Journal of
Mathematical
Inequalities
Volume 1, Number 2 (2007), 243–258

IMPLICIT ITERATION SCHEME WITH PERTURBED MAPPING FOR COMMON FIXED POINTS OF A FINITE FAMILY OF LIPSCHITZ PSEUDOCONTRACTIVE MAPPINGS

LU-CHUAN CENG [1], ADRIAN PETRUŞEL AND JEN-CHIH YAO [2]

(*communicated by R. Verma*)

Abstract. Let E be a real Banach space, $\{T_i\}_{i=1}^N$ be a finite family of continuous pseudocontractive self mappings of E and $G : E \to E$ be a mapping which is both δ-strongly accretive and λ-strictly pseudocontractive of Browder-Petryshyn type such that $\delta + \lambda \geqslant 1$. We propose a new implicit iteration scheme with perturbed mapping G for the approximation of common fixed points of $\{T_i\}_{i=1}^N$. For an arbitrary initial point $x_0 \in E$, the sequence $\{x_n\}_{n=1}^\infty$ is defined by

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1 - \alpha_n)T_n x_n$$

where $T_n = T_{n \bmod N}$, $\{\alpha_n\}_{n=1}^\infty \subset [a,b] \subset]0,1[$ and $\{\lambda_n\}_{n=1}^\infty \subset [0,1[$. We establish some weak convergence theorems for this implicit iteration scheme. Also, necessary and sufficient conditions for strong convergence of this implicit iteration scheme are obtained.

1. Introduction

Let E be a real Banach space and E^* be the dual space of E. Let I be the identity operator of E. Denote by J the normalized duality set-valued mapping from E into 2^{E^*} given by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\|^2 = \|f\|^2\}, \quad \text{for all } x \in E,$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between E and E^*. If E is a smooth Banach space, then J is single-valued. In the sequel, we shall denote the single-valued duality mapping by j and by $F(T) = \{x \in E : Tx = x\}$ the fixed point set of the mapping $T : E \to E$.

When $\{x_n\}$ is a sequence in E, then $x_n \to x$ (respectively, $x_n \rightharpoonup x$) will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$ to x.

Mathematics subject classification (2000): 47H09, 47H10, 47H17.

Key words and phrases: Continuous pseudocontractive mapping, Implicit iteration process with perturbed mapping, Strictly pseudocontractive mapping, Common fixed point, Demiclosedness principle.

[1] This research was partially supported by the Teaching and Research Award Fund for Outstanding Young Teachers in Higher Education Institutions of MOE, China and the Dawn Program Foundation in Shanghai

[2] This research was partially supported by a grant from the National Science Council.

Paper JMI-01-22

DEFINITION 1.1. Let T be a mapping with domain $D(T)$ and range $R(T)$ in E. T is called

(i) nonexpansive if

$$\|Tx - Ty\| \leqslant \|x - y\|, \quad \text{for all } x, y \in D(T);$$

(ii) Lipschitzian if there exists $L > 0$ such that

$$\|Tx - Ty\| \leqslant L\|x - y\|, \quad \text{for all } x, y \in D(T);$$

(iii) strongly accretive, if for each $x, y \in D(T)$, there exists $\delta \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \geqslant \delta \|x - y\|^2;$$

(iv) pseudocontractive, if for each $x, y \in D(T)$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leqslant \|x - y\|^2; \tag{1}$$

(v) strictly pseudocontractive in Browder-Petryshyn' sense (for short, strictly pseudocontractive), if for each $x, y \in D(T)$,there exists $k \in (0,1)$ and $j(x-y) \in J(x-y)$ such that

$$\langle Tx - Ty, j(x-y) \rangle \leqslant \|x - y\|^2 - k\|x - y - (Tx - Ty)\|^2. \tag{2}$$

REMARK 1.1. Inequality (2) can be written in the form

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geqslant k\|(I-T)x - (I-T)y\|^2. \tag{3}$$

Moreover, inequality (1) can be written in the form

$$\langle (I-T)x - (I-T)y, j(x-y) \rangle \geqslant 0. \tag{4}$$

It is easy to see from (3) that every strictly pseudocontractive map is β-Lipschitzian with $\beta \leqslant 1 + \frac{1}{k}$.

It is also well-known (see [13]) that (1) is equivalent to

$$\|x - y\| \leqslant \|x - y + s[(x - Tx) - (y - Ty)]\|, \tag{4'}$$

for any $s > 0$ and any $x, y \in D(T)$.

Let K be a nonempty convex subset of E and $T : K \to K$ be a continuous pseudocontractive mapping. For every $u \in K$ and $t \in (0,1)$, the operator $T_t : K \to K$ defined by $T_t x = tu + (1-t)Tx$, for all $x \in K$, is strongly pseudocontractive. Indeed, observe that for each $x, y \in K$, there exists $j(x-y) \in J(x-y)$ such that

$$\langle T_t x - T_t y, j(x-y) \rangle = (1-t)\langle Tx - Ty, j(x-y) \rangle \leqslant (1-t)\|x - y\|^2.$$

Since T_t is also continuous, so we know from Deimling [15], that T_t has a unique fixed point $x_t \in K$ (see also [4]), i.e.,

$$x_t = tu + (1-t)Tx_t. \tag{5}$$

Let $\{T_i\}_{i=1}^N$ be a finite family of continuous pseudocontractive self-mappings of K. Motivated by Xu and Ori's implicit iteration process (see [3]), in order to approximate a common fixed points of a finite family of nonexpansive mappings, Chen, Song and Zhou [5] used (5) for constructing the following implicit iteration process. For $x_0 \in K$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generalized as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)T_1 x_1, \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)T_2 x_2, \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)T_N x_N, \\ &\vdots \end{aligned}$$

The scheme is expressed in a compact form as

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_n x_n, \qquad \text{for all } n \geqslant 1, \tag{6}$$

where $T_n = T_{n \bmod N}$.

In [5], necessary and sufficient conditions for the strong convergence to a common fixed point of a finite family of continuous pseudocontractive mappings are established. Also strong and weak convergence theorem for a finite family of strictly pseudocontractive mappings of Browder-Petryshyn type are derived. Their results extended and improved some corresponding ones in [1-3].

