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GENERALIZED WEIGHTED INEQUALITY WITH NEGATIVE POWERS

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Abstract. In this paper necessary and sufficient conditions for the validity of the generalized Hardy inequality for the case $-\infty < q \leqslant p < 0$ and $0 < p \leqslant q < 1$ are derived. Furthermore, some special cases are considered.

1. Introduction

Let us consider the inequality

$$\left(\int_a^b \left(\int_a^x k(x,t)f(t)dt\right)^q dx\right)^{\frac{1}{q^*}} \leqslant C\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p^*}} \tag{1.1}$$

for functions f positive a.e. in (a,b), $-\infty \leqslant a < b \leqslant \infty$, where $k(x,t)$ is a kernel, i.e. a non-negative function defined in $D = \{(x,t),\ a < t \leqslant x < b\}$, and p, q, p^*, q^* are real parameters.

If we consider a kernel k of the form $K(x,t)u^{\frac{1}{q}}(x)v^{-\frac{1}{p}}(t)$ with u, v weight functions (i.e. measurable, positive and finite a.e. in (a,b)), then we can (1.1) easily rewrite into the form

$$\left(\int_a^b \left(\int_a^x K(x,t)F(t)dt\right)^q u(x)dx\right)^{\frac{1}{q^*}} \leqslant C\left(\int_a^b F^p(x)v(x)dx\right)^{\frac{1}{p^*}} \tag{1.2}$$

which is a Hardy-type inequality for the function F $(= f v^{-\frac{1}{p}})$ with weights u, v. But for simplicity, we will deal here with the "non-weighted" case (1.1).

If $q = q^* > 1$, $p = p^* > 1$, then some necessary and sufficient condition for the validity of (1.2) can be found in [2, Chapter 2]. Here, we are interested in the case of *negative powers* inside the integrals, more precisely, in the case

$$q = -q^*, \quad p = -p^*; \quad q^*, p^* > 0. \tag{1.3}$$

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The corresponding inequality

$$\left(\int_a^b \left(\int_a^x k(x,t)f(t)dt\right)^{-q^*} dx\right)^{\frac{1}{q^*}} \leqslant C\left(\int_a^b f^{-p^*}(x)dx\right)^{\frac{1}{p^*}} \tag{1.4}$$

can be easily rewritten as

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_a^x k(x,t)f(t)dt\right)^q dx\right)^{\frac{1}{q}} \tag{1.5}$$

which is the so-called *reverse* inequality to (1.1), this time with p, $q < 0$. Indeed: Taking (1.4) to the power (-1), we obtain (1.5) due to (1.3).

Together with inequality (1.5), we will consider also its counterpart

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_x^b k(x,t)f(t)dt\right)^q dx\right)^{\frac{1}{q}}. \tag{1.6}$$

In this paper, we obtain a whole scale of conditions for (1.5) and (1.6) to hold for the case

$$-\infty < q \leqslant p < 0.$$

REMARK 1.1. In [4, Theorem 3], it is shown that inequalities (1.5) and (1.6) hold if and only if the *dual inequalities*

$$\left(\int_a^b f^{q'}(x)dx\right)^{\frac{1}{q'}} \leqslant C\left(\int_a^b \left(\int_x^b k(t,x)f(t)dt\right)^{p'} dx\right)^{\frac{1}{p'}} \tag{1.7}$$

and

$$\left(\int_a^b f^{q'}(x)dx\right)^{\frac{1}{q'}} \leqslant C\left(\int_a^b \left(\int_a^x k(t,x)f(t)dt\right)^{p'} dx\right)^{\frac{1}{p'}} \tag{1.8}$$

hold, respectively, with $p' = \frac{p}{p-1}$, $q' = \frac{q}{q-1}$, and the constants C in (1.5) and (1.7), (1.6) and (1.8) are equal. Since for $p, q \in (0,1)$ we have $p', q' < 0$, we can also formulate results for the case

$$0 < p \leqslant q < 1,$$

using results for the corresponding dual inequality with negative parameters p', q' satisfying $-\infty < q' \leqslant p' < 0$. The formulation is left to the reader. Let us emphasize that in (1.7) and (1.8), we have to deal with the kernel $k(t,x)$ instead of $k(x,t)$.

The paper is organized as follows: In the next section we present and discuss our results while Section 3 contains detailed proofs.

