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FUNCTIONAL INEQUALITIES FOR GALUÉ'S GENERALIZED MODIFIED BESSEL FUNCTIONS

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Abstract. Let

$$ {}_aI_p(x) = \sum_{n\geqslant 0} \frac{(x/2)^{2n+p}}{n!\Gamma(p+an+1)} $$

be the Galué's generalized modified Bessel function depending on parameters $a = 0, 1, 2, \ldots$ and $p > -1$. Consider the function ${}_a\mathcal{I}_p : \mathbb{R} \to \mathbb{R}$, defined by ${}_a\mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}{}_aI_p(x)$. Motivated by the inequality of Lazarević, namely

$$ \cosh x < \left(\frac{\sinh x}{x}\right)^3 $$

for $x \neq 0$, in order to generalize this inequality we prove that the Turán-type, Lazarević-type inequalities

$$ [{}_a\mathcal{I}_{p+1}(x)]^2 \leqslant {}_a\mathcal{I}_p(x){}_a\mathcal{I}_{p+2}(x), \qquad [{}_a\mathcal{I}_p(x)]^{p+1} \leqslant [{}_a\mathcal{I}_{p+1}(x)]^{p+a+1} $$

hold for all $x \in \mathbb{R}$. Moreover, we prove that the functions

$$ p \mapsto {}_a\mathcal{I}_{p+1}(x)/{}_a\mathcal{I}_p(x), \qquad p \mapsto [{}_a\mathcal{I}_p(x)]^{(p+1)(p+2)\ldots(p+a)} $$

are increasing on $(-1, \infty)$.

1. Introduction and Preliminaries

Let us consider the well-known second-order Bessel differential equation [20, p. 38]

$$ x^2y''(x) + xy'(x) + (x^2 - p^2)y(x) = 0. \tag{1} $$

The function J_p, which is called the Bessel function of the first kind of order p, is defined as a particular solution of (1). This function has the form [20, p. 40]

$$ J_p(x) = \sum_{n\geqslant 0} \frac{(-1)^n}{n!\Gamma(p+n+1)}\left(\frac{x}{2}\right)^{2n+p}, \; x \in \mathbb{R}. \tag{2} $$

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The differential equation [20, p. 77]

$$x^2y''(x)+xy'(x)-(x^2+p^2)y(x)=0, \tag{3}$$

which differs from Bessel's equation only in the coefficient of y, is of frequent occurrence in problems of mathematical physics. The particular solution of (3) is called the modified Bessel function of the first kind of order p, and is defined by the formula [20, p. 77]

$$I_p(x)=\sum_{n\geqslant 0}\frac{1}{n!\Gamma(p+n+1)}\left(\frac{x}{2}\right)^{2n+p},\ \in\mathbb{R}. \tag{4}$$

Recently, Galué [13] introduced a generalized Bessel function of the form

$$_aJ_p(x)=\sum_{n\geqslant 0}\frac{(-1)^n}{n!\Gamma(p+an+1)}\left(\frac{x}{2}\right)^{2n+p},\ x\in\mathbb{R}, \tag{5}$$

where a is a natural number and for this generalized Bessel function he derived some differential properties and integral representations. Observe that $_1J_p=J_p$. We define Galué's generalized modified Bessel function analogously with (5) by the following formula:

$$_aI_p(x)=\sum_{n\geqslant 0}\frac{1}{n!\Gamma(p+an+1)}\left(\frac{x}{2}\right)^{2n+p},\ x\in\mathbb{R},\ a\in\mathbb{N}. \tag{6}$$

Let us consider the function $_a\mathcal{I}_p(x)=2^p\Gamma(p+1)x^{-p}{}_aI_p(x)$ and denote $_1\mathcal{I}_p$ simply with $\mathcal{I}_p$. In 1951 Thiruvenkatachar and Nanjundiah [19] have proved the Turán-type inequality

$$[\mathcal{I}_p(x)]^2\leqslant\mathcal{I}_{p-1}(x)\mathcal{I}_{p+1}(x)\iff I_p^2(x)-I_{p-1}(x)I_{p+1}(x)\leqslant\frac{I_p^2(x)}{p+1}, \tag{7}$$

by comparing the coefficients in the Cauchy product

$$I_p(x)I_q(x)=\sum_{n\geqslant 0}\binom{p+q+2n}{n}\frac{(x/2)^{p+q+2n}}{\Gamma(p+n+1)\Gamma(q+n+1)}.$$

For more details about the Turán type inequalities the interested reader is referred to the most recent papers [1], [7], [10], [15] on this topic and to the references therein.

