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A VISCOSITY RELAXED–EXTRAGRADIENT METHOD FOR MONOTONE VARIATIONAL INEQUALITIES AND FIXED POINT PROBLEMS

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Abstract. In this paper we introduce a viscosity relaxed-extragradient method for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for a monotone, Lipschitz-continuous mapping in a real Hilbert space H. The viscosity relaxed-extragradient method is based on two methods: extragradient-like approximation method and viscosity approximation method. We derive a weak convergence theorem for two sequences generated by this method. Utilizing this theorem we also construct an iterative process for finding a common zero of two mappings, one of which is a monotone, Lipschitz continuous mapping of H into itself and the other taken from the more general class of maximal monotone mappings of H into 2^H.

1. Introduction

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C. When $\{x_n\}$ is a sequence in H, then $x_n \to x$ (resp. $x_n \rightharpoonup x$) will denote strong (resp. weak) convergence of the sequence $\{x_n\}$ to x.

DEFINITION 1.1. Let $A : C \to H$ be a mapping. Then A is called

(i) monotone if

$$\langle Au - Av, u - v \rangle \geqslant 0 \quad \forall u, v \in C;$$

(ii) α-inverse-strongly-monotone (see [1,2]) if there exists a positive constant α such that

$$\langle Au - Av, u - v \rangle \geqslant \alpha \|Au - Av\|^2 \quad \forall u, v \in C;$$

(iii) β-strongly-monotone if there exists a positive constant β such that

$$\langle Au - Av, u - v \rangle \geqslant \beta \|u - v\|^2 \quad \forall u, v \in C;$$

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(iv) k-Lipschitz-continuous if there exists a positive constant k such that

$$\|Au - Av\| \leqslant k\|u - v\| \quad \forall u, v \in C.$$

Obviously, it is easy to see that every α-inverse-strongly-monotone mapping A is monotone and Lipschitz-continuous. Let $S : C \to C$ be a self-mapping on C. Then S is called nonexpansive if for all $u, v \in C$

$$\|Su - Sv\| \leqslant \|u - v\|.$$

We denote by $F(S)$ the set of fixed points of S, i.e., $F(S) = \{u \in C : Su = u\}$.

Let $A : C \to H$ be a mapping. The variational inequality problem is to find a $u \in C$ such that

$$\langle Au, v - u\rangle \geqslant 0 \quad \forall v \in C.$$

Following the notations in [4], the set of solutions of the variational inequality problem is denoted by $VI(C,A)$. It is well known that, if A is a strongly monotone and Lipschitz-continuous mapping on C, then the variational inequality problem has a unique solution. How to actually find a solution of the variational inequality problem is one of the best important topics in the study of the variational inequality problem. Indeed, there are a lot of different approaches towards solving this problem in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. A great deal of effort has gone into this problem; see e.g., [2–5, 9, 11, 12, 14–18, 20].

Recently, for finding an element of $F(S) \cap VI(C,A)$ under the assumption that a set $C \subset H$ is closed and convex, a mapping S of C into itself is nonexpansive, and a mapping A of C into H is α-inverse-strongly-monotone, Takahashi and Toyoda [4] introduced the following iterative scheme:

$$\begin{cases} x_0 = x \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n) S P_C(x_n - \lambda_n A x_n) \ \forall n \geqslant 0, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\lambda_n\}$ is a sequence in $(0,2\alpha)$. They proved that if $F(S) \cap VI(C,A) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to some $z \in F(S) \cap VI(C,A)$.

In 1976, for finding a solution of the nonconstrained variational inequality problem in the finite-dimensional Euclidean space $\mathcal{R}^n$ under the assumption that a set $C \subset \mathcal{R}^n$ is closed and convex and a mapping A of C into $\mathcal{R}^n$ is monotone and k-Lipschitz-continuous, Korpelevich [5] introduced the following so-called extragradient method:

$$\begin{cases} x_0 = x \in C, \\ \bar{x}_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A \bar{x}_n) \ \forall n \geqslant 0, \end{cases} \tag{1.2}$$

where $\lambda \in (0, 1/k)$. He proved that if $VI(C,A) \neq \emptyset$, then the sequences $\{x_n\}$ and $\{\bar{x}_n\}$, generated by (1.2), converge to the same point $z \in VI(C,A)$.

Recently, motivated by the idea of Korpelevich's extragradient method [5], Nadezhkina and Takahashi [3] introduced the following iterative scheme for finding an element of $F(S) \cap VI(C,A)$ and proved the following weak convergence result.

THEOREM 1.1. [3, Theorem 3.1]. *Let C be a closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz-continuous mapping of C into H and S be a nonexpansive mapping of C into itself such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$, $\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_n + (1-\alpha_n) S P_C(x_n - \lambda A y_n) \ \ \forall n \geqslant 0, \end{cases} \tag{1.3}$$

where $\{\lambda_n\} \subset [a,b]$ for some $a,b \in (0,1/k)$ and $\{\alpha_n\} \subset [c,d]$ for some $c,d \in (0,1)$. Then the sequences $\{x_n\}$, $\{y_n\}$ converge weakly to the same point $z \in F(S) \cap VI(C,A)$ where $z = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

At the same time, the idea of the extragradient iterative process introduced by Korpelevich was successively generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see e.g., [11,12,15,19].