Very recently, Zhou [11] extended the above results of Chen, Song and Zhou to a more general class of real reflexive Banach space, as well as, the results of Xu and Ori to a more general class of Lipschitzian pseudocontractive mappings.

On the other hand, let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive self mappings of a real Hilbert space H and $G : H \to H$. Suppose that there exists some constants $\kappa, \eta > 0$ such that the mapping G is κ-Lipschitzian and η-strongly monotone. Let $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$, $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$ and take a fixed number $\mu \in (0, \frac{2\eta}{\kappa^2})$. For the approximation of common fixed points of $\{T_i\}_{i=1}^N$, Zeng and Yao [6] introduced and studied the following implicit iteration process with perturbed mapping G. For an arbitrary initial point $x_0 \in H$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{aligned} x_1 &= \alpha_1 x_0 + (1-\alpha_1)[T_1 x_1 - \lambda_1 \mu G(T_1 x_1)], \\ x_2 &= \alpha_2 x_1 + (1-\alpha_2)[T_2 x_2 - \lambda_2 \mu G(T_2 x_2)] \\ &\vdots \\ x_N &= \alpha_N x_{N-1} + (1-\alpha_N)[T_N x_N - \lambda_N \mu G(T_N x_N)], \\ &\vdots \end{aligned}$$

This scheme can be expressed in a concise form as follows

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)[T_n x_n - \lambda_n \mu G(T_n x_n)], \qquad \text{for all } n \geqslant 1. \tag{7}$$

It is clear that if $\lambda_n = 0,\;$ for all $n \geqslant 1$, then the implicit iteration scheme (7) reduces to the implicit iteration process (6).

Inspired by the above implicit iteration schemes (6) and (7), we will use (5) to propose a new implicit iteration scheme with perturbed mapping for approximation of common fixed points of a finite family of continuous pseudocontractive mappings. Let $\{T_i\}_{i=1}^N$ be a finite family of continuous pseudocontractive self mappings of E and $G : E \to E$ be a perturbed map which is both λ-strictly pseudocontractive and δ-strongly accretive with $\lambda + \delta \geqslant 1$. For $x_0 \in E$ and $\{\alpha_n\}_{n=1}^\infty \subset (0,1)$ and $\{\lambda_n\}_{n=1}^\infty \subset [0,1)$, the sequence $\{x_n\}_{n=1}^\infty$ is generated as follows:

$$\begin{aligned} x_1 &= \alpha_1(x_0 - \lambda_1 G(x_0)) + (1-\alpha_1)T_1x_1, \\ x_2 &= \alpha_2(x_1 - \lambda_2 G(x_1)) + (1-\alpha_2)T_2x_2, \\ &\vdots \\ x_N &= \alpha_N(x_{N-1} - \lambda_N G(x_{N-1})) + (1-\alpha_N)T_Nx_N, \\ &\vdots \end{aligned}$$

The scheme could be expressed in a compact form as follows

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_nx_n, \quad \text{for all } n \geqslant 1, \tag{8}$$

where $T_n = T_{n \bmod N}$. Clearly, if $\lambda_n = 0$, for all $n \geqslant 1$, the implicit iteration scheme (8) reduces to the implicit iteration process (6).

Let E be a real uniformly convex Banach space with a Fréchet differentiable norm. The aim of this paper is to extend the results for continuous pseudocontractive mappings [5, Theorem 2.3], as well as, that for Lipschitzian pseudocontractive mappings [10, Theorem 3.1], to the case of implicit iteration process (8), where the perturbation G satisfies the strict pseudocontractivity and the strong accretivity condition. We will obtain weak convergence theorems for a finite family of Lipschitzian pseudocontractive self mappings, as well as, necessary and sufficient conditions for the strong convergence to a common fixed point of a finite family of continuous pseudocontractive self mappings in E. As consequences, we derive other results which extend and improve some theorems in [1-3, 5-6, 11].

2. Preliminaries

In the sequel, we shall need the following definitions and results. We refer to Takahashi [13] for details and related results.

DEFINITION 2.1. A Banach space E is called uniformly convex if for each $\varepsilon > 0$ there is a $\delta > 0$ such that for each $x, y \in E$ with $\|x\|, \|y\| \leqslant 1$ and $\|x - y\| \geqslant \varepsilon$ we have $\|x + y\| \leqslant 2(1-\delta)$. The modulus of uniform convexity of E is defined by

$$\delta_E(\varepsilon) := \inf\{1 - \|\frac{1}{2}(x+y)\| : \|x\|, \|y\| \leqslant 1, \|x-y\| \geqslant \varepsilon\}, \text{ for all } \varepsilon \in [0,2].$$

Hence, E is uniformly convex if $\delta_E(0) = 0$ and $\delta_E(\varepsilon) > 0$ for $\varepsilon \in [0,2]$.

The following result is useful in the sequel.

PROPOSITION 2.1. (Bruck [12]) *Let E be a uniformly convex Banach space with modulus of uniform convexity δ_E. Then $\delta_E : [0,2] \to [0,1]$ is continuous, monotone increasing and satisfies the following assertions*

(*a*) $\delta_E(0) = 0$ *and* $\delta_E(\varepsilon) > 0$ *for* $\varepsilon > 0$*;*

(*b*) $\|\alpha u + (1-\alpha)v\| \leqslant 1 - 2\min\{\alpha, 1-\alpha\} \cdot \delta_E(\|u-v\|)$*, for each* $\alpha \in [0,1]$ *and* $u, v \in E$ *with* $\|u\|, \|v\| \leqslant 1$.

DEFINITION 2.2. Let K be a closed subset of a Banach space E. A mapping $T : K \to K$ is said to to be semi-compact, if for any bounded sequence $\{x_n\}$ in K such that $\|x_n - Tx_n\| \to 0$ as $n \to +\infty$, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in K$ as $i \to +\infty$.

