alpha_n}{2(1-\alpha)}M_2] \leqslant 0.$$

Finally apply Lemma 2.3 to (25) and conclude that $x_n \to p$ as $n \to \infty$. This completes the proof.

COROLLARY 3.2. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $u \in C$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$, the following conditions are satisfied:*

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$*;*

(*ii*) $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 1$*;*

(*iii*) $\lim_{n\to\infty}(\gamma_{n+1}-\gamma_n) = 0$ *and* $\lim_{n\to\infty}(\delta_{n+1}-\delta_n) = 0$*;*

(*iv*) $\beta_n + (1-\beta_n)(1-\gamma_n)(2-\delta_n) \in [0,a)$ *for some* $a \in (0,1)$*.*

Then, for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (3) strongly converges to a fixed point $p \in F(T)$.

REMARK 3.3. Taking $f(x) \equiv u$ in (10), we immediately obtain (3), that is to say, our iteration scheme (10) includes (3) as a special case.

3.2. Modified Noor-Halpern iteration

In this section, we consider and analyze another iteration for finding the approximate point of the nonexpansive mapping and this is the main motivation of this section.

For given $x_0 \in C$, find the approximate solution x_n by the iterative scheme:

$$\begin{cases} z_n = \gamma_n f(x_n) + (1-\gamma_n)x_n, \\ y_n = \beta_n f(x_n) + (1-\beta_n)Tz_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n \end{cases} \tag{26}$$

which is called the modified Noor-Halpern iteration. Now we state and study the convergence result of iteration scheme (26).

THEOREM 3.4. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Given sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$, suppose the following conditions are satisfied:*

(*i*) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$*;*

(*ii*) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$*;*

(*iii*) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\frac{\gamma_n}{\beta_n} \leqslant M$ *for some* $M > 0$*.*

Then, for arbitrary $x_0 \in C$, *the sequence* $\{x_n\}$ *defined by (26) converges strongly to a fixed point* $p \in F(T)$ *which is the unique solution of the variational inequality (6).*

Proof. First we prove that $\{x_n\}$ is bounded. Take $p \in F(T)$, from (26), we have

$$\begin{aligned}\|z_n - p\| &\leqslant \gamma_n\|f(x_n) - p\| + (1-\gamma_n)\|x_n - p\| \\ &\leqslant \gamma_n\|f(x_n) - f(p)\| + \gamma_n\|f(p) - p\| + (1-\gamma_n)\|x_n - p\| \\ &\leqslant \alpha\gamma_n\|x_n - p\| + \gamma_n\|f(p) - p\| + (1-\gamma_n)\|x_n - p\| \\ &= [1-(1-\alpha)\gamma_n]\|x_n - p\| + \gamma_n\|f(p) - p\|,\end{aligned}$$

and hence

$$\begin{aligned}\|y_n - p\| &\leqslant \beta_n\|f(x_n) - p\| + (1-\beta_n)\|Tz_n - p\| \\ &\leqslant \beta_n\|f(x_n) - f(p)\| + \beta_n\|f(p) - p\| + (1-\beta_n)\|z_n - p\| \\ &\leqslant \alpha\beta_n\|x_n - p\| + \beta_n\|f(p) - p\| + (1-\beta_n)\gamma_n\|f(p) - p\| \\ &\quad + (1-\beta_n)[1-(1-\alpha)\gamma_n]\|x_n - p\| \\ &= \{\alpha\beta_n + (1-\beta_n)[1-(1-\alpha)\gamma_n]\}\|x_n - p\| \\ &\quad + [\beta_n + (1-\beta_n)\gamma_n]\|f(p) - p\| \\ &= \{1-(1-\alpha)[\beta_n + (1-\beta_n)\gamma_n]\}\|x_n - p\| \\ &\quad + [\beta_n + (1-\beta_n)\gamma_n]\|f(p) - p\|.\end{aligned}$$

Therefore

$$\begin{aligned}\|x_{n+1} - p\| &\leqslant \alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\| \\ &\leqslant \alpha_n\|x_n - p\| + (1-\alpha_n)\{1-(1-\alpha)[\beta_n + (1-\beta_n)\gamma_n]\}\|x_n - p\| \\ &\quad + (1-\alpha_n)[\beta_n + (1-\beta_n)\gamma_n]\|f(p) - p\| \\ &= \{1-(1-\alpha)(1-\alpha_n)[\beta_n + (1-\beta_n)\gamma_n]\}\|x_n - p\| \\ &\quad + (1-\alpha_n)[\beta_n + (1-\beta_n)\gamma_n]\|f(p) - p\| \\ &\leqslant \max\{\|x_n - p\|, \|f(p) - p\|/(1-\alpha)\}.\end{aligned}$$

By induction

$$\|x_n - p\| \leqslant \max\{\|x_0 - p\|, \|f(p) - p\|/(1-\alpha)\}, \quad n \geqslant 0,$$

that is, $\{x_n\}$ is bounded, so are $\{f(x_n)\}$, $\{Tx_n\}$ and $\{z_n\}$.