In 1991 Joshi and Bissu [14] examined an alternate derivation of (7) and slightly extended this inequality. Recently in [8] using a different approach we rediscovered the inequality (7), moreover we improved the extension of Joshi and Bissu, proving that the function $p\mapsto\mathcal{I}_{p+1}(x)/\mathcal{I}_p(x)$ is increasing, i.e. if $p\geqslant q>-1$, then we have that [8, Theorem 1.8]

$$\mathcal{I}_{p+1}(x)\mathcal{I}_q(x)\geqslant\mathcal{I}_p(x)\mathcal{I}_{q+1}(x). \tag{8}$$

On the other hand there is another inequality of special interest, namely the Lazarević inequality [16, p. 270]

$$\mathcal{I}_{-1/2}(x)=\cosh x\leqslant\left(\frac{\sinh x}{x}\right)^3=[\mathcal{I}_{1/2}(x)]^3,$$

or equivalently the inequality

$$[\mathcal{I}_{-1/2}(x)]^{-1/2+1} \leqslant [\mathcal{I}_{1/2}(x)]^{1/2+1}, \tag{9}$$

where $x \in \mathbb{R}$ and the exponent 3 is the least possible. In the same paper [8, Theorem 1.8] using (7) we generalized (9) proving that if $p > -1,$ then

$$[\mathcal{I}_p(x)]^{p+1} \leqslant [\mathcal{I}_{p+1}(x)]^{p+2},\ x \in \mathbb{R}. \tag{10}$$

Moreover using (8) we proved that [8, Theorem 1.8] for $x \in \mathbb{R},\ p > -1$ the function $p \mapsto [\mathcal{I}_p(x)]^{p+1}$ is increasing, i.e. we have $\partial[\mathcal{I}_p(x)]^{p+1}/\partial p \geqslant 0.$ Our aim in Section 2 is to extend these results to Galué's generalized modified Bessel functions. Theorem 1, 2 are the main results of Section 2. In Section 3 we derive some other properties of the functions ${}_aI_p$ and ${}_a\mathcal{I}_p.$

2. Generalization of Lazarević's inequality

Our first main result reads as follows.

THEOREM 1. *If $a = 0,1,2,\dots$ and $p > -1,$ then the inequalities*

$$[{}_a\mathcal{I}_{p+1}(x)]^2 \leqslant {}_a\mathcal{I}_p(x)\,{}_a\mathcal{I}_{p+2}(x), \qquad [{}_a\mathcal{I}_p(x)]^{p+1} \leqslant [{}_a\mathcal{I}_{p+1}(x)]^{p+a+1} \tag{11}$$

hold for all $x \in \mathbb{R}.$ In both of inequalities equality holds if $x = 0$ or $a = 0.$ Moreover here the exponent p is the best possible in the sense that $\alpha = (p+a+1)/(p+1)$ is the smallest value of α for which ${}_a\mathcal{I}_p(x) \leqslant [{}_a\mathcal{I}_{p+1}(x)]^{\alpha}$ holds.

Proof. First observe that when $a = 0$ the function ${}_a\mathcal{I}_p$ becomes $e^{x^2/2},$ thus in both inequalities of (11) we have equality. In what follows suppose that $a \neq 0.$ Let us consider the function ${}_a\gamma_p$ defined by the relation ${}_a\mathcal{I}_p(x) = {}_a\gamma_p(x^2),\ x \in \mathbb{R}.$ Due to definitions it is enough to show that ${}_a\gamma_{p+1}(x) \leqslant \sqrt{{}_a\gamma_p(x)\,{}_a\gamma_{p+2}(x)}$ holds for all $x \geqslant 0.$ For convenience let us introduce the following notations:

$$Q_n^p(x) := \sum_{i=0}^{n} a_i(p)x^i,\ \text{where}\ a_i(p) := \frac{(1/4)^i\Gamma(p+1)}{\Gamma(p+ai+1)i!},\ i \in \{0,1,\dots,n\}. \tag{12}$$