Products of the form $0 \cdot \infty$ are taken to be zero.

2. The Main Results

We will consider inequality (1.5); inequality (1.6) can be considered analogously (see Remark 2.5 below).

Let us denote

$$K(x,t) := \int_a^t k^{p'}(x,\tau)d\tau, \quad a < t \leqslant x < b,$$

and

$$B_s(t) := \left(\int_t^b K^{\frac{(1-s)q}{p}}(x,t) K^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)dx \right)^{-\frac{1}{q}}.$$

In what follows, we will assume that

$$0 < K(x,t) < \infty, \qquad a < t \leqslant x < b.$$

Our first result reads:

THEOREM 2.1. *Let* $-\infty < q \leqslant p < 0$ *and* $s \in (-\infty, 2-p)$. *Suppose that*

$$B_s := \sup_{a<t<b} B_s(t) < \infty. \tag{2.1}$$

Then inequality (1.5) holds, and for the best constant C, *we have*

$$C \leqslant \left(\frac{p}{p-(1-s)p'} \right)^{-\frac{1}{p'}} B_s.$$

Condition (2.1) is only *sufficient* for inequality (1.5) to hold. To find necessary and sufficient conditions, we need some additional assumptions about $k(x,t)$.

Let

$$k(x,t) = h(x,t)u^{\frac{1}{q}}(x),$$

where $h(x,t)$ and $u(x)$ are positive and finite functions and $h(x,t)$ satisfies the following condition:

- If we define

$$H(x,t) = \int_a^t h^{p'}(x,\tau)d\tau, \quad a < t \leqslant x < b$$

then $H(x,x)$ is an *absolutely continuous* function in (a,b) and

$$H_p := \sup_{a<t<b} H_p(t) = \sup_{a<t<b} \left(-\int_t^b H^{-\frac{q}{p'}}(x,t)\, dH^{\frac{q}{p'}}(x,x) \right) < \infty. \tag{2.2}$$

Then our next results reads:

THEOREM 2.2. *Let* $-\infty < q \leqslant p < 0$, $s \in [p,1)$. *Suppose that* $h(x,t)$ *is nondecreasing in* x *and satisfies (2.2). Then inequality (1.5)* [*or the equivalent inequality*

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_a^x h(x,t)f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}] \tag{2.3}$$

holds for all positive measurable functions f *if and only if*

$$A_s := \sup_{a<t<b} A_s(t) = \sup_{a<t<b} H^{\frac{1-s}{p}}(t,t)\left(\int_a^t H^{\frac{(p-s)q}{p}}(x,x)u(x)dx\right)^{-\frac{1}{q}} < \infty. \tag{2.4}$$

Moreover, if C *is the best possible constant in (2.3), then* $C \approx A_s$.

THEOREM 2.3. *Under the assumptions of Theorem 2.2, condition (2.1), i.e. the equivalent condition*

$$B_s := \sup_{a<t<b} B_s(t) = \sup_{a<t<b}\left(\int_t^b H^{\frac{(1-s)q}{p}}(x,t)H^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)u(x)dx\right)^{-\frac{1}{q}} < \infty \tag{2.5}$$

is necessary and sufficient for inequality (2.3) to hold.

REMARK 2.4. Suppose that $h(x,t)$ depends only on t, $h(x,t) = v(t)$, and denote

$$V(x) = \int_a^x v^{p'}(t)dt.$$

Then condition (2.2) is satisfied, since

$$H_s(t) = 1 - V^{\frac{q(1-s)}{p}}(t)\lim_{x\to b-} V^{\frac{q(s-1)}{p}}(x) \leqslant 1$$

and inequality (2.3) as well as condition (2.4) take the form:

$$\left(\int_a^b f^p(x)dx\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_a^x v(t)f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}}$$

and

$$A_s = \sup_{a<t<b} V^{\frac{1-s}{p}}(t)\left(\int_a^t u(x)V^{\frac{(p-s)q}{p}}(x)dx\right)^{-\frac{1}{q}} < \infty. \tag{2.6}$$

In this case, observe that our result generalizes the results of [1] and [4] for the case $-\infty < q \leqslant p < 0$.

