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ON THE STABILITY OF HOMOMORPHISMS IN QUASI–BANACH ALGEBRAS ASSOCIATED TO THE PEXIDERIZED JENSEN FUNCTIONAL EQUATION

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(*communicated by Th. Rassias*)

Abstract. In this paper, we prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Jensen functional equation. This is applied to investigate homomorphisms between quasi-Banach algebras. The concept of Hyers-Ulam-Rassias stability originated from the Th. M. Rassias' stability theorem that appeared in the paper: On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. **72** (1978), 297-300.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [34] concerning the stability of group homomorphisms : Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations when the homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a true homomorphism near it. In 1941, Hyers [9] considered the case of approximately additive mappings in Banach spaces and satisfying the well-known weak Hyers inequality controlled by a positive constant. In 1978, Th. M. Rassias [25] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

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THEOREM. (Th. M. Rassias) *Let $f: E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y)-f(x)-f(y)\| \leqslant \epsilon(\|x\|^p+\|y\|^p) \qquad (\heartsuit)$$

for all $x,y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L: E \to E'$ is the unique additive mapping which satisfies

$$\|f(x)-L(x)\| \leqslant \frac{2\epsilon}{2-2^p}\|x\|^p \qquad (\diamondsuit)$$

for all $x \in E$. If $p < 0$ then inequality $(\heartsuit)$ holds for $x,y \neq 0$ and $(\diamondsuit)$ for $x \neq 0$. Also, if the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}$-linear.

In 1990, Th. M. Rassias [27] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [7] following the same approach as in Th. M. Rassias [25], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [7], as well as by Th. M. Rassias and P. Šemrl [31] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [7], as well as of Th. M. Rassias and P. Šemrl [31] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [8], S. Jung [15], who among others studied the Hyers-Ulam-Rassias stability of functional equations.

The inequality $(\heartsuit)$ that was introduced for the first time by Th. M. Rassias [25] provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of P. Czerwik [5], S. Czerwik [6], D.H. Hyers, G. Isac and Th. M. Rassias [11], S.-M. Jung [16]).

J. M. Rassias [23] following the spirit of the innovative approach of Th. M. Rassias [25] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p,q \in \mathbb{R}$ with $p+q \neq 1$.

P. Găvruta [8] provided a further generalization of Th. M. Rassias' Theorem. In 1996, G. Isac and Th. M. Rassias [13] applied the Hyers-Ulam-Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [12], D. H. Hyers, G. Isac and Th. M. Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. During past few years several mathematicians have published on various generalizations and applications of Hyers-Ulam stability and Hyers-Ulam-Rassias stability to a number of functional equations and mappings, for example : quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation, Navier-Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: C. Park [19]-[21],Th. M. Rassias [26]-[30], F. Skof [33].

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [2, 8, 19, 22, 28]).

We recall some basic facts concerning quasi-Banach spaces and some preliminary results.

DEFINITION 1.1. [4, 32] Let X be a real linear space. A *quasi-norm* is a real-valued function on X satisfying the following:
(i) $\|x\| \geqslant 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.
(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in X$.
(iii) There is a constant $K \geqslant 1$ such that $\|x+y\| \leqslant K(\|x\| + \|y\|)$ for all $x, y \in X$.

It follows easily from condition (iii) that

$$\Big\|\sum_{i=1}^{2n} x_i\Big\| \leqslant K^n \sum_{i=1}^{2n} \|x_i\|, \qquad \Big\|\sum_{i=1}^{2n+1} x_i\Big\| \leqslant K^{n+1} \sum_{i=1}^{2n+1} \|x_i\|$$

for all integers $n \geqslant 1$ and all $x_1, x_2, \dots, x_{2n+1} \in X$.

The pair $(X, \|.\|)$ is called a *quasi-normed space* if $\|.\|$ is a quasi-norm on X. The smallest possible K is called the *modulus of concavity* of $\|.\|$. A *quasi-Banach space* is a complete quasi-normed space.

A quasi-norm $\|.\|$ is called a p*-norm* $(0 < p \leqslant 1)$ if

$$\|x+y\|^p \leqslant \|x\|^p + \|y\|^p$$

for all $x, y \in X$. In this case, a quasi-Banach space is called a p*-Banach space.*

By the Aoki-Rolewicz theorem [32] (see also [4]), each quasi-norm is equivalent to some p-norm. Since it is much easier to work with p-norms than quasi-norms, henceforth we restrict our attention mainly to p-norms.

DEFINITION 1.2. [1] Let $(A, \|.\|)$ be a quasi-normed space. The quasi-normed space $(A, \|.\|)$ is called a *quasi-normed algebra* if A is an algebra and there is a constant $C > 0$ such that $\|xy\| \leqslant C\|x\|\|y\|$ for all $x, y \in A$.

A *quasi-Banach algebra* is a complete quasi-normed algebra. If the quasi-norm $\|.\|$ is a p-norm then the quasi-Banach algebra is called a p*-Banach algebra.*

2. Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras associated to the Pexiderized Jensen functional equation

Throughout this section, assume that A is a quasi-normed algebra with quasi-norm $\|.\|_A$ and that B is a p-Banach algebra with p-norm $\|.\|_B$. Let K be the modulus of concavity of $\|.\|_B$.

The stability of homomorphisms in quasi-Banach algebras, associated to the Jensen functional equation, has been investigated in [21]. We prove the Hyers-Ulam-Rassias stability of homomorphisms in quasi-Banach algebras, associated to the Pexiderized Jensen functional equation. The stability of homomorphisms in quasi-Banach algebras, associated to the Pexiderized Cauchy functional equation, has been investigated in [18].

THEOREM 2.1. [21] *Let* $r < \frac{1}{2}$ *and* θ *be positive real numbers, and let* $f : A \to B$ *be a mapping with* $f(0) = 0$ *satisfying*

$$\left\| 2f\left(\frac{x+y}{2}\right) - f(x) - f(y) \right\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r \tag{2.1}$$

and

$$\|f(xy) - f(x)f(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r \tag{2.2}$$

for all $x, y \in A$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique homomorphism* $T : A \to B$ *such that*

$$T(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x), \quad \|f(x) - T(x)\|_B \leqslant \frac{K(1+3^r)\theta}{(3^p - 9^{pr})^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$.

REMARK 2.2. Theorem 2.1 is true when $r = 0$ (by putting $\|.\|_A^0 = 1$).