Very recently, inspired by Nadezhkina and Takahashi's iterative scheme [3], Zeng and Yao [12] introduced another iterative scheme for finding an element of $F(S) \cap VI(C,A)$ and established the following strong convergence theorem.

THEOREM 1.2. [12, Theorem 3.1]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone, k-Lipschitz-continuous mapping and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n) S P_C(x_n - \lambda_n A y_n) \ \ \forall n \geqslant 0, \end{cases} \tag{1.4}$$

where $\{\lambda_n\}$ and $\{\alpha_n\}$ satisfy the conditions:

(a) $\{\lambda_n k\} \subset (0, 1-\delta)$ *for some* $\delta \in (0,1)$*;*

(b) $\{\alpha_n\} \subset (0,1)$, $\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$.

Then the sequences $\{x_n\}, \{y_n\}$ converge strongly to the same point $P_{F(S)\cap VI(C,A)}(x_0)$ provided $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$.

On the other hand, in 2004, Xu [21] also considered so-called viscosity approximation method for finding a fixed point of a nonexpansive self-mapping on C which solves some variational inequality. Motivated by Nadezhkina and Takahashi's extragradient method [3] and Xu's viscosity approximation method [21], Ceng and Yao [11] introduced an extragradient-like approximation method and proved the following strong convergence theorem.

THEOREM 1.3. [11, Theorem 3.1]. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$, $A : C \to H$ be a monotone, k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = (1-\gamma_n) x_n + \gamma_n P_C(x_n - \lambda_n A x_n), \\ x_{n+1} = (1-\alpha_n-\beta_n) x_n + \alpha_n f(y_n) + \beta_n S P_C(x_n - \lambda_n A y_n) \ \ \forall n \geqslant 0, \end{cases} \tag{1.5}$$

where $\{\lambda_n\}$ *is a sequence in* $(0,1)$ *with* $\sum_{n=0}^{\infty}\lambda_n < \infty$, *and* $\{\alpha_n\},\{\beta_n\},\{\gamma_n\}$ *are three sequences in* $[0,1]$ *satisfying the conditions:*

(i) $\alpha_n+\beta_n \leqslant 1$ *for all* $n \geqslant 0$*;*

(ii) $\lim_{n\to\infty}\alpha_n = 0,\ \sum_{n=0}^{\infty}\alpha_n = \infty$*;*

(iii) $0 < \liminf_{n\to\infty}\beta_n \leqslant \limsup_{n\to\infty}\beta_n < 1$.

(iv) $\{\lambda_n\}\subset[a,b]$ *for some* $a,b\in(0,1/k)$.

Then the sequences $\{x_n\},\{y_n\}$ *converge strongly to the same point* $q = P_{F(S)\cap VI(C,A)}f(q)$ *if and only if* $\{Ax_n\}$ *is bounded and* $\liminf_{n\to\infty}\langle Ax_n, y-x_n\rangle \geqslant 0$ *for all* $y\in C$.

In this paper, we introduce a viscosity relaxed-extragradient method which is based on the above extragradient-like approximation method and viscosity approximation method, i.e.,

$$\begin{cases} x_0 = x\in C,\\ y_n = P_C(x_n-\lambda_n\mu_nAx_n-\lambda_n(1-\mu_n)Ay_n),\\ t_n = P_C(x_n-\lambda_nAy_n-\lambda_n(1-\mu_n)At_n),\\ x_{n+1} = (1-\alpha_n-\beta_n)x_n+\alpha_nf(t_n)+\beta_nSt_n \ \ \forall n\geqslant 0,\end{cases}$$

where $\{\lambda_n\},\{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\},\{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n+\beta_n\leqslant\tau<1\ \forall n\geqslant 0$ for some $\tau\in(0,1)$;

(ii) $\sum_{n=0}^{\infty}\alpha_n<\infty$ and $0<\sigma\leqslant\beta_n\ \forall n\geqslant 0$ for some $\sigma\in(0,1)$;

(iii) $\lim_{n\to\infty}\mu_n = 1$;

(iv) $\{\lambda_n\}\subset[a,b]$ for some $a,b\in(0,1/k)$.

It is shown that the sequences $\{x_n\},\{y_n\}$ generated by the above method converge weakly to the same point $u\in F(S)\cap VI(C,A)$. Utilizing this result we also construct an iterative process for finding a common zero of two mappings, one of which is a monotone, k-Lipschitz continuous mapping of H into itself and the other taken from the more general class of maximal monotone mappings of H into 2^H.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. For every point $x\in H$ there exists a unique nearest point in C, denoted by P_Cx, such that $\|x-P_Cx\|\leqslant\|x-y\|$ for all $y\in C$. P_C is called the metric projection of H onto C. It is known that P_C is a nonexpansive mapping from H onto C. It is also known that $P_Cx\in C$ and

$$\langle x-P_Cx, P_Cx-y\rangle\geqslant 0 \tag{2.1}$$

for all $x\in H,\ y\in C$; see [13] for more details. It is easy to see that (2.1) is equivalent to

$$\|x-y\|^2\geqslant\|x-P_Cx\|^2+\|y-P_Cx\|^2 \tag{2.2}$$

for all $x\in H,\ y\in C$.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.1) implies

$$u \in VI(C,A) \Leftrightarrow u = P_C(u - \lambda Au) \; \forall \lambda > 0.$$

It is also known that H satisfies Opial's condition [10], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$ the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

The following results will be used in the rest of this paper.