We observe that

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|(\beta_{n+1} - \beta_n)f(x_{n+1}) + \beta_n(f(x_{n+1}) - f(x_n)) \\ &\quad + (1-\beta_n)(Tz_{n+1} - Tz_n) + (\beta_n - \beta_{n+1})Tz_{n+1} \\ &\leqslant |\beta_{n+1} - \beta_n|(\|f(x_{n+1})\| + \|Tz_{n+1}\|) + \beta_n\|f(x_{n+1}) - f(x_n)\| \\ &\quad + (1-\beta_n)\|z_{n+1} - z_n\| \\ &\leqslant |\beta_{n+1} - \beta_n|(\|f(x_{n+1})\| + \|Tz_{n+1}\|) + \alpha\beta_n\|x_{n+1} - x_n\| \\ &\quad + (1-\beta_n)\|z_{n+1} - z_n\|,\end{aligned} \tag{27}$$

and

$$\begin{aligned}\|z_{n+1}-z_n\| &= \|(\gamma_{n+1}-\gamma_n)f(x_{n+1})+\gamma_n(f(x_{n+1})-f(x_n))\\&\quad+(1-\gamma_n)(x_{n+1}-x_n)+(\gamma_n-\gamma_{n+1})x_{n+1}\|\\&\leqslant|\gamma_{n+1}-\gamma_n|(\|f(x_{n+1})\|+\|x_{n+1}\|)+\alpha\gamma_n\|x_{n+1}-x_n\|\\&\quad+(1-\gamma_n)\|x_{n+1}-x_n\|\\&\leqslant|\gamma_{n+1}-\gamma_n|(\|f(x_{n+1})\|+\|x_{n+1}\|)+\|x_{n+1}-x_n\|.\end{aligned}\tag{28}$$

Substituting (28) into (27) that

$$\begin{aligned}\|y_{n+1}-y_n\| \leqslant |\beta_{n+1}-\beta_n|(\|f(x_{n+1})\|+\|Tz_{n+1}\|)\\+|\gamma_{n+1}-\gamma_n|(\|f(x_{n+1})\|+\|x_{n+1}\|)+\|x_{n+1}-x_n\|.\end{aligned}\tag{29}$$

It follows from $\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\gamma_n=0$ and (29) that

$$\limsup_{n\to\infty}(\|y_{n+1}-y_n\|-\|x_{n+1}-x_n\|)\leqslant 0.$$

Hence, by Lemma 2.2, we have

$$\lim_{n\to\infty}\|y_n-x_n\|=0.\tag{30}$$

Then

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\alpha_n)\|y_n-x_n\|=0.\tag{31}$$

From (26), we obtain

$$\begin{aligned}\|x_n-Tx_n\| &\leqslant \|x_{n+1}-x_n\|+\|x_{n+1}-Tx_n\|\\&\leqslant\|x_{n+1}-x_n\|+\alpha_n\|x_n-Tx_n\|+(1-\alpha_n)\|y_n-Tx_n\|\\&\leqslant\|x_{n+1}-x_n\|+\alpha_n\|x_n-Tx_n\|+(1-\alpha_n)\beta_n\|f(x_n)-Tx_n\|\\&\quad+(1-\alpha_n)(1-\beta_n)\|Tz_n-Tx_n\|\\&\leqslant\|x_{n+1}-x_n\|+\alpha_n\|x_n-Tx_n\|+(1-\alpha_n)\beta_n\|f(x_n)-Tx_n\|\\&\quad+(1-\alpha_n)(1-\beta_n)\gamma_n\|f(x_n)-x_n\|.\end{aligned}\tag{32}$$

We note that $\lim_{n\to\infty}\beta_n=\lim_{n\to\infty}\gamma_n=0$ and, $\{x_n\}$, $\{f(x_n)\}$ and $\{Tx_n\}$ are all bounded, therefore from (32) with (31), we have

