Journal of
Mathematical
Inequalities
Volume 1, Number 2 (2007), 259–268

ON MERIT FUNCTIONS FOR QUASIVARIATIONAL INEQUALITIES

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(*communicated by A. Čižmešija*)

Abstract. It is well known that the quasivariational inequalities are equivalent to the fixed point problems. We use this equivalent alternative formulation to construct some merit functions for quasivariational inequalities and obtain error bounds under some conditions. Since quasivariational inequalities include the classical variational inequalities and the complementarity problems as special cases, our results continue to hold for these problems.

1. Introduction

Variational inequality theory, introduced by Stampacchia [23] in the early 1960's, has witnessed an explosive growth in theoretical advances, algorithmic developments and applications across all disciplines of pure and applied sciences. Variational inequalities have been generalized and extended in various directions using innovative techniques. A useful and significant generalization of variational inequalities is called the quasivariational inequality, which enables us to study the free-moving, unilateral and equilibrium problems arising in elasticity, fluid flow through porous media, finance, economics, transportation, circuit and structural analysis in a unified framework, see [1–24]. As a result of interaction among different branches of mathematical and engineering sciences, there exist now a variety of techniques including the projection method and its variant forms, auxiliary principle, Wiener-Hopf equations, to suggest and analyze various iterative algorithms for solving variational inequalities and related optimization problems. It has been shown that the quasivariational inequalities are equivalent to the fixed points essentially using the projection lemma. This equivalence has been used to develop several iterative methods for solving quasivariational inequalities and to study the sensitivity analysis, see [1–23] and the references therein.

In recent years, much attention has been given to reformulate the variational inequality as an optimization problem. A function which can constitute an equivalent optimization problem is called a merit (gap) function. Merit functions turn out to be very useful in designing new globally convergent algorithms and in analyzing the rate of convergence of some iterative methods. Various merit (gap) functions for variational inequalities and complementarity problems have been suggested and proposed by many

Mathematics subject classification (2000): 49J40, 90C30.

Key words and phrases: Quasivariational inequalities, merit functions, error bounds, residue vector.

This research is supported by the Higher Education Commission, Pakistan, through research grant No: 1-28/HEC/HRD/2005/90.

Paper JMI-01-23

authors, see [4,5,16–22] and the references therein. Error bounds are functions which provide a measure of the distance between a solution set and an arbitrary point. Therefore, error bounds play an important role in the analysis of global or local convergence analysis of algorithms for solving variational inequalities. To the best of our knowledge, very few merit functions have been considered for quasivariational inequalities.

In this paper, we construct some merit functions for the quasivariational inequalities using the equivalence between the fixed-point and the quasivariational inequalities coupled with the auxiliary principle technique. We also obtain error bounds for the solutions of the quasi variational inequalities under some weaker conditions. Proofs of our results are simple and straightforward as compared with other methods. Since the quasi variational inequalities include variational inequalities and the implicit (quasi) complementarity problems as special cases, our results continue to hold for these problems. In this respect, our results can be considered as refinement of the previously known results for variational inequalities and related optimization problems.

2. Formulations and Basic Facts

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $K(u)$ be a closed and convex-valued set in H and $T : H \longrightarrow H$ be a nonlinear operator.

A *quasivariational inequality* consists in finding $u \in K(u)$, such that

$$\langle Tu, v-u \rangle \geqslant 0, \quad \forall v \in K(u). \tag{1}$$

It is well known [2, 10–18] that a large class of obstacle, unilateral, contact, free, moving, and equilibrium problems arising in economics, finance, physical, mathematical, engineering and applied sciences can be studied in the unifying and general framework of (1).

To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'' \geqslant f(x) & \text{on } \Omega = [a,b] \\ u \geqslant M(u) & \text{on } \Omega = [a,b] \\ [-u'' - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a,b] \\ u(a) = 0, \quad u(b) = 0 & \end{array}\right\} \tag{2}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. For the prototype encountered in applications, see [24]. We study the problem (2) in the framework of variational inequality approach. To do so, we first define the set K as

$$K(u) = \{v : v \in H_0^1(\Omega) : v \geqslant M(u), \quad \text{on } \Omega\}, \tag{3}$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (2) is

$$
\begin{aligned}
I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right) v dx - 2\int_a^b f(x)(v)\,dx, \quad \forall v \in K(u) \\
&= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x)(v)\,dx \\
&= \langle Tv, v\rangle - 2\langle f, v\rangle \qquad (4)
\end{aligned}
$$

where

$$
\begin{aligned}
\langle Tu, v\rangle &= \int_a^b \left(\frac{d^2u}{dx^2}\right)(v)\,dx = \int_a^b \frac{du}{dx}\frac{dv}{dx}dx \qquad (5) \\
\langle f, v\rangle &= \int_a^b f(x)(v)dx.
\end{aligned}
$$