REMARK 2.5. All the considerations above can be repeated for inequality (1.6). The counterpart of Theorem 2.1 reads as:

- Let $-\infty < q \leqslant p < 0$ and $s \in (-\infty, 2-p)$. Denote

$$K(x,t) = \int_t^b k^{p'}(\tau,x)d\tau \quad a < x \leqslant t < b.$$

Then inequality (1.6) holds provided

$$B_s := \sup_{a<t<b} B_s(t) = \sup_{a<t<b} \left(\int_a^t K^{\frac{(1-s)q}{p}}(x,t) K^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)dx \right)^{-\frac{1}{q}} < \infty.$$

The counterpart of Theorems 2.2 and 2.3 reads as:

- Let $-\infty < q \leqslant p < 0$, $s \in [0,p)$. Suppose that $k(x,t) = h(x,t)u^{\frac{1}{q}}(t)$, where $h(x,t)$ is positive and *nonincreasing* in t, and satisfies the conditions

$$H_s := \sup_{a<t<b} H_s(t) = \sup_{a<t<b} \left(\int_a^t H^{\frac{q(1-s)}{p}}(x,t)\, dH^{\frac{q(s-1)}{p}}(x,x) \right) < \infty,$$

with

$$H(x,t) = \int_t^b h^{p'}(\tau,x)\, d\tau, \quad a < x \leqslant t < b$$

and $H(x,x)$ is *absolutely continuous* in (a,b).

Then inequality (1.6) holds for all positive functions f if and only if

$$A_s := \sup_{a<t<b} A_s(t) = \sup_{a<t<b} H^{\frac{1-s}{p}}(t,t) \left(\int_t^b u(x) H^{\frac{(p-s)q}{p}}(x,x)dx \right)^{-\frac{1}{q}} < \infty$$

or

$$B_s := \sup_{a<t<b} B_s(t) = \sup_{a<t<b} \left(\int_a^t H^{\frac{(1-s)q}{p}}(x,t) H^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)u(x)dx \right)^{-\frac{1}{q}} < \infty.$$

Moreover, if C is the best possible constant in (1.6), then $C \approx A_s \approx B_s$.

REMARK 2.6. As already mentioned in Remark 1.1, Theorems 2.1, 2.2, 2.3 and Remark 2.5 allow us to obtain necessary and sufficient conditions for the case $0 < p \leqslant q < 1$ via the dual inequalities (1.7) and (1.8). For details see Theorem 3 in [4].

Now, let us consider inequality (2.3) with the very special kernel

$$h(x,t) = (V(x) - V(t))^{\alpha} v(t),$$

$\alpha > -1$, where $v(t)$ is a weight function and

$$V(t) = \int_a^t v^{p'}(\tau)d\tau.$$

This case is interesting since – in comparison with the assumptions of Theorem 2.2 – the kernel $h(x,t)$ need not to be nondecreasing in x and the function $H(x,x)$ need not to be absolutely continuous in (a,b). Our result reads:

THEOREM 2.7. *Let $-\infty < q \leqslant p < 0$, $\alpha > -1$ and u, v be positive and finite weight functions. Then the inequality*

$$\left(\int_a^b f^p(t)dt\right)^{\frac{1}{p}} \leqslant C\left(\int_a^b \left(\int_a^x (V(x)-V(t))^{\alpha} v(t) f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \tag{2.7}$$

holds for all functions $f > 0$ if and only if the function

$$A(x) := V(x)^{-(\alpha+\frac{1}{p'})}\left(\int_a^x u(t)dt\right)^{-\frac{1}{q}} \tag{2.8}$$

is bounded on (a,b). Moreover, if C is the best possible constant in (2.7), then

$$C \approx A := \sup_{a<x<b} A(x).$$

3. Proofs

Proof of Theorem 2.1.

Assume that (2.1) holds and let $f^p(x) = g(x)$ in (1.5). Then inequality (1.5) can be rewritten as

$$\int_a^b \left(\int_a^x k(x,t) g^{\frac{1}{p}}(t)dt\right)^q dx \leqslant C^{-q}\left(\int_a^b g(x)dx\right)^{\frac{q}{p}}. \tag{3.1}$$