THEOREM 2.3. [21] *Let* $r > 1$ *and* θ *be positive real numbers, and let* $f : A \to B$ *be a mapping with* $f(0) = 0$ *satisfying* (2.1) *and* (2.2). *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique homomorphism* $T : A \to B$ *such that*

$$T(x) = \lim_{n\to\infty} 3^n f\left(\frac{x}{3^n}\right), \quad \|f(x) - T(x)\|_B \leqslant \frac{K(1+3^r)\theta}{(9^{pr} - 3^p)^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$.

The proofs of the following results are similar to the proofs of Theorems 2.1 and 2.2 of [21] and we refer to [21].

THEOREM 2.4. *Let* θ, r, s *be positive real number* $r > \frac{1}{2}$ *and* $s > 1$. *Assume that* $f : A \to B$ *is a mapping such that*

$$\|f(x+y) - f(x) - f(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r \tag{2.3}$$

$$\|f(xy) - f(x)f(y)\|_B \leqslant \theta \|x\|_A^s \|y\|_A^s \tag{2.4}$$

for all $x, y \in A$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique homomorphism* $T : A \to B$ *such that*

$$T(x) = \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right), \quad \|f(x) - T(x)\|_B \leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$.

THEOREM 2.5. *Let* θ, r, s *be a positive real number such that* $0 \leqslant r < \frac{1}{2}$ *and* $0 \leqslant s < 1$. *Assume that* $f : A \to B$ *is a mapping satisfies* (2.3) *and* (2.4) *for all* $x, y \in A$. *If* $f(tx)$ *is continuous in* $t \in \mathbb{R}$ *for each fixed* $x \in A$, *then there exists a unique homomorphism* $T : A \to B$ *such that*

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x), \quad \|f(x) - T(x)\|_B \leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$ *(we put* $\|.\|_A^0 = 1$ *).*

The following theorem shows that the mappings $f : A \to B$ in Theorems 2.1 and 2.3 are homomorphisms.

THEOREM 2.6. *Let r, s and θ be positive real numbers with $r \neq \frac{1}{2}$ and $s \neq 1$, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying (2.1) and (2.4). If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then f is a homomorphism.*

Proof. Letting $y = 0$ in (2.1), we get $2f(x/2) = f(x)$ for all $x \in A$. By induction we infer that

$$2^n f\left(\frac{x}{2^n}\right) = f(x) \tag{2.5}$$

for all $x \in A$ and all $n \in \mathbb{Z}$. Therefore, the mapping $f : A \to B$ satisfies (2.3) and (2.4). By Theorems 2.4 and 2.5, there exists a homomorphism $T : A \to B$ such that $T(x) = f(x)$ for all $x \in A$. □

THEOREM 2.7. *Let r, s and θ be positive real numbers, and let $f, g, h : A \to B$ be mappings satisfying*

$$\left\|2f\left(\frac{x+y}{2}\right) - g(x) - h(y)\right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r \tag{2.6}$$

and

$$\|f(xy) - g(x)h(y)\|_B \leqslant \theta\|x\|_A^s\|y\|_A^s \tag{2.7}$$

for all $x, y \in A$. Suppose that at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ or $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$.

(*i*) *If $0 < r < \frac{1}{2}$ and $0 < s < 1$, then there exists a unique homomorphism $T : A \to B$ such that*

$$\begin{aligned}
\|f(x) - f(0) - T(x)\|_B &\leqslant \frac{\theta 2^{2r-1}}{(2^p - 4^{pr})^{1/p}}\|x\|_A^{2r} \\
\|g(x) - g(0) - T(x)\|_B &\leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}}\|x\|_A^{2r} \\
\|h(x) - h(0) - T(x)\|_B &\leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}}\|x\|_A^{2r}
\end{aligned} \tag{2.8}$$

for all $x \in A$.

(*ii*) *If $r > \frac{1}{2}$ and $s > 1$, then there exists a unique homomorphism $T : A \to B$ such that*

$$\begin{aligned}
\|f(x) - f(0) - T(x)\|_B &\leqslant \frac{\theta 2^{2r-1}}{(4^{pr} - 2^p)^{1/p}}\|x\|_A^{2r} \\
\|g(x) - g(0) - T(x)\|_B &\leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}}\|x\|_A^{2r} \\
\|h(x) - h(0) - T(x)\|_B &\leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}}\|x\|_A^{2r}
\end{aligned} \tag{2.9}$$

for all $x \in A$.

Proof. Letting $x = 0$ in (2.6) and (2.7), we get that

$$2f\left(\frac{y}{2}\right) = g(0) + h(y), \quad f(0) = g(0)h(y) \tag{2.10}$$

for all $y \in A$. Once again letting $y = 0$ in (2.6) and (2.7), we get that

$$2f\left(\frac{x}{2}\right) = g(x) + h(0), \quad f(0) = g(x)h(0) \tag{2.11}$$

for all $x \in A$. It follows from (2.10) and (2.11) that

$$g(x) - g(0) = h(x) - h(0), \tag{2.12}$$

$$\begin{aligned} 2f\left(\frac{x}{2}\right) - 2f(0) &= g(x) + h(0) - 2f(0) \\ &= g(x) + h(0) - g(0) - h(0) \\ &= g(x) - g(0) \end{aligned} \tag{2.13}$$

for all $x \in A$. Let $H : A \to B$ be a mapping defined by

$$H(x) := g(x) - g(0)$$

for all $x \in A$. It follows from (2.12) and (2.13) that

$$\begin{aligned} H(x+y) - H(x) - H(y) &= 2f\left(\frac{x+y}{2}\right) - 2f(0) - g(x) + g(0) - h(y) + h(0) \\ &= 2f\left(\frac{x+y}{2}\right) - g(x) - h(y) \end{aligned}$$

for all $x, y \in A$. Therefore, we obtain from (2.6) the following inequality

$$\|H(x+y) - H(x) - H(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^r,$$

for all $x, y \in A$.