The power series ${}_a\gamma_p(x)$ is convergent for all $x \in \mathbb{R},$ so from (12) it is clear that $\lim_{n\to\infty} Q_n^p(x) = {}_a\gamma_p(x),$ thus it is enough to show that

$$Q_n^{p+1}(x) \leqslant \sqrt{Q_n^p(x)Q_n^{p+2}(x)} \tag{13}$$

holds for all $x \geqslant 0$ and $p > -1.$ Using the well-known Cauchy–Buniakowsky–Schwarz inequality we obtain that

$$Q_n^p(x)Q_n^{p+2}(x) = \left[\sum_{i=0}^{n} a_i(p)x^i\right]\left[\sum_{i=0}^{n} a_i(p+2)x^i\right] \geqslant \left[\sum_{i=0}^{n} \sqrt{a_i(p)a_i(p+2)}x^i\right]^2.$$

Taking into account this inequality in order to prove (13) we just need to show that the following inequality holds

$$\sum_{i=0}^{n} \sqrt{a_i(p)a_i(p+2)}x^i \geqslant Q_n^{p+1}(x) = \sum_{i=0}^{n} a_i(p+1)x^i.$$

Now for this we prove that $\sqrt{a_i(p)a_i(p+2)} \geqslant a_i(p+1)$ holds for all $i \in \{0, 1, \ldots, n\}$. Using (12) and the hypotheses ($p > -1$) this relation is equivalent to the inequality $(p+ai+1)(p+2) \geqslant (p+ai+2)(p+1)$, which clearly holds, because $ai \geqslant 0$. Thus the first inequality in (11) holds.

It remains to prove that the first inequality in (11) implies the second inequality in (11). Observe that the function ${}_a\mathcal{I}_p$ is even, thus in fact it is enough to show that the inequality $[{}_a\gamma_p(x)]^{p+1} \leqslant [{}_a\gamma_{p+1}(x)]^{p+a+1}$, or equivalently

$$\frac{[{}_a\gamma_{p+1}(x)]^{\frac{p+a+1}{p+1}}}{{}_a\gamma_p(x)} \geqslant 1. \tag{14}$$

holds for all $x \geqslant 0$. Taking the logarithm (due to the hypotheses we have ${}_a\gamma_p(x) > 0$) of both sides of (14) we just need to prove that

$$\varphi(x) := \frac{p+a+1}{p+1} \log[{}_a\gamma_{p+1}(x)] - \log[{}_a\gamma_p(x)] \geqslant 0. \tag{15}$$

In order to show (15) we prove that the function $\varphi : [0, \infty) \to \mathbb{R}$ is increasing, and consequently $\varphi(x) \geqslant \varphi(0) = 0$. It results that

$$\varphi'(x) = \frac{p+a+1}{p+1} \cdot \frac{{}_a\gamma'_{p+1}(x)}{{}_a\gamma_{p+1}(x)} - \frac{{}_a\gamma'_p(x)}{{}_a\gamma_p(x)}.$$

Since $(n+1)a_{n+1}(p) = a_1(p)a_n(p+a)$ holds for all $a, n \in \mathbb{N}$, $p > -1$, it is easy to verify that

$${}_a\gamma'_p(x) = a_1(p)\,{}_a\gamma_{p+a}(x), \tag{16}$$

for all $a \in \mathbb{N}$, $x \in \mathbb{R}$, $p \in \mathbb{R}$ such that $p+1 \neq 0, -1, -2, \ldots$. Thus applying (16) for p and $p+1$ together with (12), we obtain

$$\begin{aligned} \varphi'(x) =& \frac{p+a+1}{p+1} a_1(p+1) \frac{{}_a\gamma_{p+a+1}(x)}{{}_a\gamma_{p+1}(x)} - a_1(p) \frac{{}_a\gamma_{p+a}(x)}{{}_a\gamma_p(x)} \\ =& a_1(p) \left[\frac{{}_a\gamma_{p+a+1}(x)}{{}_a\gamma_{p+1}(x)} - \frac{{}_a\gamma_{p+a}(x)}{{}_a\gamma_p(x)} \right]. \end{aligned}$$