LEMMA 2.1. (Tan and Xu [6, p. 303]). *Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant a_n + b_n \quad \forall n \geqslant 0.$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

LEMMA 2.2. (Demiclosedness Principle [13]). *Assume that S is a nonexpansive self-mapping of a closed convex subset C of a Hilbert space H. If S has a fixed point, then $I-S$ is demiclosed; that is, whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-S)x_n\}$ converges strongly to some $y \in H$, it follows that $(I-S)x = y$. Here I is the identity operator of H.*

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geqslant 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geqslant 0$ for all $(y,g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz-continuous mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v-u, w\rangle \geqslant 0 \text{ for all } u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in VI(C,A)$; see [7].

Throughout the rest of the paper, we shall use the following notation: for a given sequence $\{x_n\} \subset H$, $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$; that is,

$$\omega_w(x_n) := \{x \in H : \{x_{n_j}\} \text{ converges weakly to } x \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

3. Weak Convergence Theorem

We are now in a position to prove our main results in this paper. To prove it, we need two lemmas. The first lemma was proved by Schu [8] in a uniformly convex Banach space.

LEMMA 3.1. *Let H be a real Hilbert space, let $\{\varrho_n\}$ be a sequence of real numbers such that $0 < a \leqslant \varrho_n \leqslant b < 1$ for all $n \geqslant 0$, and let $\{v_n\}$ and $\{w_n\}$ be sequences in H such that*

$$\limsup_{n\to\infty} \|v_n\| \leqslant c,$$
$$\limsup_{n\to\infty} \|w_n\| \leqslant c,$$
$$\lim_{n\to\infty} \|\varrho_n v_n + (1-\varrho_n)w_n\| = c,$$

for some $c \geqslant 0$. Then,

$$\lim_{n\to\infty} \|v_n - w_n\| = 0.$$

THEOREM 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to C$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$, $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n), \\ t_n = P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n St_n \ \ \forall n \geqslant 0, \end{cases} \tag{3.1}$$

where $\{\lambda_n\}, \{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1 \ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*
(ii) $\sum_{n=0}^{\infty} \alpha_n < \infty$ *and* $0 < \sigma \leqslant \beta_n \ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*
(iii) $\lim_{n\to\infty} \mu_n = 1$*;*
(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$.

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $u \in F(S) \cap VI(C,A)$.

REMARK 3.1. First, observe that for all $x,y \in C$ and all $n \geqslant 0$

$$\begin{aligned} &\|P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu_n)Ax) - P_C(x_n - \lambda_n\mu Ax_n - \lambda_n(1-\mu_n)Ay)\| \\ &\quad \leqslant \|(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ax) - (x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay)\| \\ &\quad = \lambda_n(1-\mu_n)\|Ax - Ay\| \\ &\quad \leqslant \lambda_n k\|x-y\|. \end{aligned}$$

Thus, by Banach Contraction Principle, we know that for each $n \geqslant 0$ there exists a unique $y_n \in C$ such that

$$y_n = P_C(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n). \tag{3.2}$$

Also, observe that for all $x, y \in C$ and all $n \geqslant 0$

$$\begin{aligned}
&\|P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)Ax) - P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)Ay)\| \\
&\quad \leqslant \|(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)Ax) - (x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)Ay)\| \\
&\quad = \lambda_n(1-\mu_n)\|Ax - Ay\| \\
&\quad \leqslant \lambda_n k\|x - y\|.
\end{aligned}$$

Utilizing Banach Contraction Principle, we know that for each $n \geqslant 0$ there exists a unique $t_n \in C$ such that

$$t_n = P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n). \tag{3.3}$$

Proof of Theorem 3.1. We divide the proof into several steps.

Step 1. We claim that $\{x_n\}, \{y_n\}$ and $\{t_n\}$ are bounded. Indeed, note that $t_n = P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n)$ for all $n \geqslant 0$. Let $u \in F(S) \cap VI(C,A)$ be an arbitrary element. From (2.2), monotonicity of A, and $u \in VI(C,A)$, we have

$$\begin{aligned}
\|t_n - u\|^2 &\leqslant \|(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n) - u\|^2 \\
&\qquad -\|(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n) - t_n\|^2 \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 \\
&\qquad -\|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 + 2\lambda_n\langle Ay_n, u - t_n\rangle \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\qquad +2\lambda_n(\langle Ay_n, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&= \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\qquad +2\lambda_n(\langle Ay_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle Ay_n, y_n - t_n\rangle) \\
&\leqslant \|x_n - \lambda_n(1-\mu_n)At_n - u\|^2 - \|x_n - \lambda_n(1-\mu_n)At_n - t_n\|^2 \\
&\qquad +2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - t_n\|^2 - 2\lambda_n(1-\mu_n)\langle At_n, t_n - u\rangle \\
&\qquad +2\lambda_n\langle Ay_n, y_n - t_n\rangle \\
&= \|x_n - u\|^2 - \|x_n - y_n\|^2 - 2\langle x_n - y_n, y_n - t_n\rangle - \|y_n - t_n\|^2 \\
&\qquad +2\lambda_n\langle Ay_n, y_n - t_n\rangle - 2\lambda_n(1-\mu_n)(\langle At_n - Au, t_n - u\rangle \\
&\qquad +\langle Au, t_n - u\rangle) \\
&\leqslant \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle.
\end{aligned}$$