(i). By Theorem 2.5, there exists a unique $\mathbb{R}$-linear mapping $T : A \to B$ such that

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} H(2^n x), \quad \|H(x) - T(x)\|_B \leqslant \frac{\theta}{(2^p - 4^{pr})^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$. Therefore, the mappings $f, g, h : A \to B$ satisfy in (2.8). To complete the proof of (i), we show that T is a homomorphism. It is clear that

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} g(2^n x) = \lim_{n\to\infty} \frac{1}{2^n} h(2^n x)$$

for all $x \in A$. Therefore, we have from (2.7)

$$\begin{aligned} \|T(xy) - T(x)T(y)\|_B &= \lim_{n\to\infty} \frac{1}{4^n} \|f(4^n xy) - g(2^n x)h(2^n y)\|_B \\ &\leqslant \theta \lim_{n\to\infty} \left(\frac{4^s}{4}\right)^n \|x\|_A^s \|y\|_A^s = 0 \end{aligned}$$

for all $x, y \in A$. It proves (i).

(ii). By Theorem 2.4, there exists a unique $\mathbb{R}$-linear mapping $T : A \to B$ such that

$$T(x) = \lim_{n\to\infty} 2^n H\Big(\frac{x}{2^n}\Big), \quad \|H(x) - T(x)\|_B \leqslant \frac{\theta}{(4^{pr} - 2^p)^{1/p}} \|x\|_A^{2r}$$

for all $x \in A$. Therefore, the mappings $f, g, h : A \to B$ satisfy in (2.9). Since

$$T(x) = \lim_{n\to\infty} 2^n \Big[f\Big(\frac{x}{2^n}\Big) - f(0)\Big] = \lim_{n\to\infty} 2^n \Big[g\Big(\frac{x}{2^n}\Big) - g(0)\Big] = \lim_{n\to\infty} 2^n \Big[h\Big(\frac{x}{2^n}\Big) - h(0)\Big]$$

for all $x \in A$, we get from (2.7), (2.10) and (2.11)

$$\begin{aligned} \|T(xy) - T(x)T(y)\|_B \\ &= \lim_{n\to\infty} 4^n \Big\|f\Big(\frac{xy}{4^n}\Big) - f(0) - \Big[g\Big(\frac{x}{2^n}\Big) - g(0)\Big]\Big[h\Big(\frac{y}{2^n}\Big) - h(0))\Big]\Big\|_B \\ &= \lim_{n\to\infty} 4^n \Big\|f\Big(\frac{xy}{4^n}\Big) - g\Big(\frac{x}{2^n}\Big)h\Big(\frac{y}{2^n}\Big)\Big\|_B \\ &\leqslant \theta \lim_{n\to\infty} \Big(\frac{4}{4^s}\Big)^n \|x\|_A^s \|y\|_A^s = 0 \end{aligned}$$

for all $x, y \in A$. It proves that T is a homomorphism. $\square$

For $r = s = 0$, we have the following theorem.

THEOREM 2.8. *Let θ be a positive real number and let $f, g, h : A \to B$ be mappings satisfying*

$$\Big\|2f\Big(\frac{x+y}{2}\Big) - g(x) - h(y)\|_B \leqslant \theta, \tag{2.14}$$

and

$$\|f(xy) - g(x)h(y)\|_B \leqslant \theta \tag{2.15}$$

for all $x, y \in A$. If at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ or $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, then there exists a unique homomorphism $T : A \to B$ such that

$$\begin{aligned} \|f(x) - f(0) - T(x)\|_B &\leqslant \frac{4K^3\theta}{(3^p - 1)^{1/p}}, \\ \|g(x) - g(0) - T(x)\|_B &\leqslant \frac{8K^3\theta}{(3^p - 1)^{1/p}}, \\ \|h(x) - h(0) - T(x)\|_B &\leqslant \frac{8K^3\theta}{(3^p - 1)^{1/p}} \end{aligned} \tag{2.16}$$

for all $x \in A$.

Proof. Let $H : A \to B$ be a mapping defined by

$$H(x) := 2f\Big(\frac{x}{2}\Big)$$

for all $x \in A$. Then

$$\|H(x+y) - g(x) - h(y)\|_B \leqslant \theta \tag{2.17}$$

for all $x, y \in A$. Since

$$2H\Big(\frac{x+y}{2}\Big) - H(x) - H(y)$$
$$= \Big[H\Big(\frac{x+y}{2}\Big) - g\Big(\frac{x}{2}\Big) - h\Big(\frac{y}{2}\Big)\Big] + \Big[H\Big(\frac{x+y}{2}\Big) - g\Big(\frac{y}{2}\Big) - h\Big(\frac{x}{2}\Big)\Big]$$
$$+ \Big[g\Big(\frac{x}{2}\Big) + h\Big(\frac{x}{2}\Big) - H(x)\Big] + \Big[g\Big(\frac{y}{2}\Big) + h\Big(\frac{y}{2}\Big) - H(y)\Big]$$

for all $x, y \in A$, then we have

$$\Big\|2H\Big(\frac{x+y}{2}\Big) - H(x) - H(y)\Big\|_B \leqslant 4K^2\theta$$

for all $x, y \in A$. Hence, by Theorem 2.1, There exists a unique additive mapping $T_1 : A \to B$ satisfying

$$T_1(x) = \lim_{n\to\infty} \frac{1}{3^n} H(3^n x), \quad \|H(x) - H(0) - T_1(x)\|_B \leqslant \frac{8K^3\theta}{(3^p - 1)^{1/p}} \tag{2.18}$$

for all $x \in A$. Similarly, since

$$2g\Big(\frac{x+y}{2}\Big) - g(x) - g(y)$$
$$= \Big[g\Big(\frac{x+y}{2}\Big) - H\Big(\frac{y}{2}\Big) + h\Big(\frac{-x}{2}\Big)\Big] + \Big[g\Big(\frac{x+y}{2}\Big) - H\Big(\frac{x}{2}\Big) + h\Big(\frac{-y}{2}\Big)\Big]$$
$$+ \Big[H\Big(\frac{x}{2}\Big) - g(x) - h\Big(\frac{-x}{2}\Big)\Big] + \Big[H\Big(\frac{y}{2}\Big) - g(y) - h\Big(\frac{-y}{2}\Big)\Big]$$

for all $x, y \in A$, then we have

$$\Big\|2g\Big(\frac{x+y}{2}\Big) - g(x) - g(y)\Big\|_B \leqslant 4K^2\theta$$

for all $x, y \in A$. Similarly,

$$\Big\|2h\Big(\frac{x+y}{2}\Big) - h(x) - h(y)\Big\|_B \leqslant 4K^2\theta$$

for all $x, y \in A$. Hence, by Theorem 2.1, There exist unique additive mappings $T_2, T_3 : A \to B$ satisfying