DEFINITION 2.3. A Banach space E is said to satisfy Opial's condition, if whenever $\{x_n\}$ is a sequence in E which converges weakly to x, as $n \to +\infty$, then

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \qquad \text{for all } y \in E,\ y \neq x.$$

It is well known that every Hilbert space satisfies Opial's condition (see for instance [9]). Throughout the paper, for the weak ω-limit set of a sequence $\{x_n\}$ in E we shall use the following notation

$$\omega_\omega(x_n) := \{x \in E | \{x_{n_j}\} \rightharpoonup x,\ \text{for some subsequence } \{n_j\} \text{ of } \{n\}\}.$$

Let $U = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. The norm of E is said to be Gâteaux differentiable if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U$. In this case E is said to be smooth. We know (see [10]) that if E is smooth then the normalized duality mapping J is single-valued and continuous from the strong topology to the weak* topology. The norm of E is called Fréchet differentiable if for each $x \in U$ the above limit is attained uniformly for $y \in U$. The norm of E is called uniformly Fréchet differentiable if the above limit is attained uniformly for $x, y \in U$.

DEFINITION 2.4. A Banach space E is said to be

(*i*) uniformly smooth if $\frac{\rho_E(t)}{t} \to 0$ as $t \to 0$, where $\rho_E(t)$ is the modulus of smoothness of E defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1,\ \|y\| = t\}, \qquad \text{for all } t > 0;$$

(*ii*) q-uniformly smooth ($q > 1$), if there exists a constant $c > 0$ such that $\rho_E(t) \leqslant ct^q$.

Typical examples of both uniformly convex and uniformly smooth Banach spaces are the L^P spaces with $p > 1$. It is also known that a Banach space is uniformly smooth if and only if the norm of E is uniformly Fréchet differentiable. If E is q-uniformly smooth then the norm of E is Fréchet differentiable.

In E is a Banach space, $x \in E$ and F a nonempty subset of E, then, throughout the paper, we will denote $d(x,F) := \inf_{p \in F} \|x-p\|$.

The following results will be used in the sequel.

THEOREM 2.1. (Osilike, Udomene [2]) *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a strictly pseudocontractive mapping in the terminology of Browder-Petryshyn. Then $(I-T)$ is demiclosed at zero, i.e., $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I-T)x_n\}$ converges strongly to 0, then $x - Tx = 0$.*

THEOREM 2.2. (Zhou [11]) *Let E be a real reflexive Banach space which satisfies Opial's condition. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a continuous pseudocontractive mapping. Then $(I-T)$ is demiclosed at zero.*

THEOREM 2.3. (Zhou [11]) *Let E be a real uniformly convex Banach space. Let K be a nonempty closed convex subset of E and $T : K \to K$ be a continuous pseudocontractive mapping. Then $(I-T)$ is demiclosed at zero.*

THEOREM 2.4. (Tan, Xu [14]) *Let E be a real uniformly convex Banach space whose norm is Fréchet differentialble. Let K be a nonempty closed convex subset of E and $T_i : K \to K$, $i \in \{1,2,\cdots\}$ be a family of L_n-Lipschitzian mappings such that $\sum_{n=1}^{\infty}(L_n - 1) < +\infty$ and $F := \bigcap_{i=1}^{\infty} F(T_i) \neq \emptyset$. For arbitrary $x_1 \in K$ define the sequence $\{x_n\}_{n=1}^{\infty}$ by $x_{n+1} = T_n x_n$, for all $n \in \{1,2,\cdots\}$. Then $\lim_{n\to\infty} \langle x_n, j(p-q)\rangle$ exists for all $p,q \in F$ and for all $u,v \in \omega_\omega(x_n)$ and all $p,q \in F$ we have $\langle u-v, j(p-q)\rangle = 0$.*

LEMMA 2.1. *If $J : E \to 2^{E^*}$ is a normalized duality mapping, then for all $x,y \in E$,*

$$\|x+y\|^2 \leqslant \|x\|^2 + 2\langle y, j(x+y)\rangle, \text{ for all } j(x+y) \in J(x+y).$$

LEMMA 2.2. *Let E be a smooth Banach space and $F : E \to E$ be both λ-strictly pseudocontractive and δ-strongly accretive with $\lambda + \delta \geqslant 1$. Then*

(i) $I-F$ is nonexpansive;

(ii) For each $t \in [0,1]$ the mapping $S_t : E \to E$ defined by $S_t x = x - tG(x)$, for all $x \in E$, is nonexpansive.

Proof. Since E is smooth, J is single-valued. Utilizing the λ-strict pseudocontractivity and δ-strong accretivity of F, we have for all $x,y \in E$

$$\begin{aligned} \lambda\|(I-F)x-(I-F)y\|^2 &\leqslant \langle (I-F)x-(I-F)y, j(x-y)\rangle \\ &= \|x-y\|^2 - \langle Fx-Fy, j(x-y)\rangle \\ &\leqslant (1-\delta)\|x-y\|^2. \end{aligned}$$

Note that $\lambda + \delta \geqslant 1$. So, we derive for all $x,y \in E$

$$\|(I-F)x-(I-F)y\| \leqslant \sqrt{\frac{1-\delta}{\lambda}}\|x-y\| \leqslant \|x-y\|$$

and hence $I-F$ is nonexpansive. On the other hand, since

$$S_t x = x - tG(x) = (1-t)x + t(I-F)x, \quad \forall x \in E,$$

it follows from the nonexpansivity of $I-F$ that for all $x, y \in E$

$$\begin{aligned} \|S_t x - S_t y\| &= \|(1-t)(x-y) + t((I-F)x - (I-F)y)\| \\ &\leqslant (1-t)\|x-y\| + t\|(I-F)x - (I-F)y\| \\ &\leqslant (1-t)\|x-y\| + t\|x-y\| \\ &= \|x-y\|. \end{aligned}$$

This shows that $S_t : E \to E$ is nonexpansive. The proof is complete. □

LEMMA 2.3. (Osilike et al. [7, p. 80]) *Let* $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant (1+\delta_n)a_n + b_n, \quad \textit{for all } n \geqslant 1.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *then* $\lim_{n\to\infty} a_n$ *exists. If in addition* $\{a_n\}_{n=1}^{\infty}$ *has a subsequence which converges to zero, then* $\lim_{n\to\infty} a_n = 0$.