By applying the reverse Hölder inequality and Minkowski's integral inequality to the left hand side of (3.1), one obtains

$$\begin{aligned}
&\int_a^b \left(\int_a^x k(x,t) g^{\frac{1}{p}}(t)dt\right)^q dx \\
&= \int_a^b \left(\int_a^x g^{\frac{1}{p}}(t) K^{\frac{1-s}{p}}(x,t) K^{-\frac{(1-s)}{p}}(x,t) k(x,t)dt\right)^q dx \\
&\leqslant \int_a^b \left(\int_a^x g(t) K^{1-s}(x,t)dt\right)^{\frac{q}{p}} \left(\int_a^x K^{-\frac{(1-s)p'}{p}}(x,t) k^{p'}(x,t)dt\right)^{\frac{q}{p'}} dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_a^b \left(\int_a^x g(t) K^{1-s}(x,t)dt \right)^{\frac{q}{p}} \left(\int_a^x K^{-\frac{(1-s)p'}{p}}(x,t) dK(x,t) \right)^{\frac{q}{p'}} dx \\
&= \left(\frac{p}{p-(1-s)p'} \right)^{\frac{q}{p'}} \int_a^b \left(\int_a^x g(t) K^{1-s}(x,t)dt \right)^{\frac{q}{p}} K^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)dx \\
&= \left(\frac{p}{p-(1-s)p'} \right)^{\frac{q}{p'}} \left[\left(\int_a^b \left(\int_a^x g(t) K^{1-s}(x,t) K^{\frac{p-(1-s)p'}{p}\frac{p}{p'}}(x,x)dt \right)^{\frac{q}{p}} dx \right)^{\frac{p}{q}} \right]^{\frac{q}{p}} \\
&\leqslant \left(\frac{p}{p-(1-s)p'} \right)^{\frac{q}{p'}} \left[\int_a^b g(t) \left(\int_t^b K^{\frac{(1-s)q}{p}}(x,t) K^{\frac{p-(1-s)p'}{p}\frac{q}{p'}}(x,x)dx \right)^{\frac{p}{q}} dt \right]^{\frac{q}{p}} \\
&\leqslant \left(\frac{p}{p-(1-s)p'} \right)^{\frac{q}{p'}} \left[\int_a^b g(t) B_s(t)^{-p} dt \right]^{\frac{q}{p}} \\
&\leqslant \left(\frac{p}{p-(1-s)p'} \right)^{\frac{q}{p'}} B_s^{-q} \Big(\int_a^b g(t)dt \Big)^{\frac{q}{p}}.
\end{aligned}
\tag{3.2}
$$

The theorem is proved. □

Proof of Theorem 2.2.

First we consider the case when $s = p$.

Sufficiency: Let condition (2.4) be satisfied. The definition of the functions K and H, integration by parts and the monotonicity of h yield

$$
\begin{aligned}
B_p^{-q}(t) &= \int_t^b K^{-\frac{q}{p'}}(x,t) K^{2\frac{q}{p'}}(x,x)dx = \int_t^b H^{-\frac{q}{p'}}(x,t) H^{2\frac{q}{p'}}(x,x)u(x)dx \\
&= \int_t^b H^{2\frac{q}{p'}}(x,x)\, d\left(\int_t^x u(\tau) H^{-\frac{q}{p'}}(\tau,t)d\tau \right) \\
&\leqslant \lim_{x\to b-} H^{2\frac{q}{p'}}(x,x) \int_t^x u(\tau) H^{-\frac{q}{p'}}(\tau,t)d\tau \\
&\quad - \int_t^b \left(\int_t^x u(\tau) H^{-\frac{q}{p'}}(\tau,t)d\tau \right) d\left(H^{2\frac{q}{p'}}(x,x) \right)
\end{aligned}
$$

$$\leqslant \lim_{x\to b-} H^{-\frac{q}{p'}}(x,t)H^{\frac{q}{p'}}(x,x)A_p^{-q}(x) -2\int_t^b A_p^{-q}(x)H^{-\frac{q}{p'}}(x,t)\,dH^{\frac{q}{p'}}(x,x).$$

Consequently

$$B_p^{-q}(t) \leqslant \sup_{a<x<b} A_p^{-q}(x)\left[\lim_{x\to b-} H^{-\frac{q}{p'}}(x,t)H^{\frac{q}{p'}}(x,x) - 2\int_t^b H^{-\frac{q}{p'}}(x,t)\,dH^{\frac{q}{p'}}(x,x)\right],$$

i.e.

$$B_p \leqslant (1+2H_p)^{-\frac{1}{q}} A_p \tag{3.3}$$

and sufficiency follows from Theorem 2.1.