$$T_2(x) = \lim_{n\to\infty} \frac{1}{3^n} g(3^n x), \quad \|g(x) - g(0) - T_2(x)\|_B \leqslant \frac{8K^3\theta}{(3^p - 1)^{1/p}}, \tag{2.19}$$

$$T_3(x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x), \quad \|h(x) - h(0) - T_3(x)\|_B \leqslant \frac{8K^3\theta}{(3^p - 1)^{1/p}} \tag{2.20}$$

for all $x \in A$. It follows from (2.17) that $T_1 = T_2 = T_3$. Let $T = T_1$, by the same reasoning as in the proof of Theorem of [25], the mapping $T : A \to B$ is $\mathbb{R}$-linear. Since

$$T(x) = \lim_{n\to\infty} \frac{1}{3^n} f(3^n x) = \lim_{n\to\infty} \frac{1}{3^n} g(3^n x) = \lim_{n\to\infty} \frac{1}{3^n} h(3^n x) \tag{2.21}$$

for all $x \in A$, then it follows from (2.15) that

$$\begin{aligned} \|T(xy) - T(x)T(y)\|_B &= \lim_{n\to\infty} \frac{1}{9^n} \|f(9^n xy) - g(3^n x)h(3^n y)\|_B \\ &\leqslant \lim_{n\to\infty} \frac{\theta}{9^n} = 0 \end{aligned}$$

for all $x, y \in A$. Hence, $T(xy) = T(x)T(y)$ for all $x, y \in A$. Therefore T is a homomorphism satisfying (2.16). □

THEOREM 2.9. *Let $r, s < 0$ and θ be positive real numbers, and let $f : A \to B$ be a mapping with $f(0) = 0$ satisfying (2.1) and (2.4) for all $x, y \in A \setminus \{0\}$. If the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to B is continuous at zero for each fixed $x \in A$, then $f : A \to B$ is a homomorphism.*

Proof. Let $y \in A \setminus \{0\}$. Replacing x and y in (2.1) by $y - ny$ and $y + ny$, respectively, we get that

$$f(y) = \frac{1}{2} \lim_{n\to\infty} [f(y + ny) + f(y - ny)]. \tag{2.22}$$

It is clear that (2.22) holds for all $y \in A$. Let $x, y \in A \setminus \{0\}$. It follows from (2.1) and (2.22) that

$$\begin{aligned} &\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_B \\ &= \frac{1}{2} \lim_{n\to\infty} \left\|2\left[f\left(\frac{x+y+n(x+y)}{2}\right) + f\left(\frac{x+y-n(x+y)}{2}\right)\right]\right. \\ &\quad \left. - [f(x+nx) + f(x-nx)] - [f(y+ny) + f(y-ny)]\right\|_B \\ &\leqslant \frac{K}{2} \limsup_{n\to\infty} \left\|2f\left(\frac{(x+nx)+(y+ny)}{2}\right) - f(x+nx) - f(y+ny)\right\|_B \\ &\quad + \frac{K}{2} \limsup_{n\to\infty} \left\|2f\left(\frac{(x-nx)+(y-ny)}{2}\right) - f(x-nx) - f(y-ny)\right\|_B \\ &\leqslant \frac{K\theta}{2} \left[\limsup_{n\to\infty} (n+1)^{2r} + \limsup_{n\to\infty} (n-1)^{2r}\right] \|x\|_A^r \|y\|_A^r = 0. \end{aligned}$$

Therefore, we get that

$$2f\left(\frac{x+y}{2}\right) = f(x) + f(y) \tag{2.23}$$

for all $x, y \in A \setminus \{0\}$. Since $f(0) = 0$, (2.23) implies that f is odd. Let $x \in A \setminus \{0\}$. It follows from (2.23) that

$$\begin{aligned} 2f(x) = 2f\left(\frac{3x}{2} + \frac{-x}{2}\right) &= f(3x) + f(-x) \\ &= f(x + 2x) + f(x - 2x) \\ &= \frac{1}{2}[f(2x) + f(4x)] + \frac{1}{2}[f(2x) + f(-4x)] \\ &= f(2x). \end{aligned}$$

Since $f(0)=0$, we get that $f(2x)=2f(x)$ for all $x\in A$. Therefore, (2.23) implies that

$$f(x+y)=f(x)+f(y)$$

for all $x,y\in A$. So f is $\mathbb{Q}$-linear. Since the mapping $t\mapsto f(tx)$ is continuous in zero for each fixed $x\in A$, then the mapping $t\mapsto f(tx)$ is continuous in $t\in\mathbb{R}$. Therefore f is $\mathbb{R}$-linear. Also, it follows from (2.4) and (2.22) that

$$\begin{aligned}
\|f(xy)-&f(x)f(y)\|_B\\
&=\frac{1}{2}\lim_{n\to\infty}\Big\|[f(xy+nxy)+f(xy-nxy)]-f(x)[f(y+ny)+f(y-ny)]\Big\|_B\\
&\leqslant\frac{K}{2}\limsup_{n\to\infty}\|f(xy+nxy)-f(x)f(y+ny)\|_B\\
&\quad+\frac{K}{2}\limsup_{n\to\infty}\|f(xy-nxy)-f(x)f(y-ny)\|_B\\
&\leqslant\frac{K\theta}{2}\Big[\limsup_{n\to\infty}(n+1)^s+\limsup_{n\to\infty}(n-1)^s\Big]\|x\|_A^s\|y\|_A^s=0
\end{aligned}$$

for all $x,y\in A\setminus\{0\}$. Since $f(0)=0$, then $f(xy)=f(x)f(y)$ for all $x,y\in A$. So the mapping $f:A\to B$ is a homomorphism. □

THEOREM 2.10. *Let $r,s<0$ and θ be positive real numbers, and let $f,g:A\to B$ be mappings with $f(0)=g(0)=0$ satisfying*

$$\Big\|2f\Big(\frac{x+y}{2}\Big)-f(x)-g(y)\Big\|_B\leqslant\theta\|x\|_A^r\|y\|_A^r \tag{2.24}$$

$$\|f(xy)-f(x)g(y)\|_B\leqslant\theta\|x\|_A^s\|y\|_A^s \tag{2.25}$$

for all $x,y\in A\setminus\{0\}$. If the mapping $t\mapsto f(tx)$ from $\mathbb{R}$ to B is continuous at zero for each fixed $x\in A$, then $f:A\to B$ is a homomorphism. Moreover $f=g$.