Due to the inequality $[{}_a\gamma_{p+1}(x)]^2 \leqslant {}_a\gamma_p(x)\,{}_a\gamma_{p+2}(x)$, we have

$$\frac{{}_a\gamma_{p+1}(x)}{{}_a\gamma_p(x)} \leqslant \frac{{}_a\gamma_{p+2}(x)}{{}_a\gamma_{p+1}(x)} \leqslant \frac{{}_a\gamma_{p+3}(x)}{{}_a\gamma_{p+2}(x)} \leqslant \cdots \leqslant \frac{{}_a\gamma_{p+m+1}(x)}{{}_a\gamma_{p+m}(x)}$$

for all $p > -1$, $m = 1, 2, 3, \ldots$ and $x \geqslant 0$. Thus choosing $m = a$ it is easy to verify that

$${}_a\gamma_{p+a+1}(x)\,{}_a\gamma_p(x) \geqslant {}_a\gamma_{p+a}(x)\,{}_a\gamma_{p+1}(x)$$

holds for all $a = 1,2,\ldots,\ x \geqslant 0,\ p > -1$. It follows that $\varphi'(x) \geqslant 0$, hence the asserted result follows. Finally observe that the second inequality in (11) can be written as

$$ {}_a\mathcal{I}_p(x) \leqslant [{}_a\mathcal{I}_{p+1}(x)]^{(p+a+1)/(p+1)}. \tag{17}$$

All that remains is to note that both members of (17) are even functions. Since

$$ {}_a\mathcal{I}_p(x) = 1 + a_1(p)x^2 + \ldots, \quad \text{and} \quad [{}_a\mathcal{I}_{p+1}(x)]^{\alpha} = 1 + \alpha \cdot a_1(p+1)x^2 + \ldots,$$

we infer that $\alpha = a_1(p)/a_1(p+1) = (p+a+1)/(p+1)$ is the smallest value of α for which ${}_a\mathcal{I}_p(x) \leqslant [{}_a\mathcal{I}_{p+1}(x)]^{\alpha}$ holds. Thus the proof is complete. □

The next result is a generalization of Theorem 1.

THEOREM 2. *If $a = 0,1,2,\ldots$ and $p > -1$, then the function $p \mapsto {}_a\mathcal{I}_{p+1}(x)/{}_a\mathcal{I}_p(x)$ is increasing, i.e. if $p \geqslant q > -1$, then*

$$ {}_a\mathcal{I}_{p+1}(x){}_a\mathcal{I}_q(x) \geqslant {}_a\mathcal{I}_p(x){}_a\mathcal{I}_{q+1}(x). \tag{18}$$

Moreover, the function $p \mapsto [{}_a\mathcal{I}_p(x)]^{(p+1)(p+2)\ldots(p+a)}$ is increasing too.

Proof. There is nothing to prove when $a = 0$. So suppose that $a \neq 0$. Observe that to prove (18) it is enough to show that

$$ {}_a\gamma_{p+1}(x){}_a\gamma_q(x) \geqslant {}_a\gamma_p(x){}_a\gamma_{q+1}(x), \tag{19}$$

which is equivalent to

$$\sum_{n\geqslant 0} a_n(p+1)x^n \cdot \sum_{n\geqslant 0} a_n(q)x^n \geqslant \sum_{n\geqslant 0} a_n(p)x^n \cdot \sum_{n\geqslant 0} a_n(q+1)x^n.$$

And this is true if

$$a_i(p+1)x^i a_j(q)x^j + a_j(p+1)x^j a_i(q)x^i \geqslant a_j(p)x^j a_i(q+1)x^i + a_i(p)x^i a_j(q+1)x^j$$

holds for all $i,j \in \mathbb{N}$. This can be verified by adding up the corresponding parts of these inequalities for i and then summing for j. In what follows we want to prove that the inequality

$$a_i(p+1)a_j(q) + a_j(p+1)a_i(q) \geqslant a_j(p)a_i(q+1) + a_i(p)a_j(q+1) \tag{20}$$

holds for all $i,j \in \mathbb{N}$. Using the notations

$$\beta_1 := \Gamma(p+ai+2)\Gamma(q+aj+2),$$

$$\beta_2 := \Gamma(q+ai+2)\Gamma(p+aj+2),$$

we get that (20) is equivalent to the inequality

$$(p+1)\left[\beta_2(q+aj+1) + \beta_1(q+ai+1)\right] \geqslant (q+1)\left[\beta_1(p+aj+1) + \beta_2(p+ai+1)\right].$$