Necessity: Assume that inequality (1.5), i.e. the equivalent inequality (2.3), holds. Using the test function

$$g_\tau(t) = h^{p'-1}(\tau,t)\chi_{(a,\tau)}(t) + \infty\chi_{(\tau,b)}(t) \quad t\in(a,b),$$

where $\tau\in(a,b)$ is fixed, the right hand side of (2.3) reads

$$\int_a^\tau \left(\int_a^x h(x,t)h^{p'-1}(\tau,t)\,dt\right)^q u(x)dx \geqslant \int_a^\tau \left(\int_a^\tau h(\tau,t)h^{p'-1}(\tau,t)\,dt\right)^q u(x)dx = H^q(\tau,\tau)\int_a^\tau u(x)dx. \tag{3.4}$$

Similarly, the left hand side of (2.3) becomes

$$C^{-q}\left(\int_a^\tau h^{p'}(\tau,x)dx\right)^{\frac{q}{p}} = C^{-q}H^{\frac{q}{p}}(\tau,\tau). \tag{3.5}$$

Consequently, from (3.4) and (3.5), inequality (2.3) yields

$$H^q(\tau,\tau)\int_a^\tau u(x)dx \leqslant C^{-q}H^{\frac{q}{p}}(\tau,\tau),$$

i.e.

$$A_p(\tau) = H^{-\frac{1}{p'}}(\tau,\tau)\left(\int_a^\tau u(x)dx\right)^{-\frac{1}{q}} \leqslant C.$$

The necessity is proved.

So we have proved the sufficiency and necessity of the condition (2.4) for the case $s = p$. The case $s \in (p,1)$ follows from Theorem 2.1 in [1] (see also Remark 2.4). Here, we choose

$$V(x) := H(x,x).$$

Applying now Theorem 2.1 in [1], we obtain that

$$C_s' A_p \leqslant A_s \leqslant C_s'' A_p,$$

where C_s', C_s'' depend on $s \in (p,1)$.

The proof is complete. □

Proof of Theorem 2.3.

The proof immediately follows by applying Theorems 2.1 and 2.2.

Necessity: Let inequality (2.3) hold. Then, by Theorem 2.2, A_s is finite. Since we can easily derive an analogue of (3.3) with p replaced by s, we have that also $B_s < \infty$.

Sufficiency: Let $B_s < \infty$. Then according to Theorem 2.1, inequality (1.5), i.e. (2.3), holds.

The proof is complete. □

Proof of Theorem 2.7.

Necessity: Assume that inequality (2.7) holds. Let us choose for f the function

$$f_\tau(t) = v(t)^{p'-1}\chi_{(a,\tau)}(t) + \infty\chi_{(\tau,b)}(t). \tag{3.6}$$

Substituting (3.6) into (2.7), the left hand side of (2.7) yields

$$\int_a^b f^p(t)dt = \int_a^\tau v(t)^{(p'-1)p}\,dt = V(\tau). \tag{3.7}$$

Similarly, by substituting (3.6) into the right hand side of (2.7) we obtain

$$\begin{aligned}
&\int_a^b u(x)\left(\int_a^x (V(x)-V(t))^\alpha v(t) f_\tau(t)dt\right)^q dx \\
&= \int_a^\tau u(x)\left(\int_a^x (V(x)-V(t))^\alpha dV(t)\right)^q dx \\
&= \frac{1}{(1+\alpha)^q}\left(\int_a^\tau u(x)dx\right) V(\tau)^{(\alpha+1)q}.
\end{aligned} \tag{3.8}$$

Now, by substituting (3.7) and (3.8) into (2.7) we have

$$V(\tau)^{\frac{1}{p}} \leqslant C\left[\frac{1}{(1+\alpha)^q}\left(\int_a^\tau u(x)dx\right)V(\tau)^{(\alpha+1)q}\right]^{\frac{1}{q}},$$

hence

$$(\alpha+1)\left(\int_a^\tau u(x)dx\right)^{-\frac{1}{q}} V(\tau)^{-\alpha-\frac{1}{p'}} \leqslant C.$$

The necessity part is proved.