It is clear that the operator T defined by (5) is linear, symmetric and positive. Using the technique of Noor [15], one can show that the minimum of the functional $I[v]$ defined by (4) associated with the problem (2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of type (1). See also [1–18] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

If the convex-valued set $K(u)$ is independent of the solution u, that is, $K(u) = K$, a closed convex set, then problem (1) is equivalent to finding $u \in K$, such that

$$\langle Tu, v-u\rangle \geqslant 0, \quad \forall v \in K, \qquad (6)$$

which is known as the classic variational inequality introduced and studied by Stampacchia [23] in 1964. For the state of the art in this theory; see [1–24].

We also need the following well-known concepts and results.

DEFINITION 2.1. The operator $T : H \longrightarrow H$ is said to be

(a) *strongly monotone,* iff, there exists a constant $\alpha > 0$, such that

$$\langle Tu - Tv, u-v\rangle \geqslant \alpha\|u-v\|^2. \quad \forall u, v \in H.$$

(b) *Lipschitz continuous,* iff, there exists a constant $\beta > 0$ such that

$$\|Tu - Tv\| \leqslant \beta\|u-v\|, \quad \forall \quad u, v \in H.$$

In particular, from (a) and (b), it follows that $\alpha \leqslant \beta$.

DEFINITION 2.2. A function $M : H \longrightarrow R \cup \{+\infty\}$ is called a merit (gap) function for the quasivariational inequalities (1), if and only if,

(i) $M(u) \geqslant 0, \quad \forall u \in K(u)$.

(ii) $M(\bar{u}) = 0$, iff, $\bar{u} \in K(u)$ solves (1).

LEMMA 2.1. *Let K be a nonempty closed convex set in a real Hilbert space. For a given $z \in H$, $u \in K$ satisfies the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in K, \tag{7}$$

if and only if

$$u = P_K z,$$

where P_K is the projection of H onto the closed convex set K.

It is well known that the projection operator P_K is a nonexpansive operator, that is,

$$\|P_K u - P_K v\| \leqslant \|u - v\|, \quad \forall u, v \in H.$$

We also need the following condition .

ASSUMPTION 2.1. *$\forall u, v, w \in H$, the operator $P_{K(u)}$ satisfies the condition*

$$\|P_{K(u)} w - P_{K(v)} w\| \leqslant \nu \|u - v\|, \tag{8}$$

where $\nu > 0$ is a constant. For the applications and the examples of Assumption 2.1*, see* [11–13, 15].

3. Main Results

In this section, we consider three merit functions for the quasivariational inequalities (1) and obtain error bounds for the solution of the quasivariational inequalities (1).

We need the following result, which can be proved by using Lemma 2.1.

LEMMA 3.1. *The quasivariational inequality (1) has a solution $u \in K(u)$ if and only if $u \in K(u)$ satisfies the relation*

$$u = P_{K(u)}[u - \rho T u], \tag{9}$$

where $\rho > 0$ is a constant.

From Lemma 3.1, we conclude that the quasivariational inequalities are equivalent to the fixed point problem. This alternative equivalent formulation plays an important part in suggesting and analyzing several iterative methods for solving variational inequalities. This fixed-point formulation has been used to suggest and analyze several iterative methods for solving the quasivariational inequalities (1).

We now consider the residue vector

$$R_\rho(u) \equiv R(u) := u - P_{K(u)}[u - \rho T u]. \tag{10}$$

It is clear from Lemma 3.1 that (1) has a solution $u \in K(u)$, iff, $u \in K(u)$ is a root of the equation

$$R(u) = 0. \tag{11}$$

It is known that the normal residue vector $\|R(u)\|$ is a merit function for the quasivariational inequalities (1). We use the relation (11) to derive the error bound for the solution of (1).