Since $y_n = P_C(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}
&\langle x_n - \lambda_n Ay_n - y_n, t_n - y_n\rangle \\
&\quad = \langle x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n - y_n, t_n - y_n\rangle + \lambda_n\mu_n\langle Ax_n - Ay_n, t_n - y_n\rangle \\
&\quad \leqslant \lambda_n\mu_n\langle Ax_n - Ay_n, t_n - y_n\rangle \\
&\quad \leqslant \lambda_n k\|x_n - y_n\|\|t_n - y_n\|.
\end{aligned}$$

So, we have

$$\begin{aligned}
\|t_n - u\|^2 &\leqslant \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + 2\lambda_n k\|x_n - y_n\|\|t_n - y_n\| \\
&\leqslant \|x_n - u\|^2 - \|x_n - y_n\|^2 - \|y_n - t_n\|^2 + \lambda_n^2 k^2\|x_n - y_n\|^2 + \|y_n - t_n\|^2 \\
&= \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\
&\leqslant \|x_n - u\|^2.
\end{aligned} \tag{3.4}$$

Therefore, from (3.4), $x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n St_n$, and $u = Su$, we have

$$\begin{aligned}
\|x_{n+1}-u\| &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n St_n - u\| \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\| + \alpha_n\|f(t_n)-u\| + \beta_n\|St_n-u\| \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\| + \alpha_n\{\|f(t_n)-f(u)\| + \|f(u)-u\|\} + \beta_n\|t_n-u\| \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\| + \alpha_n\{\alpha\|t_n-u\| + \|f(u)-u\|\} + \beta_n\|t_n-u\| \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\| + \alpha_n\{\alpha\|x_n-u\| + \|f(u)-u\|\} + \beta_n\|x_n-u\| \\
&= (1-(1-\alpha)\alpha_n)\|x_n-u\| + (1-\alpha)\alpha_n\cdot\tfrac{1}{1-\alpha}\|f(u)-u\| \\
&\leqslant \max\{\|x_n-u\|, \tfrac{1}{1-\alpha}\|f(u)-u\|\}
\end{aligned} \tag{3.5}$$

for all $n \geqslant 0$. Obviously, it is easy to see that

$$\|x_n-u\| \leqslant \max\{\|x_0-u\|, \frac{1}{1-\alpha}\|f(u)-u\|\} \quad \forall n \geqslant 0.$$

This shows that $\{x_n\}$ is bounded and so are $\{t_n\},\{y_n\}$ due to (3.4).

Step 2. We claim that the following statements hold:

(i) $\lim_{n\to\infty}\|x_n-u\|$ exists for each $u \in F(S)\cap VI(C,A)$;

(ii) $\lim_{n\to\infty}\|x_n-y_n\| = 0$;

(iii) $\lim_{n\to\infty}\|x_n-t_n\| = 0$.

Indeed, let $u \in F(S)\cap VI(C,A)$ be an arbitrary element. Utilizing (3.4) we know that

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|(1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n St_n - u\|^2 \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\|^2 + \alpha_n\|f(t_n)-u\|^2 + \beta_n\|St_n-u\|^2 \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\|^2 + \alpha_n\|f(t_n)-u\|^2 + \beta_n\|t_n-u\|^2 \\
&\leqslant (1-\alpha_n-\beta_n)\|x_n-u\|^2 + \alpha_n\|f(t_n)-u\|^2 \\
&\qquad +\beta_n[\|x_n-u\|^2 + (\lambda_n^2k^2-1)\|x_n-y_n\|^2] \\
&\leqslant \|x_n-u\|^2 + \alpha_n\|f(t_n)-u\|^2 + \beta_n(\lambda_n^2k^2-1)\|x_n-y_n\|^2 \\
&\leqslant \|x_n-u\|^2 + \alpha_n\|f(t_n)-u\|^2.
\end{aligned} \tag{3.6}$$

Since $\sum_{n=0}^{\infty}\alpha_n < \infty$ and $\{f(t_n)-u\}$ is bounded, we deduce from Lemma 2.1 that $\lim_{n\to\infty}\|x_n-u\|$ exists. From the last relations, we obtain also

$$\begin{aligned}
\sigma(1-\lambda_n^2k^2)\|x_n-y_n\|^2 &\leqslant \beta_n(1-\lambda_n^2k^2)\|x_n-y_n\|^2 \\
&\leqslant \|x_n-u\|^2 - \|x_{n+1}-u\|^2 + \alpha_n\|f(t_n)-u\|^2.
\end{aligned}$$

So we have

$$\|x_n-y_n\|^2 \leqslant \frac{1}{\sigma(1-\lambda_n^2k^2)}\{\|x_n-u\|^2 - \|x_{n+1}-u\|^2 + \alpha_n\|f(t_n)-u\|^2\}.$$