Hence it is enough to show that

$$(p+1)(i\beta_1 + j\beta_2) - (q+1)(i\beta_2 + j\beta_1) \geqslant 0 \tag{21}$$

holds for all $i,j \in \mathbb{N}$. Because $p \geqslant q$, to deduce (21) it remains to prove that $(i-j)(\beta_1-\beta_2) \geqslant 0$. Finally observe that this above inequality clearly holds because by assumptions we have that if $i \geqslant j$ ($i \leqslant j$ respectively) then $\beta_1 \geqslant \beta_2$ ($\beta_1 \leqslant \beta_2$ respectively). This is justified by the fact that the function

$$x \mapsto \frac{\Gamma(ax+p+2)}{\Gamma(ax+q+2)}$$

is increasing on $[0,\infty)$, and from this it remains to prove that

$$\frac{\Gamma'(ax+p+2)}{\Gamma(ax+p+2)} \geqslant \frac{\Gamma'(ax+q+2)}{\Gamma(ax+q+2)} \tag{22}$$

holds for all $x \geqslant 0$ and $p \geqslant q > -1$. It is well-known that the function $x \mapsto \Gamma(ax)$ is log-convex, therefore $x \mapsto \Gamma'(ax)/\Gamma(ax)$ is increasing, which implies (22).

In what follows we proceed exactly as in the proof of Theorem 1. Suppose that $p \geqslant q > -1$, and define the function $\phi : [0,\infty) \to \mathbb{R}$ with the relation

$$\phi(x) = \frac{(p+1)_a}{(q+1)_a}\log[{}_a\gamma_p(x)] - \log[{}_a\gamma_q(x)]. \tag{23}$$

Application of (16) for p and q, together with (23) implies that

$$\phi'(x) = a_1(q)\left[\frac{{}_a\gamma_{p+a}(x)}{{}_a\gamma_p(x)} - \frac{{}_a\gamma_{q+a}(x)}{{}_a\gamma_q(x)}\right].$$

Now from (19) we obtain that

$$\frac{{}_a\gamma_q(x)}{{}_a\gamma_p(x)} \geqslant \frac{{}_a\gamma_{q+1}(x)}{{}_a\gamma_{p+1}(x)} \geqslant \frac{{}_a\gamma_{q+2}(x)}{{}_a\gamma_{p+2}(x)} \geqslant \cdots \geqslant \frac{{}_a\gamma_{q+m}(x)}{{}_a\gamma_{p+m}(x)},$$

where $p \geqslant q > -1$, $m = 1,2,\ldots$ and $x \geqslant 0$. Thus choosing again $m = a$ it is clear that we have

$${}_a\gamma_{p+a}(x)\,{}_a\gamma_q(x) \geqslant {}_a\gamma_p(x)\,{}_a\gamma_{q+a}(x), \tag{24}$$

and consequently $\phi'(x) \geqslant 0$, for all $x \geqslant 0$. Thus ϕ is increasing and consequently $\phi(x) \geqslant \phi(0) = 0$, i.e. $[{}_a\gamma_p(x)]^{(p+1)_a} \geqslant [{}_a\gamma_q(x)]^{(q+1)_a}$, for all $x \geqslant 0$, where $(p+1)_a = (p+1)(p+2)\ldots(p+a) = \Gamma(p+a+1)/\Gamma(p+1)$, $(p+1)_0 = 1$ is the well-known Pochhammer symbol defined in terms of Euler's gamma function. This implies that $[{}_a\mathcal{I}_p(x)]^{(p+1)_a} \geqslant [{}_a\mathcal{I}_q(x)]^{(q+1)_a}$, for all $x \in \mathbb{R}$, therefore the proof is complete. $\square$

There is another notion of generalized Bessel function, which was elaborated by the author. Namely, the generalized Bessel function of the first kind v_p is defined [9] as a particular solution of the differential equation

$$x^2v''(x) + bxv'(x) + \left[cx^2 - p^2 + (1-b)p\right]v(x) = 0, \tag{25}$$

where $b,p,c \in \mathbb{R}$, and v_p has the infinite series representation

$$v_p(x) = \sum_{n\geqslant 0}\frac{(-1)^n c^n}{n!\,\Gamma\left(p+n+\frac{b+1}{2}\right)}\cdot\left(\frac{x}{2}\right)^{2n+p},\ x \in \mathbb{R}.$$