THEOREM 3.1. *Let* $\overline{u} \in K(u)$ *be a solution of* (1) *and let Assumption 2.1 hold. Let the operator* T *be both strongly monotone and Lipschitz continuous with constants* $\alpha > 0$ *and and* $\beta > 0$ *respectively. If* $\overline{u} \in K(u)$ *satisfies*

$$\langle T\overline{u}, v - \bar{u}\rangle \geqslant 0, \forall v \in K(\bar{u}), \tag{12}$$

then

$$k_1\|R(u)\| \leqslant \|u - \bar{u}\| \leqslant k_2\|R(u)\|, \quad \forall u \in K(u), \tag{13}$$

where k_1, k_2 *are generic constants.*

Proof. Let $\overline{u} \in K(u)$ be solution of (1). Then, taking $v = P_{K(u)}[u - \rho Tu]$ in (12), we have

$$\langle T\overline{u}, P_{K(u)}[u - \rho Tu] - \overline{u}\rangle \geqslant 0. \tag{14}$$

Letting $u = P_{K(u)}[u - \rho Tu], \quad z = u - \rho Tu$ and $v = \overline{u}$ in (7), we have

$$\langle \rho Tu + P_{K(u)}[u - \rho Tu] - u, \overline{u} - P_{K(u)}[u - \rho Tu]\rangle \geqslant 0. \tag{15}$$

Adding (14) and (15), we obtain

$$\langle T\overline{u} - Tu + (1/\rho)(u - P_{K(u)}[u - \rho Tu]), P_{K(u)}[u - \rho Tu] - \overline{u}\rangle \geqslant 0. \tag{16}$$

Since T is a strongly monotone, there exists a constant $\alpha > 0$, such that

$$\begin{aligned}
\alpha\|\overline{u} - u\|^2 &\leqslant \langle T\overline{u} - Tu, \overline{u} - u\rangle \\
&= \langle T\overline{u} - Tu, \overline{u} - P_{K(u)}[u - \rho Tu]\rangle + \langle T\overline{u} - Tu, P_{K(u)}[u - \rho Tu] - u\rangle \\
&\leqslant (1/\rho)\langle u - P_{K(u)}[u - \rho Tu], P_{K(u)}[u - \rho Tu] - u + u - \overline{u}\rangle \\
&\quad + \langle T\overline{u} - Tu, P_{K(u)}[u - \rho Tu] - u\rangle \\
&\leqslant -(1/\rho)\|R(u)\|^2 + (1/\rho)\|R(u)\|\|u - \overline{u}\| + \|T\overline{u} - Tu\|\|R(u)\| \\
&\leqslant (1/\rho)(1 + \beta\rho)\|R(u)\|\|\overline{u} - u\|,
\end{aligned}$$

which implies that

$$\|\overline{u} - u\| \leqslant k_2\|R(u)\|, \tag{17}$$

the right-hand inequality in (13) with $k_2 = (1/\alpha\rho)(1 + \rho\beta)$.

Now from Assumption 2.1 and Lipschitz continuity of T, we have

$$\begin{aligned}
\|R(u)\| &= \|u - P_{K(u)}[u - \rho Tu]\| \\
&= \|u - \bar{u} + P_{K(\bar{u})}[\bar{u} - \rho T\bar{u}] - P_{K(u)}[u - \rho Tu]\| \\
&\leqslant \|u - \bar{u}\| + \|P_{K(\bar{u})}[\bar{u} - \rho T\bar{u}] - P_{K(\bar{u})}[u - \rho Tu]\| \\
&\quad + \|P_{K(\bar{u})}[u - \rho Tu] - P_{K(u)}[u - \rho Tu]\| \\
&\leqslant \|u - \bar{u}\| + \nu\|u - \bar{u}\| + \|u - \bar{u} + \rho(Tu - T\bar{u})\| \\
&\leqslant \{2 + \nu + \rho\beta\}\|u - \bar{u}\| = k_1\|u - \bar{u}\|,
\end{aligned}$$

from which we have

$$(1/k_1)\|R(u)\| \leqslant \|u - \bar{u}\|, \tag{18}$$

the left-most inequality in (13) with $k_1 = (2 + \nu + \rho\beta)$.

Combining (17) and (18), we obtain the required (13). □

Letting $u = 0$ in (13), we have

$$(1/k_1)\|R(0)\| \leqslant \|\bar{u}\| \leqslant k_2\|R(0)\|. \tag{19}$$

Combining (13) and (19), we obtain a relative error bound for any point $u \in K(u)$.

THEOREM 3.2. *Assume that all the assumptions of Theorem 3.1 hold. If* $0 \neq \bar{u} \in K(u)$ *is a solution of (1), then*

$$c_1\|R(u)\|/\|R(0)\| \leqslant \|u - \bar{u}\|/\|\bar{u}\| \leqslant c_2\|R(u)\|/\|R(0)\|.$$

Note that the normal residue vector (merit function) $R(u)$ defined by (10) is nondifferentiable. To overcome the nondifferentiability, which is a serious drawback of the residue merit function, we consider another merit function associated with problem (1). This merit function can be viewed as a regularized merit function, see [18–21]. We consider the function

$$\begin{aligned} M_\rho(u) &= \langle Tu, u - P_{K(u)}[u - \rho Tu]\rangle \\ &\quad -(1/2\rho)\|u - P_{K(u)}[u - \rho Tu]\|^2, \quad \forall u \in K(u). \end{aligned} \tag{20}$$

from which it follows that $M_\rho(u) \geqslant 0, \quad \forall u \in K(u)$.