LEMMA 2.4. (Tan, Xu [8, p. 303]) *Let* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leqslant a_n + b_n, \quad \textit{for all } n \geqslant 1.$$

If $\sum_{n=1}^{\infty} b_n$ *converges, then* $\lim_{n\to\infty} a_n$ *exists.*

3. Main Results

We start with the following lemma.

LEMMA 3.1. *Let* E *be a real smooth Banach space,* $G : E \to E$ *be* δ*-strongly accretive and* λ*-strictly pseudocontractive with* $\delta + \lambda \geqslant 1$ *and let* $T_i : E \to E$, $i \in \{1, 2, ..., N\}$, *be a finite family of continuous pseudocontractive mappings, such that* $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. *Suppose* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *are real sequences satisfying the following conditions*

(i) $\alpha_n \in [a, b]$ *for some* $a, b \in (0, 1)$;

(ii) $\lambda_n \in [0, 1)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ *and let* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \textit{for all } n \geqslant 1 \qquad (8),$$

where $T_n = T_{n \bmod N}$. *Then*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists, for all* $p \in F$;

(b) $\lim_{n\to\infty} d(x_n, F)$ *exists.*

Proof. Since E is smooth, the normalized duality mapping J is single-valued. Let $p \in F$ and $n \geqslant 1$. Then, utilizing (1) and Lemma 2.2 (ii) we have

$$\begin{aligned} \|x_n - p\|^2 &= \langle \alpha_n(x_{n-1} - \lambda_n G(x_{n-1}) - p) + (1-\alpha_n)(T_n x_n - p), j(x_n - p)\rangle \\ &= \alpha_n[\langle S_{\lambda_n}x_{n-1} - S_{\lambda_n}p, j(x_n-p)\rangle - \langle \lambda_n G(p), j(x_n - p)\rangle] \\ &\quad +(1-\alpha_n)\langle T_n x_n - T_n p, j(x_n-p)\rangle \\ &\leqslant \alpha_n[\|S_{\lambda_n}x_{n-1} - S_{\lambda_n}p\|\|x_n - p\| + \lambda_n\|G(p)\|\|x_n-p\|] \\ &\quad +(1-\alpha_n)\|x_n-p\|^2 \\ &\leqslant \alpha_n[\|x_{n-1}-p\|\|x_n-p\| + \lambda_n\|G(p)\|\|x_n-p\|] + (1-\alpha_n)\|x_n-p\|^2 \\ &= \alpha_n\|x_n-p\|[\|x_{n-1}-p\| + \lambda_n\|G(p)\|] + (1-\alpha_n)\|x_n-p\|^2, \end{aligned}$$

where $S_{\lambda_n}x_{n-1} = x_{n-1} - \lambda_n G(x_{n-1})$ and $S_{\lambda_n}p = p - \lambda_n G(p)$. So

$$\|x_n - p\|^2 \leqslant \|x_n - p\|[\|x_{n-1} - p\| + \lambda_n\|G(p)\|]. \tag{9}$$

If $\|x_n - p\| = 0$, the result is follows. Next, let $\|x_n - p\| > 0$. Then from (9) we have

$$\|x_n - p\| \leqslant \|x_{n-1} - p\| + \lambda_n\|G(p)\|. \tag{10}$$

Notice that condition (ii) implies that $\lim_{n\to\infty} \lambda_n = 0$. By Lemma 2.4 we get that $\lim_{n\to\infty} \|x_n - p\|$ exists. Thus $\{x_n\}$ is bounded and so is $\{G(x_n)\}$ due to the fact that G is β-Lipschitzian with $\beta \leqslant 1 + \frac{1}{\lambda}$. On the other hand, from (10) we obtain

$$\begin{aligned} \|x_n - p\| &\leqslant \|x_{n-1} - p\| + \lambda_n\|G(p) - G(x_{n-1}) + G(x_{n-1})\| \\ &\leqslant \|x_{n-1} - p\| + \lambda_n\beta\|x_{n-1} - p\| + \lambda_n\|G(x_{n-1})\| \\ &\leqslant (1+\lambda_n\beta)\|x_{n-1}-p\| + \lambda_n M \end{aligned}$$

where $\|G(x_n)\| \leqslant M$, for all $n \geqslant 1$, for some $M > 0$.

Taking the infimum over all $p \in F$, we have

$$d(x_n, F) \leqslant (1+\lambda_n\beta)d(x_{n-1}, F) + \lambda_n M.$$

Therefore, by applying Lemma 2.3 implies that $\lim_{n\to\infty} d(x_n, F)$ exists. The proof is complete. □

First main result of this paper is the following.

THEOREM 3.1. *Let* E *be a real uniformly convex Banach space whose norm is Fréchet differentiable and* $G : E \to E$ *be* δ*-strongly accretive and* λ*-strictly pseudocontractive mapping with* $\delta + \lambda \geqslant 1$. *Let* $T_i : E \to E$ *be a finite family of* L_i*-Lipschitzian pseudocontractive mappings, where* $i \in \{1, 2, ..., N\}$, *such that* $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$.

Suppose $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *are real sequences satisfying the following conditions*

(i) $\alpha_n \in [a,b]$ *for some* $a,b \in (0,1)$ *and* $\limsup_{n\to\infty} \alpha_n < 1$;

(ii) $\lambda_n \in [0,1)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ *and let* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \text{for all } n \geqslant 1 \tag{8},$$

where $T_n = T_{n \bmod N}$.

Then the sequence $\{x_n\}$ *converges weakly to a common fixed point of the family* $\{T_n\}_{n\in\{1,2,\cdots N\}}$.

Proof. Since the norm of E is Fréchet differentiable, the space E is also smooth, and then the normalized duality mapping J is single-valued.

Also from Lemma 3.1, $\lim_{n\to\infty} \|x_n - p\|$ exists, where $p \in F$. In particular, the sequences $\{x_n\}$ and $\{G(x_n)\}$ are bounded.

We organize the proof in three steps.

Step 1. $\|x_{n-1} - x_n\| \to 0$ as $n \to +\infty$.