Hence,

$$x_n - y_n \to 0 \quad \text{as } n\to\infty.$$

Further, we obtain

$$\begin{aligned}\|y_n - t_n\| &= \|P_C(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n) - P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n)\| \\ &\leqslant \|(x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n) - (x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n)\| \\ &= \|\lambda_n\mu_n(Ay_n - Ax_n) + \lambda_n(1-\mu_n)At_n\| \\ &\leqslant \lambda_n\mu_n\|Ay_n - Ax_n\| + \lambda_n(1-\mu_n)\|At_n\| \\ &\leqslant \lambda_n k\|y_n - x_n\| + \lambda_n(1-\mu_n)\|At_n\|.\end{aligned}$$

Since $x_n - y_n \to 0$, $\mu_n \to 1$ and $\{At_n\}$ is bounded, we get

$$y_n - t_n \to 0 \quad \text{as } n \to \infty.$$

From

$$\|x_n - t_n\| \leqslant \|x_n - y_n\| + \|y_n - t_n\|,$$

we have also

$$x_n - t_n \to 0 \quad \text{as } n \to \infty.$$

Step 3. We claim that the following statements hold:
(i) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
(ii) $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.
Indeed, according to Step 2 (i) we denotes

$$\lim_{n\to\infty} \|x_n - u\| = d.$$

Now put $\varrho_n = \alpha_n + \beta_n$ for all $n \geqslant 0$. Then we write

$$x_{n+1} = (1-\varrho_n)x_n + \varrho_n z_n,$$

where

$$z_n = \frac{\alpha_n f(t_n) + \beta_n St_n}{\alpha_n + \beta_n} = \frac{\alpha_n}{\alpha_n + \beta_n} f(t_n) + \frac{\beta_n}{\alpha_n + \beta_n} St_n.$$

Let $u \in F(S) \cap VI(C,A)$ be an arbitrary element. Since

$$\begin{aligned}\|z_n - u\| &\leqslant \tfrac{\alpha_n}{\alpha_n+\beta_n}\|f(t_n) - u\| + \tfrac{\beta_n}{\alpha_n+\beta_n}\|St_n - u\| \\ &\leqslant \tfrac{\alpha_n}{\sigma}\|f(t_n) - u\| + \|t_n - u\| \\ &\leqslant \tfrac{\alpha_n}{\sigma}\|f(t_n) - u\| + \|x_n - u\|,\end{aligned}$$

we have

$$\limsup_{n\to\infty} \|z_n - u\| \leqslant d.$$

Further, we have

$$\lim_{n\to\infty} \|(1-\varrho_n)(x_n - u) + \varrho_n(z_n - u)\| = \lim_{n\to\infty} \|x_{n+1} - u\| = d.$$

Note that conditions (i), (ii) imply that $0 < \sigma \leqslant \varrho_n \leqslant \tau < 1$ for all $n \geqslant 0$. Thus, by Lemma 3.1, we obtain

$$\lim_{n\to\infty} \|z_n - x_n\| = 0$$

which implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \varrho_n \|z_n - x_n\| = 0.$$

Observe that

$$\begin{aligned} \tfrac{\sigma}{\tau}\|St_n - x_n\| &\leqslant \tfrac{\beta_n}{\alpha_n+\beta_n}\|St_n - x_n\| \\ &= \|\tfrac{\alpha_n}{\alpha_n+\beta_n}(f(t_n) - x_n) + \tfrac{\beta_n}{\alpha_n+\beta_n}(St_n - x_n) - \tfrac{\alpha_n}{\alpha_n+\beta_n}(f(t_n) - x_n)\| \\ &\leqslant \|z_n - x_n\| + \tfrac{\alpha_n}{\alpha_n+\beta_n}\|f(t_n) - x_n\| \\ &\leqslant \|z_n - x_n\| + \tfrac{\alpha_n}{\sigma}\|f(t_n) - x_n\|, \end{aligned}$$

and hence

$$\lim_{n\to\infty} \|St_n - x_n\| = 0$$

due to the boundedness of $\{f(t_n)\}$ and $\{x_n\}$. Since

$$\|Sx_n - x_n\| \leqslant \|Sx_n - St_n\| + \|St_n - x_n\| \leqslant \|x_n - t_n\| + \|St_n - x_n\|,$$

we have from Step 2 (iii)

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Step 4. We claim that $\omega_w(x_n) \subset F(S) \cap VI(C,A)$, where $\omega_w(x_n)$ denotes the weak ω-limit set of $\{x_n\}$, i.e.,

$$\omega_w(x_n) = \{u \in H : \{x_{n_j}\} \text{ converges weakly to } u \text{ for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

Indeed, since $\{x_n\}$ is bounded, it has a subsequence which converges weakly to some point in C and hence $\omega_w(x_n) \neq \emptyset$. Let $u \in \omega_w(x_n)$ be an arbitrary point. Then there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ which converges weakly to u and hence we have $\lim_{j\to\infty} \|x_{n_j} - Sx_{n_j}\| = 0$. Note that from Lemma 2.2 it follows that $I - S$ is demiclosed at zero. Thus $u \in F(S)$. Now, we show $u \in VI(C,A)$. Let