Proof. Replacing x by nx in (2.24), we get that

$$g(y)=\lim_{n\to\infty}\Big[2f\Big(\frac{nx+y}{2}\Big)-f(nx)\Big]$$

for all $x,y\in A\setminus\{0\}$. Similarly, we have

$$f(y)=\lim_{n\to\infty}\Big[2f\Big(\frac{nx+y}{2}\Big)-g(nx)\Big]$$

for all $x,y\in A\setminus\{0\}$. Hence

$$f(y)-g(y)=\lim_{n\to\infty}[f(nx)-g(nx)] \tag{2.26}$$

for all $x,y\in A\setminus\{0\}$. Replacing x and y by nx in (2.24), we get that

$$\lim_{n\to\infty}[f(nx)-g(nx)]=0 \tag{2.27}$$

for all $x \in A \setminus \{0\}$. Since $f(0) = g(0)$, it follows from (2.26) and (2.27) that $f = g$. Therefore the result follows from Theorem 2.9. □

THEOREM 2.11. *Let $r, s < 0$ and θ be positive real numbers, and let $f, g, h : A \to B$ be mappings with $f(0) = g(0) = 0$ satisfying (2.6) and (2.7) for all $x, y \in A \setminus \{0\}$. Let h be an odd mapping and let the mapping $t \mapsto f(tx)$ from $\mathbb{R}$ to B be continuous at zero for each fixed $x \in A$, then $f : A \to B$ is a homomorphism. Moreover $f = g = h$.*

Proof. Similar to the proof of Theorem 2.10 we have

$$g(x) = \lim_{n\to\infty}\left[2f\left(\frac{x+ny}{2}\right) - h(ny)\right], \quad h(x) = \lim_{n\to\infty}\left[2f\left(\frac{x+ny}{2}\right) - g(ny)\right]$$

for all $x, y \in A \setminus \{0\}$. Hence

$$g(x) - h(x) = \lim_{n\to\infty}[g(ny) - h(ny)] \tag{2.28}$$

for all $x, y \in A \setminus \{0\}$. Replacing x and y by nx and $-nx$, respectively, in (2.6), we get that

$$\lim_{n\to\infty}[g(nx) - h(nx)] = 2f(0) = 0 \tag{2.29}$$

for all $x \in A \setminus \{0\}$. Since $h(0) = g(0) = 0$, it follows from (2.28) and (2.29) that $g = h$.

Replacing y by $y - x$ in (2.6), we get that

$$\left\|2f\left(\frac{y}{2}\right) - h(x) - h(y-x)\right\|_B \leqslant \theta\|x\|_A^r\|y-x\|_A^r$$

for all $x \in A \setminus \{0\}$ and all $y \in A \setminus \{x\}$. Therefore, we have

$$2f\left(\frac{y}{2}\right) = \lim_{n\to\infty}[h(nx) - h(nx-y)], \quad h(y) = \lim_{n\to\infty}\left[2f\left(\frac{nx}{2}\right) - h(nx-y)\right]$$

for all $x, y \in A \setminus \{0\}$. So we get that

$$2f\left(\frac{y}{2}\right) - h(y) = \lim_{n\to\infty}\left[h(nx) - 2f\left(\frac{nx}{2}\right)\right] \tag{2.30}$$

for all $x, y \in A \setminus \{0\}$. Let $y_0 \in A \setminus \{0\}$ and let $b = 2f\left(\frac{y_0}{2}\right) - h(y_0)$. It follows from (2.30) that

$$2f\left(\frac{y}{2}\right) - h(y) = b \tag{2.31}$$

for all $y \in A \setminus \{0\}$. Replacing y by y/n in (2.31), we get that

$$2f\left(\frac{y}{2n}\right) - h\left(\frac{y}{n}\right) = b \tag{2.32}$$

for all $y \in A \setminus \{0\}$. Since h is odd, replacing y by $-y$ in (2.32), we get that

$$2f\left(\frac{-y}{2n}\right) + h\left(\frac{y}{n}\right) = b \tag{2.33}$$

for all $y \in A \setminus \{0\}$. It follows from (2.32) and (2.33), that

$$f\left(\frac{y}{2n}\right)+f\left(\frac{-y}{2n}\right)=b$$

for all $y \in A \setminus \{0\}$. Since the mapping $t \mapsto f(tx)$ is continuous in zero for each fixed $x \in A$, we obtain that $b=0$. Since $f(0)=h(0)=0$, (2.31) implies that

$$2f\left(\frac{y}{2}\right)=h(y) \tag{2.34}$$

for all $y \in A$. Therefore, (2.6) implies that

$$\|h(x+y)-h(x)-h(y)\|_B \leqslant \theta\|x\|_A^r\|y\|_A^r \tag{2.35}$$

for all $x,y \in A \setminus \{0\}$. It follows from (2.35) that

$$h(x)=\lim_{n\to\infty}[h(x+ny)-h(ny)]$$

for all $x,y \in A$. Therefore we have

$$\begin{aligned}
&\|h(x+y)-h(x)-h(y)\|_B\\
&\qquad=\lim_{n\to\infty}\Big\|[h(x+y+ny)-h(ny)]-h(x)-[h(y+ny)-h(ny)]\Big\|_B\\
&\qquad=\lim_{n\to\infty}\|h(x+y+ny)-h(x)-h(y+ny)\|_B\\
&\qquad\leqslant\theta\lim_{n\to\infty}(n+1)^r\|x\|_A^r\|y\|_A^r=0
\end{aligned}$$

for all $x,y \in A\setminus\{0\}$. So $h(x+y)=h(x)+h(y)$ for all $x,y \in A\setminus\{0\}$. Since $h(0)=0$, then h is additive and we conclude that h is $\mathbb{Q}$-linear. By the continuity of the mapping $t \mapsto h(tx)$, we get that h is $\mathbb{R}$-linear. Therefore, it follows from (2.34)

$$f(y)=\frac{1}{2}h(2y)=h(y)$$

for all $y \in A$. So $f=h=g$. Hence the result follows from Theorem 2.9. □

THEOREM 2.12. *Let r,t and θ be positive real numbers and $q,s<0$ be real numbers such that $\lambda=r+s\neq 1$. Assume that $f,g,h: A \to B$ are mappings satisfying*