Sufficiency: Assume that (2.7) holds. Using integration by parts and the monotonicity of V, we can easily show that

$$\int_a^x (V(x)-V(t))^\gamma V^\beta(t)dV(t) \approx V^{\gamma+\beta+1}(x) \tag{3.9}$$

for $-\infty < p < 0$ and with $\gamma > -1$, $\beta \geqslant 0$ or $\gamma \geqslant 0$, $\beta > -1$. For easy computation, we rewrite (2.7) in the form

$$\int_a^b u(x)\left(\int_a^x (V(x)-V(t))^\alpha v(t)f(t)dt\right)^q dx \leqslant C^{-q}\left(\int_a^b f^p(t)dt\right)^{\frac{q}{p}}. \tag{3.10}$$

Let $\beta > 0$. By applying the reverse Hölder inequality, formula (3.9) and Minkowski's integral inequality to the left hand side of (3.10), we have

$$\begin{aligned}
&\int_a^b u(x)\left(\int_a^x (V(x)-V(t))^\alpha V(t)^\beta V(t)^{-\beta} v(t)f(t)dt\right)^q dx \\
&\qquad \leqslant \int_a^b u(x)\left(\int_a^x (V(x)-V(t))^{\alpha p'} V(t)^{\beta p'} dV(t)\right)^{\frac{q}{p'}} \left(\int_a^x V(t)^{-\beta p} f^p(t)dt\right)^{\frac{q}{p}} dx \\
&\qquad \leqslant C_1 \int_a^b u(x)V(x)^{(\alpha+\beta)q+\frac{q}{p'}} \left(\int_a^x V(t)^{-\beta p} f^p(t)dt\right)^{\frac{q}{p}} dx \\
&\qquad = C_1\left[\left(\int_a^b \left(\int_a^x u(x)^{\frac{p}{q}} V(x)^{(\alpha+\beta)p+\frac{p}{p'}} V(t)^{-\beta p} f^p(t)dt\right)^{\frac{q}{p}} dx\right)^{\frac{p}{q}}\right]^{\frac{q}{p}} \\
&\qquad \leqslant C_1\left[\int_a^b V(t)^{-\beta p} f^p(t)\left(\int_t^b u(x)V(x)^{(\alpha+\beta)q+\frac{q}{p'}} dx\right)^{\frac{p}{q}} dt\right]^{\frac{q}{p}} \\
&\qquad \leqslant C_1 B^{-q}\left(\int_a^b f^p(t)dt\right)^{\frac{q}{p}},
\end{aligned}$$

where

$$B^{-q} = \sup_{a<t<b} \left[V(t)^{-\beta q} \int_t^b u(x) V(x)^{(\alpha+\beta)q+\frac{q}{p'}} dx \right]. \tag{3.11}$$

Next, we show that

$$B^{-q} \leqslant C_2 A^{-q}.$$

We do this by estimating the integral on the right hand side of (3.11) as follows

$$\begin{aligned}
&\int_t^b u(x) V(x)^{(\alpha+\beta)q+\frac{q}{p'}} dx \\
&\qquad = \int_t^b V(x)^{(\alpha+\beta)q+\frac{q}{p'}} d \int_t^x u(t) dt \\
&\qquad = \lim_{x\to b-} V(x)^{(\alpha+\beta)q+\frac{q}{p'}} \int_t^x u(t) dt - \int_t^b \Big(\int_t^x u(t) dt \Big) d\, V(x)^{(\alpha+\beta)q+\frac{q}{p'}} \\
&\leqslant \lim_{x\to b-} \left[A(x)^{-q} V(x)^{\beta q} \right] - A^{-q} \Big(\alpha + \beta + \frac{1}{p'} \Big) q \int_t^b V(x)^{\beta q - 1} dV(x) \\
&\leqslant \Big(1 - \frac{\alpha+\beta+\frac{1}{p'}}{\beta} \Big) A^{-q} \lim_{x\to b-} V(x)^{\beta q} + \frac{\alpha+\beta+\frac{1}{p'}}{\beta} A^{-q} V(t)^{\beta q} \\
&\leqslant \frac{\alpha+\beta+\frac{1}{p'}}{\beta} A^{-q} V(t)^{\beta q}
\end{aligned} \tag{3.12}$$

Combining inequalities (3.11) and (3.12) we have

$$B^{-q} \leqslant \frac{\alpha+\beta+\frac{1}{p'}}{\beta} A^{-q}$$

and the sufficiency part is proved. The proof is complete. □

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