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0.\tag{33}$$

Since

$$\begin{aligned}\|y_n-Ty_n\| &\leqslant \|y_n-Tx_n\|+\|Tx_n-Ty_n\|\\&\leqslant\|y_n-x_n\|+\|x_n-Tx_n\|+\|y_n-x_n\|\\&=2\|y_n-x_n\|+\|x_n-Tx_n\|.\end{aligned}\tag{34}$$

It follows from (30), (33) and (34) that

$$\lim_{n\to\infty}\|y_n-Ty_n\|=0.\tag{35}$$

Again by similar way,

$$\begin{aligned}\|z_n - Tz_n\| &\leqslant \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Tz_n\| \\ &\leqslant \gamma_n\|f(x_n) - x_n\| + \|x_n - y_n\| + \|y_n - Tz_n\| \\ &\leqslant \gamma_n\|f(x_n) - x_n\| + \|x_n - y_n\| + \beta_n\|f(x_n) - Tz_n\|,\end{aligned}$$

which implies that (noting that (i), (iii) and (30))

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{36}$$

It follows from (35), (36) and Lemma 2.5 that

$$\limsup_{n\to\infty}\langle f(p) - p, j(y_n - p)\rangle \leqslant 0 \text{ and } \limsup_{n\to\infty}\langle f(p) - p, j(z_n - p)\rangle \leqslant 0. \tag{37}$$

From (26) and Lemma 2.1, we have

$$\begin{aligned}\|z_n - p\|^2 &= \|\gamma_n(f(x_n) - p) + (1-\gamma_n)(x_n - p)\|^2 \\ &\leqslant (1-\gamma_n)^2\|x_n - p\|^2 + 2\gamma_n\langle f(x_n) - p, j(z_n - p)\rangle \\ &\leqslant (1-\gamma_n)^2\|x_n - p\|^2 + 2\gamma_n\langle f(x_n) - f(p), j(z_n - p)\rangle \\ &\quad + 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle \\ &\leqslant (1-\gamma_n)^2\|x_n - p\|^2 + 2\alpha\gamma_n\|x_n - p\|\|z_n - p\| \\ &\quad + 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle \\ &\leqslant (1-\gamma_n)^2\|x_n - p\|^2 + \alpha\gamma_n(\|x_n - p\|^2 + \|z_n - p\|^2) \\ &\quad + 2\gamma_n\langle f(p) - p, j(z_n - p)\rangle,\end{aligned}$$

that is

$$\begin{aligned}\|z_n - p\|^2 \leqslant [1 - \frac{2(1-\alpha)\gamma_n}{1-\alpha\gamma_n}]\|x_n - p\|^2 + \frac{\gamma_n^2}{1-\alpha\gamma_n}\|x_n - p\|^2 \\ + \frac{2\gamma_n}{1-\alpha\gamma_n}\langle f(p) - p, j(z_n - p)\rangle,\end{aligned} \tag{38}$$

and

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n(f(x_n) - p) + (1-\beta_n)(Tz_n - p)\|^2 \\ &\leqslant (1-\beta_n)^2\|Tz_n - p\|^2 + 2\beta_n\langle f(x_n) - p, j(y_n - p)\rangle \\ &\leqslant (1-\beta_n)^2\|z_n - p\|^2 + 2\beta_n\langle f(x_n) - f(p), j(y_n - p)\rangle \\ &\quad + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\ &\leqslant (1-\beta_n)^2\|z_n - p\|^2 + 2\alpha\beta_n\|x_n - p\|\|y_n - p\| \\ &\quad + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\ &\leqslant (1-\beta_n)^2\|z_n - p\|^2 + \alpha\beta_n(\|x_n - p\|^2 + \|y_n - p\|^2) \\ &\quad + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle \\ &\leqslant (1-\beta_n)^2[1 - \frac{2(1-\alpha)\gamma_n}{1-\alpha\gamma_n}]\|x_n - p\|^2 + \frac{\gamma_n^2}{1-\alpha\gamma_n}\|x_n - p\|^2 \\ &\quad + \frac{2\gamma_n}{1-\alpha\gamma_n}\langle f(p) - p, j(z_n - p)\rangle + \alpha\beta_n\|x_n - p\|^2 \\ &\quad + \alpha\beta_n\|y_n - p\|^2 + 2\beta_n\langle f(p) - p, j(y_n - p)\rangle,\end{aligned}$$