For $p \in F$ and each $n \geqslant 1$, using $(4')$ with $s := \frac{1-\alpha_n}{2\alpha_n}$, we have

$$\begin{aligned}
\|x_n - p\| &\leqslant \|x_n - p + \tfrac{1-\alpha_n}{2\alpha_n}[(x_n - T_n x_n) - (p - Tp)]\| \\
&= \|x_n - p + \tfrac{1-\alpha_n}{2\alpha_n}(x_n - T_n x_n)\| \\
&= \|x_n - p + \tfrac{1-\alpha_n}{2\alpha_n}[\alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n - T_n x_n]\| \\
&= \|x_n - p + \tfrac{1-\alpha_n}{2}(x_{n-1} - \lambda_n G(x_{n-1}) - T_n x_n)\| \\
&= \|\alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n - p \tfrac{1-\alpha_n}{2}(x_{n-1} - \lambda_n G(x_{n-1}) - T_n x_n)\| \\
&= \|(\alpha_n + \tfrac{1-\alpha_n}{2})x_{n-1} - \lambda_n(\alpha_n + \tfrac{1-\alpha_n}{2})G(x_{n-1}) + (1-\alpha_n - \tfrac{1-\alpha_n}{2})T_n x_n - p\| \\
&= \|\tfrac{1+\alpha_n}{2}x_{n-1} - \lambda_n \tfrac{1+\alpha_n}{2}G(x_{n-1}) + \tfrac{1-\alpha_n}{2}T_n x_n - p\| \\
&= \tfrac{1}{2}\|x_{n-1} - x_n - \lambda_n G(x_{n-1}) - 2p\| \\
&\leqslant \tfrac{1}{2}\|x_{n-1} - x_n - 2p\| + \tfrac{\lambda_n}{2}\|G(x_{n-1})\| \\
&\leqslant \|x_{n-1} - p\| \cdot [1 - \delta(\tfrac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|})] + \tfrac{\lambda_n}{2}\|G(x_{n-1})\| \\
&\leqslant \|x_{n-1} - p\| \cdot [1 - \delta(\tfrac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|})] + \tfrac{\lambda_n}{2}M,
\end{aligned}$$

where $\|G(x_n)\| \leqslant M$, for all $n \geqslant 1$, for some $M > 0$.

Hence for any $n \geqslant 1$ we get

$$|x_{n-1} - p\| \cdot \delta(\frac{\|x_{n-1} - x_n\|}{\|x_{n-1} - p\|}) \leqslant \|x_{n-1} - p\| - \|x_n - p\| + \frac{\lambda_n}{2}M.$$

Letting $n \to +\infty$ and using the properties of δ and hypothesis (ii), we obtain

$$\|x_{n-1} - x_n\| \to 0 \text{ as } n \to +\infty.$$

Step 2. $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0$ for each $l \in \{1,2,...,N\}$.

Using by the assumptions (i) and (ii), we have

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &\leqslant \tfrac{1}{1-\alpha_n}\|x_{n-1} - x_n\| + \tfrac{\alpha_n\lambda_n}{1-\alpha_n}\|G(x_n)\| \\ &\leqslant \tfrac{1}{1-\alpha_n}\|x_{n-1} - x_n\| + \tfrac{\|G_n\|}{1-\alpha_n}\lambda_n \to 0, \end{aligned}$$

as $n \to +\infty$.

Also

$$\begin{aligned} \|x_n - T_n x_n\| &= \|\alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n - T_n x_n\| \\ &= \alpha_n\|x_{n-1} - \lambda_n G(x_{n-1}) - T_n x_n\| \\ &\leqslant \alpha_n\|x_{n-1} - T_n x_n\| + \alpha_n\lambda_n\|G(x_{n-1})\| \\ &\leqslant \alpha_n\|x_{n-1} - T_n x_n\| + \lambda_n M \to 0, \end{aligned}$$

as $n \to +\infty$.

Then, for $i \in \{1,2,\cdots,N\}$ we have

$$\|x_n - T_{n+i}x_n\| \leqslant (1+L)\|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0,$$

where $L := \max_{1\leqslant i\leqslant N} L_i$.

Letting $n \to +\infty$ in the previous relation, we obtain $\|x_n - T_{n+i}x_n\| \to 0$, for each $i \in \{1,2,\cdots,N\}$.

Without loss of generality, we can assume that $n_k = j \ (\mathrm{mod} N)$ for all k and some $j \in \{1,2,...,N\}$. For any fixed $l \in \{1,2,...,N\}$, we can find an $i \in \{1,2,...,N\}$, independent of k, such that $n_k + i = l \ (\mathrm{mod} N)$ for all k. It then follows from (16) that

$$\lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0 \text{ for each } l \in \{1,2,...,N\}.$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \text{ for each } l \in \{1,2,...,N\}.$$

Step 3. $\{x_n\} \rightharpoonup p$ as $n \to +\infty$.

Theorem 2.3 implies $\omega_\omega(x_n) \subset F$. By Theorem 2.4 we get that $\omega_\omega(x_n)$ is a singleton. Hence, from Step 2, we get $\{x_n\} \rightharpoonup p$ as $n \to +\infty$.

The proof is now complete. □

REMARK 3.1. Theorem 3.1 extends Theorem 2.6 in Chen, Song, Zhou [5] and Theorem 3.1 in Zhou [11] to the case of an implicit iteration process (see (8)) with a perturbation. Also, Theorem 3.1 holds in weaker assumptions on the space E, on the mappings T_n and on the parameters $\{\alpha_n\}$ than in Theorem 2 from Xu, Ori [3] and than in Theorem 2.6 from Chen, Song, Zhou [5].

THEOREM 3.2. *Let E be a real reflexive and smooth Banach space which satisfies Opial's condition and $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive mapping with $\delta + \lambda \geqslant 1$. Let $T_i : E \to E$ be a finite family of strictly pseudocontractive mappings, where $i \in \{1,2,...,N\}$, such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$.*

Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions

(i) $\alpha_n \in [a,b]$ *for some* $a,b \in (0,1)$*;*

(ii) $\lambda_n \in [0,1)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ and let $\{x_n\}$ be defined by

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \text{for all } n \geqslant 1 \qquad (8),$$

where $T_n = T_{n \bmod N}$.