We now show that the function $M_\rho(u)$ defined by (20) is a merit function and this is the main motivation of our next result.

THEOREM 3.3. $\forall u \in K(u)$, *we have*

$$M_\rho(u) \geqslant (1/2\rho)\|R(u)\|^2. \tag{21}$$

In particular, we have $M_\rho(u) = 0$, *iff,* $u \in K(u)$ *is a solution of (1).*

Proof. Setting $v = u, \quad u = P_{K(u)}[u - \rho Tu]$ and $z = u - \rho Tu$ in (7), we have

$$\langle Tu - (1/\rho)(u - P_{K(u)}[u - \rho Tu]), u - P_{K(u)}[u - \rho Tu]\rangle \geqslant 0.$$

which implies that

$$\langle Tu, R(u)\rangle \geqslant (1/\rho)\|R(u)\|^2. \tag{22}$$

Combining (20) and (22), we have

$$\begin{aligned} M_\rho(u) &= \langle Tu, R(u)\rangle - (1/2\rho)\|R(u)\|^2 \\ &\geqslant (1/\rho)\|R(u)\|^2 - (1/2\rho)\|R(u)\|^2 \\ &= (1/2\rho)\|R(u)\|^2, \end{aligned}$$

the required result (21). Clearly we have $M_\rho(u) \geqslant 0, \quad \forall u \in K(u)$.

Now if $M_\rho(u) = 0$, then clearly $R(u) = 0$. Hence by Lemma 3.1, we see that $u \in K(u)$ is a solution of (1). Conversely, if $u \in K(u)$ is a solution of (1), then $u = P_{K(u)}[u - \rho Tu]$ by Lemma 3.1. Consequently, from (20), we see that $M_\rho(u) = 0$, the required result. □

From Theorem 3.3, we see that the function $M_\rho(u)$ defined by (20) is a merit function for the quasivariational inequalities (1). We now derive the error bounds without using the Lipschitz continuity of the operator T.

THEOREM 3.4. *Let T be a strongly monotone with a constant $\alpha > 0$. If $\overline{u} \in K(u)$ is a solution of (1), then*

$$\|u - \bar{u}\|^2 \leqslant (2\rho)/(2\alpha\rho - 1)M_\rho(u), \quad \forall u \in H. \tag{23}$$

Proof. From (20), we have

$$\begin{aligned} M_\rho(u) &\geqslant \langle Tu, u - \bar{u}\rangle - (1/2\rho)\|u - \bar{u}\|^2 \\ &= \langle Tu - T\bar{u} + \bar{u}, u - \bar{u}\rangle - (1/2\rho)\|u - \bar{u}\|^2 \\ &\geqslant \langle T\bar{u}, u - \bar{u}\rangle + \alpha\|u - \bar{u}\|^2 - (1/2\rho)\|u - \bar{u}\|^2, \end{aligned} \tag{24}$$

where we have used the fact that the operator T is strongly monotone with a constant $\alpha > 0$. Taking $v = u$ in (13), we have

$$\langle T\bar{u}, u - \bar{u}\rangle \geqslant 0. \tag{25}$$

From (20) and (24), we have

$$\begin{aligned} M_\rho(u) &\geqslant \alpha\|u - \bar{u}\|^2 - (1/2\rho)\|u - \bar{u}\|^2 \\ &= (\alpha - 1/2\rho)\|u - \bar{u}\|^2, \end{aligned}$$

from which the result (19) follows. □

We consider another merit function associated with quasivariational inequalities (1), which can be viewed as a difference of two regularized merit functions. Such type of the merit functions functions were introduced and studied by many authors for solving variational inequalities and complementarity problems; see [18–21]. Here we define the D-merit function by a formal difference of the regularized merit function defined by (20). To this end, we consider the following function

$$\begin{aligned} D_{\rho,\mu}(u) &= \langle Tu, P_{K(u)}[u - \mu Tu] - P_{K(u)}[u - \rho Tu]\rangle \\ &\quad +(1/2\mu)\|u - P_{K(u)}[u - \mu Tu]\|^2 - (1/2\rho)\|u - P_{K(u)}[u - \rho Tu]\|^2 \\ &= \langle Tu, R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 \\ &\quad -(1/2\rho)\|R_\rho(u)\|^2, \quad u \in K(u), \quad \rho > \mu > 0. \end{aligned} \tag{26}$$

It is clear that the $D_{\rho,\mu}(u)$ is everywhere finite. We now show that the function $D_{\rho,\mu}(u)$ defined by (26) is indeed a merit function for the mixed quasivariational inequalities (1) and this is the motivation of our next result.