which implies that

$$\begin{aligned} \|y_n - p\|^2 \leqslant [1 - \frac{2(1-\alpha)\beta_n}{1-\alpha\beta_n}]\|x_n - p\|^2 + [\frac{\beta_n^2}{1-\alpha\beta_n} \\ + \frac{\gamma_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)}]\|x_n - p\|^2 \\ + \frac{2\gamma_n}{(1-\alpha\beta_n)(1-\alpha\gamma_n)}\langle f(p) - p, j(z_n - p)\rangle \\ + \frac{2\beta_n}{1-\alpha\beta_n}\langle f(p) - p, j(y_n - p)\rangle . \end{aligned} \tag{39}$$

Again from (26)

$$\begin{aligned} \|x_{n+1} - p\|^2 &= \|\alpha_n(x_n - p) + (1-\alpha_n)(y_n - p)\|^2 \\ &\leqslant [\alpha_n\|x_n - p\| + (1-\alpha_n)\|y_n - p\|]^2 \\ &\leqslant \alpha_n^2\|x_n - p\|^2 + (1-\alpha_n)^2\|y_n - p\|^2 \\ &\quad + 2\alpha_n(1-\alpha_n)\|x_n - p\|\|y_n - p\| \\ &\leqslant \alpha_n^2\|x_n - p\|^2 + (1-\alpha_n)^2\|y_n - p\|^2 \\ &\quad + \alpha_n(1-\alpha_n)(\|x_n - p\|^2 + \|y_n - p\|^2) \\ &= \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|y_n - p\|^2. \end{aligned} \tag{40}$$

Substituting (38) and (39) into (40), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 \leqslant [1 - \frac{2(1-\alpha)(1-\alpha_n)\beta_n}{1-\alpha\beta_n}]\|x_n - p\|^2 + [\frac{(1-\alpha_n)\beta_n^2}{1-\alpha\beta_n} \\ + \frac{(1-\alpha_n)\gamma_n^2}{(1-\alpha\beta_n)(1-\alpha\gamma_n)}]\|x_n - p\|^2 \\ + \frac{2(1-\alpha_n)\gamma_n}{(1-\alpha\beta_n)(1-\alpha\gamma_n)}\langle f(p) - p, j(z_n - p)\rangle \\ + \frac{2(1-\alpha_n)\beta_n}{1-\alpha\beta_n}\langle f(p) - p, j(y_n - p)\rangle \\ = (1-\delta_n)\|x_n - p\|^2 + \delta_n\sigma_n, \end{aligned} \tag{41}$$

where $\delta_n = \frac{2(1-\alpha)(1-\alpha_n)\beta_n}{1-\alpha\beta_n}$ and

$$\begin{aligned} \sigma_n = \{[\frac{\beta_n}{2(1-\alpha)} + \frac{\gamma_n^2}{2(1-\alpha)(1-\alpha\gamma_n)\beta_n}]M_3 \\ + \frac{\gamma_n}{(1-\alpha)(1-\alpha\gamma_n)\beta_n}\langle f(p) - p, j(z_n - p)\rangle \\ + \frac{1}{1-\alpha}\langle f(p) - p, j(y_n - p)\rangle \}, \end{aligned}$$

where $M_3 > 0$ is a constant such that $\|x_n - p\|^2 \leqslant M_3$ for all $n \geqslant 0$. It is easily seen from (i)-(iii) and (37) that

$$\sum_{n=0}^{\infty} \delta_n = \infty \quad \text{and} \quad \limsup_{n\to\infty} \sigma_n \leqslant 0.$$

Finally apply Lemma 2.3 to (41) and conclude that $x_n \to p$. This completes the proof.

REMARK 3.5. Taking $\gamma_n \equiv 0$ for all $n \geqslant 0$ in (26), then we have

$$\begin{cases} y_n = \beta_n f(x_n) + (1-\beta_n)Tx_n, \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n)y_n. \end{cases} \tag{42}$$

The following result is an immediate consequence of Theorem 3.4.

COROLLARY 3.6. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and $f \in \Pi_C$. Given sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in $(0,1)$, the following conditions are satisfied:*

(i) $\lim_{n\to\infty} \beta_n = 0$ *and* $\sum_{n=0}^{\infty} \beta_n = \infty$;

(ii) $0 < \liminf_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} \alpha_n < 1$.

Then, for arbitrary $x_0 \in C$, the sequence $\{x_n\}$ defined by (42) strongly converges to a fixed point $p \in F(T)$ which is the unique solution of the variational inequality (6).