Then the sequence $\{x_n\}$ converges weakly to a common fixed point of the family $\{T_n\}_{n\in\{1,2,\cdots N\}}$.

Proof. Since E is smooth, the normalized duality mapping J is single-valued. Also, since the mappings $T_i : E \to E$ are strictly pseudocontractive for each $i \in \{1,2,...,N\}$, we deduce from (3) that

$$\langle (I-T_i)x - (I-T_i)y, j(x-y)\rangle \geqslant k_i\|(I-T_i)x - (I-T_i)y\|^2, \quad \text{for all } x,y \in E$$

where $k_i \in (0,1)$, for $i \in \{1,2,...,N\}$.

Put $k = \min_{1\leqslant i\leqslant N}\{k_i\}$. Then $k \in (0,1)$ and

$$\langle (I-T_i)x - (I-T_i)y, j(x-y)\rangle \geqslant k\|(I-T_i)x - (I-T_i)y\|^2, \quad \text{for all } x,y \in E \quad (11)$$

for each $i \in \{1,2,...,N\}$. Moreover, it is easy to see that each T_i (where $1 \leqslant i \leqslant N$) is β-Lipschitzian with $\beta \leqslant 1 + \frac{1}{k}$. Hence it is obvious that T_i is a continuous pseudocontractive self mapping on E for each $i \in \{1,2,...,N\}$. Hence all the conditions of Lemma 3.1 are satisfied. Thus, by Lemma 3.1 we conclude that $\lim_{n\to\infty}\|x_n - p\|$ exists, where $p \in \Omega$. Consequently, $\{x_n\}$ and $\{G(x_n)\}$ are bounded, due to the fact that G is β-Lipschitzian with $\beta \leqslant 1 + \frac{1}{\lambda}$.

Since $x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n$, for all $n \geqslant 1$, we have

$$x_{n-1} = \frac{1}{\alpha_n}x_n + (1 - \frac{1}{\alpha_n})T_n x_n + \lambda_n G(x_{n-1}). \qquad (12)$$

It now follows from (12) that

$$\begin{aligned} x_n - x_{n-1} &= (1-\frac{1}{\alpha_n})(x_n - T_n x_n) - \lambda_n G(x_{n-1}), \\ \langle x_n - x_{n-1}, j(x_n-p)\rangle &= (1-\frac{1}{\alpha_n})\langle x_n - T_n x_n, j(x_n-p)\rangle - \lambda_n\langle G(x_{n-1}), j(x_n-p)\rangle \qquad (13) \\ &= -\frac{1-\alpha_n}{\alpha_n}\langle x_n - T_n x_n, j(x_n-p)\rangle - \lambda_n\langle F(x_{n-1}), j(x_n-p)\rangle. \end{aligned}$$

By Lemma 2.1, using (13) and (11), we get that for each $n \geqslant 1$ we have

$$\begin{aligned} \|x_n - p\|^2 &= \|x_{n-1} - p + x_n - x_{n-1}\|^2 \\ &\leqslant \|x_{n-1} - p\|^2 + 2\langle x_n - x_{n-1}, j(x_n - p)\rangle \\ &= \|x_{n-1} - p\|^2 - 2\tfrac{1-\alpha_n}{\alpha_n}\langle x_n - T_n x_n - (p - T_n p), j(x_n - p)\rangle \\ &\quad -2\lambda_n\langle G(x_{n-1}), j(x_n - p)\rangle \\ &\leqslant \|x_{n-1} - p\|^2 - 2k\tfrac{1-\alpha_n}{\alpha_n}\|x_n - T_n x_n\|^2 + 2\lambda_n\|G(x_{n-1})\|\|x_n - p\|. \end{aligned} \tag{14}$$

Thus,from (14) and condition $0 < a \leqslant \alpha_n \leqslant b < 1$, we obtain that

$$\begin{aligned} \tfrac{2k(1-b)}{b}\|x_n - T_n x_n\|^2 &\leqslant 2k\tfrac{1-\alpha_n}{\alpha_n}\|x_n - T_n x_n\|^2 \\ &\leqslant \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + 2\lambda_n\|G(x_{n-1})\|\|x_n - p\| \end{aligned}$$

and hence

$$\begin{aligned} \sum_{i=1}^{n} \frac{2k(1-b)}{b}\|x_i - T_i x_i\|^2 &\leqslant \|x_0 - p\|^2 - \|x_n - p\|^2 \\ &\quad + \sum_{i=1}^{n} 2\lambda_i\|G(x_{i-1})\|\|x_i - p\|. \end{aligned} \tag{15}$$

Since both $\{x_n\}$ and $\{G(x_n)\}$ are bounded, we may assume that $\|x_n - p\| \leqslant M$ and $\|G(x_n)\| \leqslant M$, for some $M > 0$. Thus since $\sum_{n=1}^{\infty} \lambda_n < \infty$ it follows from (15) that

$$\frac{2k(1-b)}{b}\sum_{n=1}^{\infty}\|x_n - T_n x_n\|^2 \leqslant \|x_0 - p\|^2 + 2M^2\sum_{n=1}^{\infty}\lambda_n < \infty.$$

Consequently

$$\lim_{n\to\infty}\|x_n - T_n x_n\|^2 = 0, \text{ i.e., } \lim_{n\to\infty}\|x_n - T_n x_n\| = 0.$$