$$\left\|2f\left(\frac{x+y}{2}\right)-g(x)-h(y)\right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^s \tag{2.36}$$

$$\|f(xy)-g(x)h(y)\|_B \leqslant \theta\|x\|_A^t\|y\|_A^q \tag{2.37}$$

for all $x \in A$ and all $y \in A\setminus\{0\}$. If $g(0)=0$ and the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ and $t \mapsto h(tx)$ are continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $g: A \to B$ is a homomorphism and satisfies

$$\|f(x)-g(x)\|_B \leqslant 2^{\lambda-1}C\|x\|_A^\lambda \tag{2.38}$$

for all $x \in A$, where $C=\min\left\{\theta, \frac{2K\theta}{|2^{\lambda p}-2^p|^{1/p}}\right\}$. Moreover $g: A \to B$ is a unique homomorphism satisfies (2.38).

Proof. Letting $x = 0$ in (2.37) and (2.36), we get that $f(0) = 0$ and $2f(y/2) = h(y)$ for all $y \in A \setminus \{0\}$. Since the mappings $t \mapsto f(tx)$ and $t \mapsto h(tx)$ are continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then $h(0) = 0$. So $2f(y/2) = h(y)$ for all $y \in A$. Therefore

$$\|h(x+y) - g(x) - h(y)\|_B \leqslant \theta \|x\|_A^r \|y\|_A^s \tag{2.39}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. It follows from (2.39) that

$$g(x) = \lim_{n\to\infty} [h(x+ny) - h(ny)]$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. So

$$\begin{aligned}
&\|g(x+y) - g(x) - g(y)\|_B \\
&\qquad = \lim_{n\to\infty} \Big\| [h(x+y+ny) - h(ny)] - g(x) - [h(y+ny) - h(ny)] \Big\|_B \\
&\qquad = \lim_{n\to\infty} \|h(x+y+ny) - g(x) - h(y+ny)\|_B \\
&\qquad \leqslant \theta \lim_{n\to\infty} (n+1)^s \|x\|_A^r \|y\|_A^s = 0
\end{aligned}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. So $g(x+y) = g(x) + g(y)$ for all $x \in A$ and all $y \in A \setminus \{0\}$. Since $g(0) = 0$, then g is additive and we conclude that g is $\mathbb{Q}$-linear. By the continuity of the mapping $t \mapsto g(tx)$, we get that g is $\mathbb{R}$-linear. Also we have

$$\begin{aligned}
&\|g(xy) - g(x)g(y)\|_B \\
&\qquad = \lim_{n\to\infty} \left\| 2\left[f\left(\frac{xy+nxy}{2}\right) - f\left(\frac{nxy}{2}\right)\right] - g(x)[h(y+ny) - h(ny)] \right\|_B \\
&\qquad = 2K \limsup_{n\to\infty} \left\| f\left(\frac{xy+nxy}{2}\right) - g\left(\frac{x}{2}\right) h(y+ny) \right\|_B \\
&\qquad\quad + 2K \limsup_{n\to\infty} \left\| f\left(\frac{nxy}{2}\right) - g\left(\frac{x}{2}\right) h(ny) \right\|_B \\
&\qquad \leqslant K\theta 2^{1-t} \Big[\limsup_{n\to\infty} (n+1)^q + \limsup_{n\to\infty} n^q \Big] \|x\|_A^t \|y\|_A^q = 0
\end{aligned}$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Since $g(0) = 0$, then $g(xy) = g(x)g(y)$ for all $x, y \in A$. It proves that the mapping $g : A \to B$ is a homomorphism. To continue the rest of the proof we have two cases:

Case I. Let $\lambda > 1$. Putting $y = x$ in (2.39), we get that

$$\|h(2x) - g(x) - h(x)\|_B \leqslant \theta \|x\|_A^\lambda \tag{2.40}$$

for all $x \in A \setminus \{0\}$. If we put $x = -y$ in (2.39), we have

$$\|g(y) - h(y)\|_B \leqslant \theta \|y\|_A^\lambda \tag{2.41}$$

for all $y \in A \setminus \{0\}$. It is clear that (2.40) and (2.41) hold for all $x, y \in A$. It follows from (2.40) and (2.41) that

$$\|h(2x) - 2h(x)\|_B \leqslant 2K\theta \|x\|_A^\lambda \tag{2.42}$$

for all $x \in A$. If we replace x in (2.42) by $x/2^{n+1}$ and multiply both sides of (2.42) to 2^n, then we have

$$\Big\|2^{n+1}h\Big(\frac{x}{2^{n+1}}\Big)-2^nh\Big(\frac{x}{2^n}\Big)\Big\|_B \leqslant K\theta\Big(\frac{2}{2^\lambda}\Big)^{n+1}\|x\|_A^\lambda \tag{2.43}$$

for all $x \in A$. Since B is a p-Banach algebra,

$$\begin{aligned}\Big\|2^{n+1}h\Big(\frac{x}{2^{n+1}}\Big)-2^mh\Big(\frac{x}{2^m}\Big)\Big\|_B^p &\leqslant \sum_{i=m}^{n}\Big\|2^{i+1}h\Big(\frac{x}{2^{i+1}}\Big)-2^ih\Big(\frac{x}{2^i}\Big)\Big\|_B^p\\ &\leqslant K^p\theta^p\sum_{i=m}^{n}\Big(\frac{2}{2^\lambda}\Big)^{(i+1)p}\|x\|_A^{\lambda p}\end{aligned} \tag{2.44}$$

for all non-negative integers m and n with $n \geqslant m$ and all $x \in A$. It follows from (2.44) that the sequence $\{2^nh(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^nh(\frac{x}{2^n})\}$ converges. So one can define a mapping $T : A \to B$ by

$$T(x) := \lim_{n\to\infty} 2^nh\Big(\frac{x}{2^n}\Big)$$

for all $x \in A$. Since $2f(x/2) = h(x)$ for all $x \in A$, then

$$T(x) = \lim_{n\to\infty} 2^nh\Big(\frac{x}{2^n}\Big) = \lim_{n\to\infty} 2^nf\Big(\frac{x}{2^n}\Big)$$

for all $x \in A$. It follows from (2.41)