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C,A)$; see [7]. Let $(v,w) \in G(T)$. Then we have $w \in Tv = Av + N_C v$ and hence $w - Av \in N_C v$. So, we have $\langle v - t, w - Av\rangle \geqslant 0$ for all $t \in C$. On the other hand, from $t_n = P_C(x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n)$ and $v \in C$ we have

$$\langle x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n - t_n, t_n - v\rangle \geqslant 0$$

and hence

$$\langle v - t_n, \frac{t_n - x_n}{\lambda_n} + Ay_n + (1-\mu_n)At_n\rangle \geqslant 0.$$

From $\langle v-t, w-Av\rangle \geqslant 0$ for all $t \in C$ and $t_{n_j} \in C$, we have

$$\begin{aligned}\langle v-t_{n_j}, w\rangle &\geqslant \langle v-t_{n_j}, Av\rangle \\ &\geqslant \langle v-t_{n_j}, Av\rangle - \langle v-t_{n_j}, \tfrac{t_{n_j}-x_{n_j}}{\lambda_{n_j}} + Ay_{n_j} + (1-\mu_{n_j})At_{n_j}\rangle \\ &= \langle v-t_{n_j}, Av-At_{n_j}\rangle + \langle v-t_{n_j}, At_{n_j}-Ay_{n_j}\rangle \\ &\qquad -\langle v-t_{n_j}, \tfrac{t_{n_j}-x_{n_j}}{\lambda_{n_j}}\rangle - (1-\mu_{n_j})\langle v-t_{n_j}, At_{n_j}\rangle \\ &\geqslant \langle v-t_{n_j}, At_{n_j}-Ay_{n_j}\rangle - \langle v-t_{n_j}, \tfrac{t_{n_j}-x_{n_j}}{\lambda_{n_j}}\rangle - (1-\mu_{n_j})\langle v-t_{n_j}, At_{n_j}\rangle .\end{aligned}$$

Since A is Lipschitz continuous, we have

$$At_{n_j} - Ay_{n_j} \to 0 \quad \text{as } j \to \infty.$$

So, we obtain $\langle v-u, w\rangle \geqslant 0$ as $j \to \infty$. Since T is maximal monotone, we have $u \in T^{-1}0$ and hence $u \in VI(C,A)$. Therefore, $u \in F(S) \cap VI(C,A)$. This shows that $\omega_w(x_n) \subset F(S) \cap VI(C,A)$.

Step 5. We claim that $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $u \in F(S) \cap VI(C,A)$.

Indeed, it is sufficient to show that $\omega_w(x_n)$ is a single-point set because $x_n - y_n \to 0$ as $n \to \infty$. Since $\omega_w(x_n) \neq \emptyset$, let us take two points $u, \hat{u} \in \omega_w(x_n)$ arbitrarily. Then there exist two subsequences $\{x_{n_j}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$ such that $x_{n_j} \rightharpoonup u$ and $x_{m_k} \rightharpoonup \hat{u}$, respectively. In terms of Step 4, we know that $u, \hat{u} \in F(S) \cap VI(C,A)$. Meantime, according to Step 2 (i) we also know that there exist both $\lim_{n\to\infty} \|x_n - u\|$ and $\lim_{n\to\infty} \|x_n - \hat{u}\|$. Let us show that $u = \hat{u}$. Assume that $u \neq \hat{u}$. From the Opial condition [10] it follows that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \liminf_{j\to\infty} \|x_{n_j} - u\| < \liminf_{j\to\infty} \|x_{n_j} - \hat{u}\| \\ &= \lim_{n\to\infty} \|x_n - \hat{u}\| = \liminf_{k\to\infty} \|x_{m_k} - \hat{u}\| \\ &< \liminf_{k\to\infty} \|x_{m_k} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This leads to a contradiction. Thus, we have $u = \hat{u}$. This implies that $\omega_w(x_n)$ is a single-point set. Without loss of generality, we may write $\omega_w(x_n) = \{u\}$. Consequently, $\{x_n\}$ converges weakly to $u \in F(S) \cap VI(C,A)$. Since $x_n - y_n \to 0$ as $n \to \infty$, we have also

$$y_n \rightharpoonup u \in F(S) \cap VI(C,A).$$

This completes the proof. □

The second lemma was proved by Takahashi and Toyoda [4].

LEMMA 3.2. *Let H be a real Hilbert space and let D be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H. Suppose that, for all $u \in D$,*

$$\|x_{n+1} - u\| \leqslant \|x_n - u\| \quad \forall n \geqslant 0.$$

Then, the sequence $\{P_D x_n\}$ converges strongly to some $z \in D$.

By the careful analysis of the proof of Theorem 3.1, we can state another weak convergence theorem.

THEOREM 3.2.. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a monotone and k-Lipschitz continuous mapping and $S : C \to C$ be a nonexpansive mapping such that $F(S) \cap VI(C,A) \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = P_C(x_n - \lambda_n \mu_n A x_n - \lambda_n(1-\mu_n)Ay_n), \\ t_n = P_C(x_n - \lambda_n A y_n - \lambda_n(1-\mu_n)At_n), \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n t_n + \beta_n S t_n \ \ \forall n \geqslant 0, \end{cases} \tag{3.7}$$

where $\{\lambda_n\}, \{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1 \ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*
(ii) $0 < \sigma \leqslant \beta_n \ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*
(iii) $\lim_{n\to\infty} \mu_n = 1$*;*
(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$.