This function permits us to study the Bessel function J_p -defined by (2)- and the modified Bessel function I_p -defined by (4)- together. For $c = 1$ and $b = 1$ equation (25) reduces to Bessel's equation (1) with the Bessel function of the first kind as a particular solution. Now for $c = -1$ and $b = 1$ the equation (25) reduces to the differential equation (3). The generalized and normalized Bessel function of the first kind is defined [9] as follows

$$u_p(x) = 2^p\Gamma(\kappa)\cdot x^{-p/2}v_p(x^{1/2}) = \sum_{n\geqslant 0}\frac{(-c/4)^n}{(\kappa)_n}\frac{x^n}{n!}, \tag{26}$$

where $\kappa := p+(b+1)/2 \neq 0,-1,-2,\ldots$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the well known Pochhammer symbol defined in terms of Euler's Γ-function. This function is related to an obvious transform of hypergeometric function ${}_0F_1$, i.e. $u_p(x) = {}_0F_1(\kappa,-cx/4)$ and satisfies the following differential equation

$$xu''(x) + \kappa u'(x) + (c/4)u(x) = 0.$$

For properties of the function u_p, such as differential properties, integral representations and interesting functional inequalities we refer to the papers [5], [6], [9]. Finally, let us consider the function λ_p defined by $\lambda_p(x) = u_p(x^2)$. Recall that for $c = -1$ and $b = 1$ this function reduces to the function $\mathcal{I}_p$, defined by $\mathcal{I}_p(x) = 2^p\Gamma(p+1)x^{-p}I_p(x)$.

Thus, if we combine the two notions of generalized Bessel function, then we may define the function ${}_av_p$ depending now from four parameters a,b,c,p in the following form:

$${}_av_p(x) := \sum_{n\geqslant 0}\frac{(-1)^n c^n}{n!\Gamma\left(p+an+\frac{b+1}{2}\right)}\cdot\left(\frac{x}{2}\right)^{2n+p},\ x\in\mathbb{R}.$$

Moreover, analogously as in (26) we may define the normalized form of this generalized Bessel function, i.e. for $a\in\mathbb{N}$, $b,c,p,x\in\mathbb{R}$ such that $\kappa := p+(b+1)/2 \neq 0,-1,-2,\ldots,$ let us consider the function

$${}_a\lambda_p(x) := {}_au_p(x^2) = 2^p\Gamma(\kappa)\cdot x^{-p}{}_av_p(x) = \sum_{n\geqslant 0}\frac{(-c/4)^n\Gamma(\kappa)}{\Gamma(\kappa+an)}\frac{x^{2n}}{n!}. \tag{27}$$

It is worth mentioning that when $c = -1$ and $b = 1$ then ${}_au_p$ reduces to ${}_a\gamma_p$, while ${}_a\lambda_p$ reduces to ${}_a\mathcal{I}_p$. The next result improves the results from [8].

THEOREM 3. *If $a = 0,1,2,\ldots,$ $c\leqslant 0$, $\kappa > 0$ and $x\in\mathbb{R}$, then*

$$[{}_a\lambda_{p+1}(x)]^2 \leqslant {}_a\lambda_p(x)\cdot{}_a\lambda_{p+2}(x),\ [{}_a\lambda_p(x)]^{\kappa} \leqslant [{}_a\lambda_{p+1}(x)]^{\kappa+1}.$$

In both inequalities equality holds if $a = 0$ or $c = 0$ or $x = 0$. Here the exponent κ is the best possible in the sense that $\alpha = (\kappa+1)/\kappa$ is the smallest value of α for which ${}_a\lambda_p(x) \leqslant [{}_a\lambda_{p+1}(x)]^{\alpha}$ holds. Moreover, the functions $p\mapsto {}_a\lambda_{p+1}(x)/{}_a\lambda_p(x)$, $p\mapsto [{}_a\lambda_p(x)]^{\kappa}$ are increasing.

Proof. Taking into account the relation ${}_a\lambda_p(x) = {}_a\mathcal{I}_{\kappa-1}(\sqrt{-c}x)$ the results follows from Theorem 1 and 2 just changing in every inequality of the mentioned theorems x with $\sqrt{-c}x$, and p with $\kappa-1$. □

3. Properties of the functions ${}_a\gamma_p$ and ${}_a\mathcal{I}_p$

Before we state and prove the main results of this section, let us recall a lemma due to Biernacki and Krzyż [12] which is one of the crucial facts in the proof of our results. Note that this lemma is a special case of a more general lemma in [18].

LEMMA 1. [12, 18] *Suppose that the power series* $f(x) = \sum_{n\geqslant 0} \alpha_n x^n$ *and* $g(x) = \sum_{n\geqslant 0} \beta_n x^n$ *(* $\beta_n > 0$ *for all* $n \geqslant 0$ *) both converge for* $|x| < \infty$*. Then the function* $x \mapsto f(x)/g(x)$ *is (strictly) increasing (decreasing) for* $x > 0$ *if the sequence* $\{\alpha_n/\beta_n\}_{n\geqslant 0}$ *is (strictly) increasing (decreasing).*

It is worth mentioning that this lemma for $x \in (0,1)$ was used, among other things, to prove many interesting inequalities for the zero-balanced Gaussian hypergeometric functions (see [3] and [18]) and also for the generalized Bessel functions (see [4], [5], [6] for more details).