REFERENCES

[1] F. E. BROWDER, W. V. PETRYSHYN, *Construction of fixed points of nonlinear mappings*, J. Math. Anal. Appl. **20** (1967), 197-228.

[2] C. BYRNE, *A unified treatment of some iterative algorithms in signal processing and image reconstruction*, Inverse Problems **20** (2004), 103-120.

[3] C. I. PODILCHUK, R. J. MAMMONE, *Image recovery by convex projections using a least-squares constraint*, J. Opt. Soc. Amer. **7** (1990), 517-521.

[4] B. HALPERN, *Fixed points of nonexpansive maps*, Bull. Amer. Math. Soc. **73** (1967), 957-961.

[5] S. REICH, *Strong convergence theorems for resolvents of accretive operators in Banach spaces*, J. Math. Anal. Appl. **75** (1980), 128-292.

[6] N. SHIOJI, W. TAKAHASHI, *Strong convergence of approximated sequence for nonexpansive mappings*, Proc. Amer. Math. Soc. **125** (1997), 3641-3645.

[7] R. WITTMANN, *Approximation of fixed points of nonexpansive mappings*, Arch. Math. **58** (1992), 486-491.

[8] H. K. XU, *Another control condition in an iterative method for nonexpansive mappings*, Bull. Austral. Math. Soc. **65** (2002), 109-113.

[9] H. K. XU, *Viscosity approximation methods for nonexpansive mappings*, J. Math. Anal. Appl. **298** (2004), 279-291.

[10] A. MOUDAFI, *Viscosity approximation methods for fixed points problems*, J. Math. Anal. Appl. **241** (2000), 46-55.

[11] M. ASLAM NOOR, *New approximation schemes for general variational inequalities*, J. Math. Anal. Appl. **251** (2000), 271-229.

[12] M. ASLAM NOOR, *Three-step iterative algorithms for multivalued quasi-variational inclusions*, J. Math. Anal. Appl. **255** (2001), 589-604.

[13] M. ASLAM NOOR,, *Some developments in general variational inequalities,*, Appl. Math. Computation **152** (2004), 199-277.

[14] Y. YAO AND M. ASLAM NOOR, *On viscosity methods for variational inequalities*, J. Math. Anal. Appl. **325** (2007), 776-787.
[15] Y. SU, X. QIN, *Strong convergence of modified Noor iterations*, International J. Math. Math. Sci. (2006), 1-11.
[16] T. SUZUKI, *Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals*, J. Math. Anal. Appl. **305** (2005), 227-239.
[17] M. ASLAM NOOR, *General variational inequalities and nonexpansive mappings*, J. Math. Anal. Appl. **331** (2007), 810-822.
[18] M. ASLAM NOOR AND A. BNOUHACHEM, *On an iterative algorithm for general variational inequalities,*, Appl. Math. Computation **185** (2007), 155-168.
[19] M. ASLAM NOOR AND Z. HUANG, *Three-step methods for nonexpansive mappings and variational inequalities*, Appl. Math. Computation **187** (2007), 680-685.
[20] M. ASLAM NOOR, K. INAYAT NOOR AND TH. M. RASSIAS, *Some aspects of variational inequalities*, J. Comput. Appl. Math. **47** (1993), 285-312.
[21] T. SUZUKI, *Moudafi's viscosity approximation with Meir-Keeler contraction*, J. Math. Anal. Appl. **325** (2007), 342-352.
[22] I. CHRYSSOVERGHI, I. COLETSOS AND B. KOKKINIS, *Discretization methods for noconvex control problems with state constriants*, J. Comput. Appl. Math. **191** (2006), 1-31.
[23] I. CHRYSSOVERGHI, I. COLETSOS AND B. KOKKINIS, *Discretization methods for noconvex control problems with state constriants*, Numer. Funct. Anal. Optim. **26** (2005), 321-348.
[24] I. CHRYSSOVERGHI, I. COLETSOS AND B. KOKKINIS, *Approximate relaxed descent method for optimal control problems*, Control Cyber. **30** (2001), 384-404.
[25] I. CHRYSSOVERGHI, A. BACOPOULOS I. COLETSOS AND B. KOKKINIS, *Discrete appromation of nonconvex hyperbolic optimal control problems with state constraints Approximate relaxed descent method for optimal control problems*, Control Cyber. **27** (1988), 29-50.

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