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $u \in F(S) \cap VI(C,A)$, where $u = \lim_{n\to\infty} P_{F(S)\cap VI(C,A)} x_n$.

Proof. We divide the proof into several steps.

Step 1. We claim that the following statements hold:

(i) $\lim_{n\to\infty} \|x_n - u\|$ exists for each $u \in F(S) \cap VI(C,A)$;
(ii) $\{x_n\}, \{y_n\}$ and $\{t_n\}$ are bounded.

Indeed, let $u \in F(S) \cap VI(C,A)$ be an arbitrary element. As in Step 1 of the proof of Theorem 3.1, we can derive

$$\begin{aligned} \|t_n - u\|^2 &\leqslant \|x_n - u\|^2 + (\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\leqslant \|x_n - u\|^2. \end{aligned}$$

Moreover, we can obtain

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leqslant \|x_n - u\|^2 + (\alpha_n + \beta_n)(\lambda_n^2 k^2 - 1)\|x_n - y_n\|^2 \\ &\leqslant \|x_n - u\|^2. \end{aligned} \tag{3.8}$$

This implies that $\lim_{n\to\infty} \|x_n - u\|$ exists. Hence $\{x_n\}$ is bounded and so is $\{t_n\}$. Consequently, from (3.8) it follows that $\{y_n\}$ is bounded.

Step 2. We claim that the following statements hold:

(i) $\lim_{n\to\infty} \|x_n - y_n\| = 0$;
(ii) $\lim_{n\to\infty} \|x_n - t_n\| = 0$.
Indeed, utilizing (3.8) we have

$$\|x_n - y_n\|^2 \leqslant \frac{1}{(\alpha_n + \beta_n)(1 - \lambda_n^2 k^2)} \{\|x_n - u\|^2 - \|x_{n+1} - u\|^2\},$$

and hence $x_n - y_n \to 0$ as $n \to \infty$. As in Step 2 of the proof of Theorem 3.1, we can obtain $y_n - t_n \to 0$ as $n \to \infty$. Thus, from

$$\|x_n - t_n\| \leqslant \|x_n - y_n\| + \|y_n - t_n\|$$

we get $x_n - t_n \to 0$ as $n \to \infty$.

Step 3. We claim that the following statements hold:
(i) $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$;
(ii) $\lim_{n\to\infty} \|Sx_n - x_n\| = 0$.
Indeed, put $\varrho_n = \alpha_n + \beta_n$ for all $n \geqslant 0$. Then we write

$$x_{n+1} = (1 - \varrho_n)x_n + \varrho_n z_n,$$

where

$$z_n = \frac{\alpha_n t_n + \beta_n S t_n}{\alpha_n + \beta_n} = \frac{\alpha_n}{\alpha_n + \beta_n} t_n + \frac{\beta_n}{\alpha_n + \beta_n} S t_n.$$

As in Step 3 of the proof of Theorem 3.1, we can obtain $z_n - x_n \to 0$ as $n \to \infty$. Hence, $x_{n+1} - x_n \to 0$ as $n \to \infty$.

Observe that

$$\begin{aligned}
\tfrac{\sigma}{\tau}\|St_n - x_n\| &\leqslant \tfrac{\beta_n}{\alpha_n+\beta_n}\|St_n - x_n\| \\
&= \|\tfrac{\alpha_n}{\alpha_n+\beta_n}(t_n - x_n) + \tfrac{\beta_n}{\alpha_n+\beta_n}(St_n - x_n) - \tfrac{\alpha_n}{\alpha_n+\beta_n}(t_n - x_n)\| \\
&\leqslant \|z_n - x_n\| + \tfrac{\alpha_n}{\alpha_n+\beta_n}\|t_n - x_n\| \\
&\leqslant \|z_n - x_n\| + \|t_n - x_n\|,
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \|St_n - x_n\| = 0.$$

Since

$$\|Sx_n - x_n\| \leqslant \|Sx_n - St_n\| + \|St_n - x_n\| \leqslant \|x_n - t_n\| + \|St_n - x_n\|,$$

we have

$$\lim_{n\to\infty} \|Sx_n - x_n\| = 0.$$

Step 4. We claim that $\omega_w(x_n) \subset F(S) \cap VI(C,A)$.
Indeed, the proof of this step is the same as in Step 4 of the proof of Theorem 3.1.
Step 5. We claim that the following statements hold:
(i) $\{x_n\}$ and $\{y_n\}$ converge weakly to the same point $u \in F(S) \cap VI(C,A)$;
(ii) $\{P_{F(S)\cap VI(C,A)}x_n\}$ converges strongly to such a $u \in F(S) \cap VI(C,A)$.

Indeed, the proof of statement (i) is the same as in Step 5 of the proof of Theorem 3.1. Now, put

$$u_n = P_{F(S)\cap VI(C,A)}x_n.$$

We show that

$$u = \lim_{n\to\infty} u_n.$$

From

$$u_n = P_{F(S)\cap VI(C,A)}x_n \quad \text{and} \quad u \in F(S)\cap VI(C,A),$$

we have

$$\langle u - u_n, u_n - x_n \rangle \geqslant 0.$$

By Lemma 3.2, $\{u_n\}$ converges strongly to some $z \in F(S)\cap VI(C,A)$. Then, we have

$$\langle u - z, z - u \rangle \geqslant 0$$

and hence $u = z$. This completes the proof. □

4. Applications

Utilizing Theorems 3.1 and 3.2 in the above section, we prove several weak convergence theorems in a real Hilbert space.