Therefore,

$$\begin{aligned} \|x_{n-1} - T_n x_n\| &= \|\tfrac{1}{\alpha_n}(x_n - T_n x_n) + \lambda_n G(x_{n-1})\| \\ &\leqslant \tfrac{1}{a}\|x_n - T_n x_n\| + \lambda_n M \to 0 \quad \text{as } n \to \infty \end{aligned}$$

and

$$\begin{aligned} \|x_n - x_{n-1}\| &= \|(1-\alpha_n)(T_n x_n - x_{n-1}) - \alpha_n\lambda_n G(x_{n-1})\| \\ &\leqslant (1-\alpha_n)\|x_{n-1} - T_n x_n\| + \alpha_n\lambda_n\|G(x_{n-1})\| \\ &\leqslant \|x_{n-1} - T_n x_n\| + \lambda_n M \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This implies that

$$\|x_{n+i} - x_n\| \to 0 \quad \text{as } n \to \infty,$$

for each $i \in \{1, 2, ..., N\}$. Since each T_i is β-Lipschitzian with $\beta \leqslant 1 + \frac{1}{k}$, we have for each $i \in \{1, 2, \cdots, N\}$ that

$$\|T_i x - T_i y\| \leqslant \beta\|x - y\| \quad \text{for all } x, y \in E.$$

Therefore,

$$\begin{aligned}\|x_n - T_{n+i}x_n\| &\leqslant \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \|T_{n+i}x_{n+i} - T_{n+i}x_n\| \\ &\leqslant \|x_n - x_{n+i}\| + \|x_{n+i} - T_{n+i}x_{n+i}\| + \beta\|x_{n+i} - x_n\| \\ &= (1+\beta)\|x_{n+i} - x_n\| + \|x_{n+i} - T_{n+i}x_{n+i}\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Then

$$\lim_{n\to\infty} \|x_n - T_{n+i}x_n\| = 0 \quad \text{for each } i \in \{1,2,\cdots,N\}. \tag{16}$$

Without loss of generality, we can assume that $n_k = j \pmod N$ for all k and some $j \in \{1,2,...,N\}$. For any fixed $l \in \{1,2,...,N\}$, we can find an $i \in \{1,2,...,N\}$ independent of k, such that $n_k + i = l \pmod N$ for all k. It then follows from (16) that

$$\lim_{n_k\to\infty} \|x_{n_k} - T_l x_{n_k}\| = 0 \text{ for each } l \in \{1,2,...,N\}.$$

Thus

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \text{ for each } l \in \{1,2,...,N\}.$$

The proof is complete. □

REMARK 3.2. Theorem 3.2 extends Theorem 2.6 in Chen, Song, Zhou [5] and Theorem 3.2 in Zhou [11] to the case of an implicit iteration process (see (8)) with a perturbation. Also, Theorem 3.2 holds in weaker assumptions on the space E, than in Theorem 2.6 in Chen, Song, Zhou [5], where E is supposed to be q-uniformly smooth and uniformly convex.

In a similar way to Chen, Song, Zhou [5] we can establish the following strong convergence result.

THEOREM 3.3. *Let E be a real smooth Banach space, $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geqslant 1$ and let $T_i : E \to E$, $i \in \{1,2,...,N\}$, be continuous pseudocontractive self mappings such that $F = \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions*

(i) $\alpha_n \in [a,b]$ for some $a, b \in (0,1)$;

(ii) $\lambda_n \in [0,1)$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ and let $\{x_n\}$ be defined by

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \forall n \geqslant 1,$$

where $T_n = T_{n \bmod N}$.

Then $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$ if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.

Proof. Clearly, the necessity is obvious.
We will show the sufficiency. Suppose $\liminf_{n\to\infty} d(x_n, F) = 0$. Then Lemma 3.1 implies

that $\lim_{n\to\infty} d(x_n, F) = 0$. Since $\{G(x_n)\}$ is bounded, we may assume that there exists $M > 0$ such that $\|G(x_n)\| \leqslant M$, for all $n \geqslant 1$. Hence it follows from (10) that, for all $n \geqslant 1$ and $p \in F$, we have

$$\begin{aligned} \|x_{n+m} - p\| &\leqslant \|x_{n+m-1} - p\| + \lambda_{n+m}\|G(p)\| \\ &\leqslant \|x_{n+m-2} - p\| + \lambda_{n+m-1}\|G(p)\| + \lambda_{n+m}\|G(p)\| \\ &\vdots \\ &\leqslant \|x_n - p\| + \lambda_{n+1}\|G(p)\| + \lambda_{n+2}\|G(p)\| + \cdots + \lambda_{n+m}\|G(p)\| \\ &= \|x_n - p\| + \|G(p)\| \sum_{i=n+1}^{n+m} \lambda_i, \end{aligned}$$

and so

$$\begin{aligned} \|x_{n+m} - x_n\| &\leqslant \|x_{n+m} - p\| + \|x_n - p\| \\ &\leqslant 2\|x_n - p\| + \|G(p)\| \sum_{i=n+1}^{n+m} \lambda_i \\ &\leqslant 2\|x_n - p\| + \|G(p) - G(x_n)\| \sum_{i=n+1}^{n+m} \lambda_i + \|G(x_n)\| \sum_{i=n+1}^{n+m} \lambda_i \\ &\leqslant (2 + L\| \sum_{i=n+1}^{n+m} \lambda_i)\|x_n - p\| + M \sum_{i=n+1}^{n+m} \lambda_i. \end{aligned} \tag{17}$$

Taking the infimum over all $p \in F$, we obtain from from (17) that

$$\|x_{n+m} - x_n\| \leqslant (2 + L \sum_{i=n+1}^{n+m} \lambda_i) d(x_n, F) + M \sum_{i=n+1}^{n+m} \lambda_i \to 0 \quad \text{as } n \to \infty.$$

Thus $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence. Suppose $\lim_{n\to\infty} x_n = u$. Then

$$d(u, F) = \lim_{n\to\infty} d(x_n, F) = 0.$$

As each T_i ($1 \leqslant i \leqslant N$) is continuous pseudocontractive mapping, we claim that $F(T_i)$ is closed. Indeed, note that $F(T_i) \neq \emptyset$ for each i. Let $\{p_n\}_{n=1}^{\infty} \subset F(T_i)$ such that $\lim_{n\to\infty} p_n = p$. Then we have $T_i p = \lim_{n\to\infty} T_i p_n = \lim_{n\to\infty} p_n = p$. Thus $p \in F(T_i)$ for each $i \in \{1, 2, \cdots N\}$. This shows that $F(T_i)$ is closed, for each $i \in \{1, 2, \cdots N\}$. Consequently, F is closed and hence $u \in F$. The proof is complete. □

COROLLARY 3.1. *Let E be a real smooth Banach space, let $G : E \to E$ be δ-strongly accretive and λ-strictly pseudocontractive with $\delta + \lambda \geqslant 1$, and let $T_i : E \to E$, $i = 1, 2, ..., N$, be continuous pseudocontractive self mapping such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Suppose $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\lambda_n\}_{n=1}^{\infty}$ are real sequences satisfying the following conditions:*

(i) $\alpha_n \in [a,b]$ *for some* $a,b \in (0,1)$*;*

(ii) $\lambda_n \in [0,1)$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ *and let* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \text{for all } n \geqslant 1,$$

where $T_n = T_{n \bmod N}$.