$$\begin{aligned}\|T(x)-g(x)\| &= \lim_{n\to\infty}\Big\|2^nh\Big(\frac{x}{2^n}\Big)-2^ng\Big(\frac{x}{2^n}\Big)\Big\|_B\\ &\leqslant \theta\lim_{n\to\infty}\Big(\frac{2}{2^\lambda}\Big)^n\|x\|_A^\lambda = 0\end{aligned}$$

for all $x \in A$. Therefore $T = g$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.44), we get

$$\|h(x)-g(x)\|_B \leqslant \frac{2K\theta}{(2^{\lambda p}-2^p)^{1/p}}\|x\|_A^\lambda \tag{2.45}$$

for all $x \in A$. Since the mapping $g : A \to B$ is a homomorphism and $h(2x) = 2f(x)$ for all $x \in A$, then (2.38) follows from (2.41) and (2.45). To prove the uniqueness of g, let $Q : A \to B$ be another homomorphism satisfying (2.38). We have

$$\begin{aligned}\|g(x)-Q(x)\|_B &= \lim_{n\to\infty} 2^n\Big\|f\Big(\frac{x}{2^n}\Big)-Q\Big(\frac{x}{2^n}\Big)\Big\|_B\\ &\leqslant 2^{\lambda-1}C\lim_{n\to\infty}\Big(\frac{2}{2^\lambda}\Big)^n\|x\|_A^\lambda = 0\end{aligned}$$

for all $x \in A$. So $g = Q$.

Case II. Let $\lambda < 1$. If we replace x in (2.42) by 2^nx and divide both sides of (2.42) by 2^{n+1}, then we have

$$\Big\|\frac{1}{2^{n+1}}h(2^{n+1}x)-\frac{1}{2^n}h(2^nx)\Big\|_B \leqslant K\theta\Big(\frac{2^\lambda}{2}\Big)^n\|x\|_A^\lambda \tag{2.46}$$

for all $x \in A$ $(x \in A \setminus \{0\})$. Since B is a p-Banach algebra,

$$\begin{aligned}\left\|\frac{1}{2^{n+1}}h(2^{n+1}x) - \frac{1}{2^m}h(2^m x)\right\|_B^p &\leqslant \sum_{i=m}^{n}\left\|\frac{1}{2^{i+1}}h(2^{i+1}x) - \frac{1}{2^i}h(2^i x)\right\|_B^p \\ &\leqslant K^p\theta^p\sum_{i=m}^{n}\left(\frac{2^\lambda}{2}\right)^{ip}\|x\|_A^{\lambda p}\end{aligned} \tag{2.47}$$

for all non-negative integers m and n with $n \geqslant m$ and all $x \in A$ $(x \in A \setminus \{0\})$. It follows from (2.47) that the sequence $\{\frac{1}{2^n}h(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n}h(2^n x)\}$ converges. So one can define a mapping $T : A \to B$ by

$$T(x) := \lim_{n\to\infty}\frac{1}{2^n}h(2^n x)$$

for all $x \in A$. The rest of the proof is similar to the proof of case I. □

COROLLARY 2.13. *Let r, t and θ be positive real numbers and let $q, s < 0$ be real numbers. Assume that $f : A \to B$ is a mapping satisfying*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_B \leqslant \theta\|x\|_A^r\|y\|_A^s \tag{2.48}$$

$$\|f(xy) - f(x)f(y)\|_B \leqslant \theta\|x\|_A^t\|y\|_A^q \tag{2.49}$$

for all $x \in A$ and all $y \in A\setminus\{0\}$. If $f(0) = 0$ and the mapping $t \mapsto f(tx)$ is continuous in $0 \in \mathbb{R}$ for each fixed $x \in A$, then the mapping $f : A \to B$ is a homomorphism.

In Theorem 2.12, let $0 < t < 1$ and $\lambda < 1$. If we replace x in (2.37) by nx and divide both sides of (2.37) by n, then we have

$$\left\|\frac{1}{n}f(nxy) - g(x)h(y)\right\|_B \leqslant \theta n^{t-1}\|x\|_A^t\|y\|_A^q$$

for all $x \in A$ and all $y \in A \setminus \{0\}$. Therefore

$$\lim_{n\to\infty}\frac{1}{n}f(nxy) = g(x)h(y)$$

for all $x, y \in A$. It follows from the proof of Theorem 2.12 (case II), $g(xy) = g(x)h(y)$ for all $x, y \in A$. Since the mapping $g : A \to B$ is a homomorphism, then we have

$$g(x)[g(y) - h(y)] = 0 \tag{2.50}$$

for all $x, y \in A$. Similarly, one can obtain (2.50) if $t > 1$ and $\lambda > 1$. Therefore we have the following results:

COROLLARY 2.14. *In Theorem 2.12, let $g \neq 0$, $t \neq 1$ and $B = \mathbb{C}$. Then $f, g, h : A \to B$ are homomorphisms. Moreover $f = g = h$.*

COROLLARY 2.15. *In Theorem 2.12, let A and B be unital with units e_A and e_B, respectively. If $t \neq 1$ and $g(e_A) = e_B$, then $f, g, h : A \to B$ are homomorphisms. Moreover $f = g = h$.*

3. Homomorphisms between unital quasi-Banach algebras

Throughout this section, assume that A is a unital quasi-Banach algebra with quasi-norm $\|.\|_A$ and unit e and that B is a unital p-Banach algebra with p-norm $\|.\|_B$ and unit e'.

We investigate homomorphisms between unital quasi-Banach algebras, associated to the Pexiderized Jensen functional equation. We generalize the results of [21].