We note that we can see easily that this lemma remains true if we get even and odd functions, i.e.

(1) $f(x) = \sum_{n\geqslant 0} \alpha_n x^{2n}$ and $g(x) = \sum_{n\geqslant 0} \beta_n x^{2n}$ for $x \in \mathbb{R}$,

(2) $f(x) = \sum_{n\geqslant 0} \alpha_n x^{2n+1}$ and $g(x) = \sum_{n\geqslant 0} \beta_n x^{2n+1}$ for $x > 0$.

For the reader's convenience we give a different proof of Lemma 1 to see that why we can change the condition $x \in (0,1)$ with $x > 0$, moreover the functions f and g to an even or odd function.

Proof of Lemma 1. We consider only the case when $\{\alpha_n/\beta_n\}_{n\geqslant 0}$ is strictly increasing and we prove that this implies that $x \mapsto f(x)/g(x)$ is strictly increasing too. The other cases are similar, so we omit the details. If $x > y > 0$, then inequality $f(x)/g(x) > f(y)/g(y)$ is equivalent to

$$\sum_{n\geqslant 0} \alpha_n x^n \cdot \sum_{n\geqslant 0} \beta_n y^n > \sum_{n\geqslant 0} \alpha_n y^n \cdot \sum_{n\geqslant 0} \beta_n x^n,$$

and this is true if

$$\alpha_i \beta_j x^i y^j + \alpha_j \beta_i x^j y^i > \alpha_j \beta_i x^i y^j + \alpha_i \beta_j x^j y^i \qquad (1)$$

holds for all $i, j \in \mathbb{N}$. This can be verified by adding up the corresponding parts of these inequalities for i and then summing for j. But (1) can be transformed to the inequality $(\alpha_i\beta_j - \alpha_j\beta_i)(x^i y^j - x^j y^i) > 0$, which clearly holds because by assumptions we have that if $i > j$ ($i < j$ respectively) then $\alpha_i/\beta_i > \alpha_j/\beta_j$ ($\alpha_i/\beta_i < \alpha_j/\beta_j$ respectively) and $x^i y^j - x^j y^i = (xy)^j (x^{i-j} - y^{i-j}) > 0$ ($x^i y^j - x^j y^i = (xy)^j (x^{i-j} - y^{i-j}) < 0$ respectively).

THEOREM 4. *Suppose that* $a = 0, 1, 2, \ldots$ *and* $p > -1$*. Then the following assertions are true:*

(1) *the function* $x \mapsto {}_a\gamma_p(x)$ *is log-concave on* $(0,\infty)$*;*

(2) *the function* $x \mapsto {}_a\gamma_p(e^{-x})$ *is log-convex on* $(0,\infty)$*;*

(3) *the function* $x \mapsto [{}_a\gamma_p(1-e^{-x})]^{-1}$ *is log-convex on* $(0,\infty)$*;*

(4) *the function* $x \mapsto {}_a\gamma_p(e^{-x})/{}_a\gamma_p(1-e^{-x})$ *is log-convex on* $(0,\infty)$*;*

(5) *the function* $x \mapsto {}_a\mathcal{I}_p(x)$ *is log-convex on* $\mathbb{R}$ *provided* $p \geqslant -1/2$*;*

(6) *the function* $x \mapsto {}_a\mathcal{I}_p(x)/{}_a\mathcal{I}_q(x)$ *is increasing (decreasing) on* $\mathbb{R}$ *for* $p \leqslant q$ *(* $p \geqslant q$ *).*

Proof. First let us focus on parts (1)–(5). Observe that in particular ${}_0\gamma_p(x) = e^{x/2}$ and ${}_0\mathcal{I}_p(x) = e^{x^2/2}$, thus parts (1)–(5) clearly holds for $a = 0$. When $a = 1$ then ${}_1\gamma_p = \gamma_p$ and ${}_1\mathcal{I}_p = \mathcal{I}_p$. For the function γ_p parts (1)–(4) among other things were established more generally in [5] and [6]. For the function $\mathcal{I}_p$ part (5) was established by Neuman in [17]. Suppose that for $a = k$ parts (1)–(5) hold. Now changing p with $p+k$, where k is a natural number, it is clear that parts (1)–(5) remain true. Let us record the formula

$$(p+1)_a \cdot {}_{a+1}\gamma_p(x) = {}_a\gamma_{p+a}(x), \tag{2}$$

where $p > -1$, $a = 0,1,2,\ldots$ and $x \in \mathbb{R}$. Using (2) it follows that (1)–(5) hold for $a = k+1$, hence by mathematical induction the asserted results follow.