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i\}_{i=1}^N$ *if and only if* $\{x_n\}$ *has a subsequence which converges strongly to some* $u \in F$.

As a particular case we have the following result.

THEOREM 3.4. *Let* E *be a real smooth Banach space,* $G : E \to E$ *be* δ*-strongly accretive and* λ*-strictly pseudocontractive with* $\delta + \lambda \geqslant 1$ *and let* $T_i : E \to E$, $i \in \{1,2,...,N\}$*, be strictly pseudocontractive self mappings such that* $F := \bigcap_{i=1}^N F(T_i) \neq \emptyset$. *Suppose that at least one mapping* $T \in \{T_1, T_2, ..., T_N\}$ *is semi-compact. Let* $\{\alpha_n\}_{n=1}^{\infty}$ *and* $\{\lambda_n\}_{n=1}^{\infty}$ *be real sequences satisfying the following conditions:*

(i) $\lambda_n \in [0,1)$ *and* $\alpha_n \in [a,b]$ *for some* $a,b \in (0,1)$*;*

(ii) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

Let $x_0 \in E$ *and let* $\{x_n\}$ *be defined by*

$$x_n = \alpha_n(x_{n-1} - \lambda_n G(x_{n-1})) + (1-\alpha_n)T_n x_n, \quad \text{for all } n \geqslant 1,$$

where $T_n = T_{n \bmod N}$.

Then $\{x_n\}$ *converges strongly to a common fixed point of the mappings* $\{T_i\}_{i=1}^N$.

Proof. We follow the approach given in [5, Theorem 2.5]. First, we notice that from Lemma 3.1 and Theorem 3.1 we have that $\lim_{n\to\infty} \|x_n - p\|$ exists (where $p \in F$) and $\lim_{n\to\infty} \|x_n - T_l x_n\| = 0, \quad \text{for all } l \in \{1,2,...,N\}$. Thus $\{x_n\}$ is bounded. Then, by hypothesis, there exists a semi-compact mapping $T \in \{T_1, T_2, ..., T_N\}$. We may assume, without loss of generality, that $T := T_1$ is semi-compact. Therefore, $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$ and by the definition of semi-compactness there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \to x^* \in E \quad \text{as } i \to \infty$.

Thus

$$\|x^* - T_l x^*\| = \lim_{i\to\infty} \|x_{n_i} - T_l x_{n_i}\| = 0, \quad \text{for all } l \in \{1,2,...,N\}.$$

Thus $x^* \in F$. Then

$$\liminf_{n\to\infty} d(x_n, F) \leqslant \liminf_{n\to\infty} \|x_n - x^*\| \leqslant \lim_{i\to\infty} \|x_{n_i} - x^*\| = 0.$$

By Theorem 3.3 we have that $\lim_{n\to\infty} x_n = x^* \in F$. The proof is complete. □

REFERENCES

[1] M. O. OSILIKE, *Implicit iteration process for common fixed points of a finite family of strictly pseudo-contractive maps*, J. Math. Anal. Appl. 294 (2004) 73-81.

[2] M. O. OSILIKE, A. UDOMENE, *Demiclosedness principle and convergence results for strictly pseudo-contractive mappings of Browder-Petryshyn type*, J. Math. Anal. Appl. 256 (2001) 431-445.

[3] H. K. XU, R. G. ORI, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim. 22 (2001) 767-773.

[4] S. S. CHANG, Y. J. CHO, H. ZHOU, *Iterative Methods for Nonlinear Operator Equations in Banach Spaces*, Nova Science Publishers, New York, 2002.

[5] R. D. CHEN, Y. S. SONG, H. Y. ZHOU, *Convergence theorems for implicit iteration process for a finite family of continuous pseudocontractive mappings*, J. Math. Anal. Appl. 314 (2006) 701-709.

[6] L. C. ZENG, J. C. YAO, *Implicit iteration scheme with perturbed mapping for common fixed points of a finite family of nonexpansive mappings*, Nonlinear Anal. TMA 64 (2006) 2507-2515.

[7] M. O. OSILIKE, S. C. ANIAGBOSOR, B. G. AKUCHU, *Fixed points of asymptotically demicontractive mappings in arbitrary Banach spaces*, PanAmer. Math. J. 12 (2002) 77-88.

[8] K. K. TAN, H. K. XU, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl. 178 (1993) 301-308.

[9] Z. OPIAL, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. 73 (1967) 591-597.

[10] K. DEIMLING, *Nonlinear Functional Analysis*, Springer-Verlag New York 1985.

[11] H. ZHOU, *Convergence theorems of common fixed points for a finite family of Lipschitz pseudocontractions in Banach spaces*, Nonlinear Anal. TMA (2007) doi:101016/j.na.2007.02.041

[12] R. E. BRUCK, *A simple proof of the mean ergodic theorem for nonlinear contractions in Banach spaces*, Israel J. Math. 32 (1979) 107-116.

[13] W. TAKAHASHI, *Nonlinear Functional Analysis-Fixed Point Theory and Applications*, Yokohama Publishers Inc. Yokohama 2000.

[14] K. K. TAN, H. K. XU, *Fixed point iteration process for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc. 122 (1994) 733-739.

[15] K. DEIMLING, *Zeros of accretive operators*, Manuscripta Math. 13(1974) 365-374.

(Received May 14, 2007)

Lu-Chuan Ceng
Department of Mathematics
Shanghai Normal University
Shanghai 200234
China
e-mail: zenglc@hotmail.com

Adrian Petruşel
Department of Applied Mathematics
Babeş-Bolyai University
400084 Cluj-Napoca
Romania

Jen-Chih Yao
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung
Taiwan 804
e-mail: yaojc@math.nsysu.edu.tw