THEOREM 4.1. *Let H be a real Hilbert space. Let $f : H \to H$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$, $A : H \to H$ be a monotone and k-Lipschitz continuous mapping and $S : H \to H$ be a nonexpansive mapping such that $F(S)\cap A^{-1} \neq \emptyset$. Let $\{x_n\},\{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n, \\ t_n = x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n, \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n St_n \ \ \forall n \geqslant 0, \end{cases}$$

where $\{\lambda_n\},\{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\},\{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1\ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*
(ii) $\sum_{n=0}^{\infty}\alpha_n < \infty$ *and* $0 < \sigma \leqslant \beta_n\ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*
(iii) $\lim_{n\to\infty}\mu_n = 1$*;*
(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$.

Then the sequences $\{x_n\},\{y_n\}$ converge weakly to the same point $u \in F(S)\cap A^{-1}0$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $P_H = I$. By Theorem 3.1, we obtain the desired result. □

THEOREM 4.2. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone and k-Lipschitz continuous mapping and $S : H \to H$ be a nonexpansive mapping such that $F(S) \cap A^{-1}0 \neq \emptyset$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n, \\ t_n = x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n, \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n t_n + \beta_n St_n \ \ \forall n \geqslant 0, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1 \ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*

(ii) $0 < \sigma \leqslant \beta_n \ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*

(iii) $\lim_{n\to\infty} \mu_n = 1$*;*

(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$.

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $u \in F(S) \cap A^{-1}0$, where $u = \lim_{n\to\infty} P_{F(S)\cap A^{-1}0}x_n$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $P_H = I$. By Theorem 3.2, we obtain the desired result. □

REMARK 4.1. Notice that $F(S) \cap A^{-1}0 \subset VI(F(S),A)$. See also Yamada [9] for the case when $A : H \to H$ is a strongly monotone and Lipschitz continuous mapping and $S : H \to H$ is a nonexpansive mapping.

THEOREM 4.3. *Let H be a real Hilbert space. Let $f : H \to H$ be a contractive mapping with a contractive constant $\alpha \in (0,1)$, $A : H \to H$ be a monotone and k-Lipschitz continuous mapping and $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in C, \\ y_n = x_n - \lambda_n\mu_n Ax_n - \lambda_n(1-\mu_n)Ay_n, \\ t_n = x_n - \lambda_n Ay_n - \lambda_n(1-\mu_n)At_n, \\ x_{n+1} = (1-\alpha_n-\beta_n)x_n + \alpha_n f(t_n) + \beta_n J_r^B t_n \ \ \forall n \geqslant 0, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1 \ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*

(ii) $\sum_{n=0}^{\infty} \alpha_n < \infty$ *and* $0 < \sigma \leqslant \beta_n \ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*

(iii) $\lim_{n\to\infty} \mu_n = 1$*;*

(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a,b \in (0,1/k)$.

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $u \in A^{-1}0 \cap B^{-1}0$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.1 we obtain the desired result. □

THEOREM 4.4. *Let H be a real Hilbert space. Let $A : H \to H$ be a monotone and k-Lipschitz continuous mapping and let $B : H \to 2^H$ be a maximal monotone mapping such that $A^{-1}0 \cap B^{-1}0 \neq \emptyset$. Let J_r^B be the resolvent of B for each $r > 0$. Let $\{x_n\}, \{y_n\}$ be the sequences generated by*

$$\begin{cases} x_0 = x \in H, \\ y_n = x_n - \lambda_n \mu_n A x_n - \lambda_n (1-\mu_n) A y_n, \\ t_n = x_n - \lambda_n A y_n - \lambda_n (1-\mu_n) A t_n, \\ x_{n+1} = (1-\alpha_n-\beta_n) x_n + \alpha_n t_n + \beta_n J_r^B t_n \ \ \forall n \geqslant 0, \end{cases}$$

where $\{\lambda_n\}, \{\mu_n\}$ is sequences in $(0,1]$ and $\{\alpha_n\}, \{\beta_n\}$ are sequences in $[0,1]$ satisfying the conditions:

(i) $\alpha_n + \beta_n \leqslant \tau < 1 \ \forall n \geqslant 0$ *for some* $\tau \in (0,1)$*;*
(ii) $0 < \sigma \leqslant \beta_n \ \forall n \geqslant 0$ *for some* $\sigma \in (0,1)$*;*
(iii) $\lim_{n\to\infty} \mu_n = 1$*;*
(iv) $\{\lambda_n\} \subset [a,b]$ *for some* $a, b \in (0, 1/k)$*.*

Then the sequences $\{x_n\}, \{y_n\}$ converge weakly to the same point $u \in A^{-1}0 \cap B^{-1}0$, where $u = \lim_{n\to\infty} P_{A^{-1}0 \cap B^{-1}0} x_n$.

Proof. We have $A^{-1}0 = VI(H,A)$ and $F(J_r^B) = B^{-1}0$. Putting $P_H = I$, by Theorem 3.2 we obtain the desired result. □

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