Now let us prove part (6). In view of (12) let us consider the sequence

$$r_n = \frac{(1/4)^n\Gamma(p+1)}{\Gamma(p+an+1)n!} \Bigg/ \frac{(1/4)^n\Gamma(q+1)}{\Gamma(q+an+1)n!},$$

then it is clear that $(p+an+1)_a r_{n+1} = (q+an+1)_a r_n$ for all $n \geqslant 0$ and $p,q > -1$. So we get that $r_{n+1} \geqslant r_n$ if and only if $q \geqslant p > -1$ and $r_n \geqslant r_{n+1}$ if and only if $p \geqslant q > -1$. These facts together with Lemma 1 in turn imply (6). Thus the proof is complete. □

REMARK 1.

1. First note that proceeding exactly as in the proof of Theorem 3, the results of Theorem 4 hold for the functions ${}_au_p$ and ${}_a\lambda_p$ under corresponding conditions. Parts (1)–(4) improve the results obtained in [5] for the function ${}_1u_p = u_p$. In fact, there is another argument to prove (1)–(4). Namely from [5] we know that part (2) actually holds for every power series with positive coefficients, part (3) is a consequence of part (1), and finally (4) follows from (2) and (3). Thus it remains to prove (1). In view of Lemma 1 it is enough to prove that the sequence $s_n = (n+1)a_{n+1}(p)/a_n(p)$ is decreasing. Observe that the inequality $s_n \geqslant s_{n+1}$, $n \geqslant 0$ is equivalent to $(p+an+a+1)_a \geqslant (p+an+1)_a$, which by the ascending factorial notation clearly holds. Or rewriting the last inequality in terms of Euler's $\Gamma-$ function, i.e.

$$\Gamma(p+an+1)\Gamma(p+an+2a+1) \geqslant [\Gamma(p+an+a+1)]^2, \tag{3}$$

because

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt,$$

(3) is a simple consequence of the following generalizationof Schwarz inequality

$$\int_a^b f(t)[g(t)]^i dt \cdot \int_a^b f(t)[g(t)]^j dt \geqslant \left[\int_a^b f(t)[g(t)]^{\frac{i+j}{2}} dt\right]^2,$$

where f and g are two nonnegative functions of a real variable, i and j are real numbers such that the integrals exist.

2. Many other functional inequalities can be deduced from parts (1)–(5) of Theorem 4 using the definition of log-concavity, log-convexity respectively (please see [5], [6], [11] and [17] for further results).

3. Finally, we note that from part (6) of Theorem 4 can be deduced an extension of the asymptotic formula [8, Theorem 1.8] $[I_p(x)]^2 \sim I_{p-1}(x)I_{p+1}(x)$. Namely if $x > 0$ and $a = 0, 1, 2, \ldots$ are fixed and $p \to \infty$, then

$$ {}_aI_p(x)\,{}_aI_{p+a-1}(x) \sim {}_aI_{p-1}(x)\,{}_aI_{p+a}(x). $$

In order to prove the asserted result, we show that for $p > 0$ and $x > 0$ we have

$$ 1 \leqslant \frac{{}_aI_p(x)\,{}_aI_{p+a-1}(x)}{{}_aI_{p-1}(x)\,{}_aI_{p+a}(x)} \leqslant 1 + \frac{a}{p}. \tag{4} $$

From part (6) of Theorem 4 it is clear that if $p \geqslant q > -1$, then $({}_a\mathcal{I}_p(x)/{}_a\mathcal{I}_q(x))' \leqslant 0$, i.e. we have for $x > 0$

$$ {}_aI_{p+a}(x)\,{}_aI_q(x) - {}_aI_p(x)\,{}_aI_{q+a}(x) \leqslant 0. \tag{5} $$

Taking in (5) $q = p - 1$ we get the left hand side of (4), which is in fact the extension of the well-known Amos inequality [2, p. 243] for the function ${}_1I_p = I_p$. The right hand side of (4) can be deduced easily from the proof of Theorem 2 using the inequality (24) for $q = p - 1$.

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