THEOREM 3.5. *$\forall u \in K(u), \rho > \mu > 0,$ we have*

$$(\rho-\mu)\|R_\rho(u)\|^2 \geqslant 2\rho\mu D_{\rho,\mu}(u) \geqslant (\rho-\mu)\|R_\mu(u)\|^2. \tag{27}$$

In particular, $D_{\rho,\mu}(u)=0,$ iff $u \in K(u)$ solves problem (1).

Proof. Taking $v = P_{K(u)}[u-\mu Tu], u = P_{K(u)}[u-\rho Tu]$ and $z = u-\rho Tu$ in (7), we have

$$\langle P_{K(u)}[u-\rho Tu] - u + \rho Tu, P_{K(u)}[u-\mu Tu] - P_{K(u)}[u-\rho Tu]\rangle \geqslant 0.$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \geqslant (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle . \tag{28}$$

From (26) and (28), we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geqslant (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 - (1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\geqslant 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2, \end{aligned} \tag{29}$$

which implies the right-most inequality in (27).

In a similar way, by taking $u = P_{K(u)}[u-\mu Tu], z = u-\mu Tu$ and $v = P_{K(u)}[u-\mu Tu]$ in (7), we have

$$\langle P_{K(u)}[u-\mu Tu] - u + \mu Tu, P_{K(u)}[u-\mu Tu] - P_{K(u)}[u-\mu Tu]\rangle \geqslant 0,$$

which implies that

$$\langle Tu, R_\rho(u) - R_\mu(u)\rangle \leqslant (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle . \tag{30}$$

Consequently, from (26) and (30), we obtain

$$\begin{aligned} D_{\rho,\mu}(u) &\leqslant (1/\mu)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle + (1/2\mu)\|R_\mu(u)\|^2 - (1/2\rho)\|R_\rho(u)\|^2 \\ &= 1/2(1/\mu - 1/\rho)\|R_\mu(u)\|^2 + (1/\rho)\langle R_\rho(u), R_\rho(u) - R_\mu(u)\rangle \\ &\quad -(1/2\rho)\|R_\rho(u) - R_\mu(u)\|^2 - (1/\rho)\langle R_\mu(u), R_\rho(u) - R_\mu(u)\rangle \\ &= 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2 - (1/2\mu)\|R_\rho(u) - R_\mu(u)\|^2 \\ &\leqslant 1/2(1/\mu - 1/\rho)\|R_\rho(u)\|^2, \end{aligned} \tag{31}$$

which implies the left-most inequality in (27).

Combining (29) and (31), we obtain (27), the required result. □

Using essentially the technique of Theorem 3.4, we can obtain the following result.

THEOREM 3.6. *Let $\bar{u} \in K(u)$ be a solution of (1). If the operator T is strongly monotone with constant $\alpha > 0$, then*

$$\|u-\bar{u}\|^2 \leqslant (2\rho\mu)/(\rho(2\mu\alpha+1)-\mu)D_{\rho,\mu}, \quad \forall u \in K(u). \tag{32}$$

Proof. Let $\bar{u} \in K(u)$ be a solution of (1). Then, taking $v = u$ in (12), we have

$$\langle T\bar{u}, u-\bar{u}\rangle \geqslant 0. \tag{33}$$

Also from (26), (33) and strongly monotonicity of T, we have

$$\begin{aligned} D_{\rho,\mu}(u) &\geqslant \langle Tu, u-\bar{u}\rangle + (1/2\mu)\|u-\bar{u}\|^2 - (1/2\rho)\|u-\bar{u}\|^2 \\ &\geqslant \langle T\bar{u}, u-\bar{u}\rangle + \alpha\|u-\bar{u}\|^2 + (1/2\mu)\|u-\bar{u}\|^2 - (1/2\rho)\|u-\bar{u}\|^2 \\ &\geqslant (\alpha + (1/2\mu) - (1/2\rho))\|u-\bar{u}\|^2, \end{aligned}$$

from which the required result (32) follows. □

Acknowledgement. The author would like thank the referee for his/her very constructive and useful suggestions and comments.

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(Received February 27, 2007)

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