THEOREM 3.1. *Let r,s and θ be positive real numbers such that $0<r<\frac{1}{2}$ and $0<s<1$, and let $f,g,h:A\to B$ be mappings satisfying (2.6) and (2.7) for all $x,y\in A$. Suppose that at least one of the mappings $t\mapsto f(tx)$, $t\mapsto g(tx)$ or $t\mapsto h(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$, and $\lim_{n\to\infty}\frac{1}{2^n}f(2^ne)=e'$. Then the mappings $f,g,h:A\to B$ are homomorphisms. Moreover $f=g=h$*

Proof. By Theorem 2.7, there exists a unique homomorphism $T:A\to B$ satisfying

$$T(x)=\lim_{n\to\infty}\frac{1}{2^n}f(2^nx)=\lim_{n\to\infty}\frac{1}{2^n}g(2^nx)=\lim_{n\to\infty}\frac{1}{2^n}h(2^nx)$$

for all $x\in A$. It follows from (2.7) that

$$\begin{aligned}\|T(x)-g(x)\|_B&=\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^nx)-g(x)\right\|_B\\&=\lim_{n\to\infty}\left\|\frac{1}{2^n}f(2^nxe)-g(x)e'\right\|_B\\&=\lim_{n\to\infty}\frac{1}{2^n}\left\|f(2^nxe)-g(x)h(2^ne)\right\|_B\\&\leqslant\theta\lim_{n\to\infty}\left(\frac{2^s}{2}\right)^n\|x\|_A^s\|e\|_A^s=0\end{aligned}$$

for all $x\in A$. So $T=g$. Similarly, one can obtain that $T=h$. Therefore $g=h$ and $g(0)=h(0)=0$. Since the mapping $h:A\to B$ is a homomorphism, it follows from (2.10) that $f=h$. $\square$

COROLLARY 3.2. *Let θ,r,s be non-negative real numbers such that $0<r<\frac{1}{2}$ and $0<s<1$. Suppose that $f:A\to B$ is a mapping satisfies (2.1) and (2.4) for all $x,y\in A$. If the mapping $t\mapsto f(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$ and $\lim_{n\to\infty}\frac{1}{2^n}f(2^ne)=e'$, then the mapping $f:A\to B$ is a homomorphism.*

THEOREM 3.3. *Let θ be a positive real number and let $f,g,h:A\to B$ be mappings satisfying (2.14) and (2.15). If at least one of the mappings $t\mapsto f(tx)$, $t\mapsto g(tx)$ or $t\mapsto h(tx)$ is continuous in $t\in\mathbb{R}$ for each fixed $x\in A$ and $\lim_{n\to\infty}\frac{1}{3^n}f(3^ne)=e'$, then the mappings $g,h:A\to B$ are homomorphisms. Moreover $g=h$ and*

$$\|f(x)-g(x)\|_B\leqslant\frac{\theta}{2}\tag{3.1}$$

for all $x\in A$.

Proof. By Theorem 2.8 and its proof, there exists a unique homomorphism $T : A \to B$ satisfying (2.21). Similar to the proof of Theorem 3.1, we get that the mappings $g,h : A \to B$ are homomorphisms and $g = h$. Letting $y = x$ in (2.14), we get (3.1). $\square$

REMARK 3.4. In Theorem 3.3, we can not infer that f is a homomorphism. Let A be a unital algebra with unit e, and let $f,g,h : A \to A$ be mappings defined by

$$f(x) = x + \frac{\theta}{4\|e\|}e, \quad g(x) = h(x) = x$$

for all $x \in A$. It is clear that the conditions of Theorem 3.3 hold (with $A = B$), but the mapping $f : A \to A$ is not a homomorphism.

COROLLARY 3.5. *Let θ be a non-negative real number. Suppose that $f : A \to B$ is a mapping satisfies*

$$\left\|2f\left(\frac{x+y}{2}\right) - f(x) - f(y)\right\|_B \leqslant \theta,$$
$$\|f(xy) - f(x)f(y)\|_B \leqslant \theta$$

for all $x,y \in A$. If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} \frac{1}{3^n}f(3^n e) = e'$, then the mapping $f : A \to B$ is a homomorphism.

THEOREM 3.6. *Let r,s and θ be positive real numbers such that $r > \frac{1}{2}$ and $s > 1$, and let $f,g,h : A \to B$ be mappings satisfying (2.6) and (2.7) for all $x,y \in A$. Suppose that at least one of the mappings $t \mapsto f(tx)$, $t \mapsto g(tx)$ or $t \mapsto h(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$, and $\lim_{n\to\infty} 2^n[f(\frac{e}{2^n}) - f(0)] = e'$. Then the mappings $f,g,h : A \to B$ are homomorphisms. Moreover $f = g = h$.*

Proof. By Theorem 2.7, there exists a unique homomorphism $T : A \to B$ satisfying

$$T(x) = \lim_{n\to\infty} 2^n\left[f\left(\frac{x}{2^n}\right) - f(0)\right] = \lim_{n\to\infty} 2^n\left[g\left(\frac{x}{2^n}\right) - g(0)\right] = \lim_{n\to\infty} 2^n\left[h\left(\frac{x}{2^n}\right) - h(0)\right]$$

for all $x \in A$. Since $f(0) = g(x)h(0)$ for all $x \in A$, it follows from (2.7) that

$$\begin{aligned}
\|T(x) - g(x)\|_B &= \lim_{n\to\infty} \left\|2^n\left[f\left(\frac{x}{2^n}\right) - f(0)\right] - g(x)\right\|_B \\
&= \lim_{n\to\infty} \left\|2^n\left[f\left(\frac{xe}{2^n}\right) - f(0)\right] - g(x)e'\right\|_B \\
&= \lim_{n\to\infty} 2^n\left\|\left[f\left(\frac{xe}{2^n}\right) - f(0)\right] - g(x)\left[h\left(\frac{e}{2^n}\right) - h(0)\right]\right\|_B \\
&= \lim_{n\to\infty} 2^n\left\|f\left(\frac{xe}{2^n}\right) - g(x)h\left(\frac{e}{2^n}\right)\right\|_B \\
&\leqslant \theta \lim_{n\to\infty} \left(\frac{2}{2^s}\right)^n \|x\|_A^s\|e\|_A^s = 0
\end{aligned}$$

for all $x \in A$. So $T = g$. Similarly, one can obtain that $T = h$. Therefore $g = h$ and $g(0) = h(0) = 0$. Since the mapping $h : A \to B$ is a homomorphism, it follows from (2.10) that $f = h$. $\square$

COROLLARY 3.7. *Let θ, r, s be positive real numbers such that $r > \frac{1}{2}$ and $s > 1$. Suppose that $f : A \to B$ is a mapping satisfies (2.1) and (2.4) for all $x, y \in A$. If the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in A$ and $\lim_{n\to\infty} 2^n[f(\frac{e}{2^n}) - f(0)] = e'$, then the mapping $f : A \to B$ is a homomorphism.*

REMARK 3.8. In Corollaries 3.2, 3.5 and 3.7, the results hold if e and e' are left (right) units for A and B, respectively.

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