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VECTOR VARIATIONAL–LIKE INEQUALITIES WITH RELAXED η-α PSEUDOMONOTONE MAPPINGS IN BANACH SPACES

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Abstract. In this paper, we introduce two new concepts of relaxed η-α pseudomonotonicity and relaxed η-α demipseudomonotonicity as well as two classes of vector variational-like inequalities with relaxed η-α peseudomonotone mappings and relaxed η-α demipseudomonotone mappings in Banach spaces. By using KKM technique, we obtain the existence of solutions for vector variational-like inequalities with relaxed η-α pseudomonotone mappings in reflexive Banach spaces. We also show the solvability of vector variational-like inequalities with relaxed η-α demipseudomonotone mappings in reflexive Banach spaces by means of Kakutani-Fan-Glicksberg fixed-point theorem. The results presented in this paper extend and improve some corresponding results of several authors.

1. Introduction and Preliminaries

Vector variational inequality was first introduced and studied by Giannessi [9] in the setting of the finite-dimensional Euclidean space. In 1990, Chen and Yang [4] considered general vector variational inequalities and vector complementary problems in infinite dimensional spaces. Recently, many existence results of solutions for some kinds of vector variational inequalities have been derived and the variational inequalities has found many of its applications in the vector optimization, set-valued optimization, approximate analysis of vector optimization problems and vector network equilibrium problems. For details, we refer to [3, 10, 11] and the references therein.

It is well known that monotonicity plays an important role in the study of the vector variational inequalities as well as scalar variational inequalities. In the recent years, many important generalizations of monotonicity, such as relaxed monotonicity, p-monotonicity, quasimonotonicity, pseudomonotonicity, dense pseudomonotonicity, and semimonotonicity, etc., have been introduced to study various classes of variational inequalities and complementary problems (see, for example, [1-8, 10-18, 20-24] and the references therein). In 1999, by using the combination of compactness and monotonicity, Chen [2] introduced the concept of semimonotonicity and studied the so-called semimonotone scalar variational inequality in Banach spaces. Recently, Fang and Huang [7] introduced a new concept of relaxed η-α monotonicity and obtained

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some existence theorems of solutions for the variational-like inequalities with relaxed η-α monotone mappings in reflexive Banach spaces. Very recently, Bai, Zhou and Ni [1] introduced a new concept of relaxed η-α pseudomonotonicity and obtained some existence results on solutions for variational-like inequalities with relaxed η-α pseudomonotone mappings in reflexive Banach spaces by using KKM technique.

Inspired and motivated by the works mentioned above, in this paper, we introduce two new concepts of relaxed η-α pseudomonotonicity and relaxed η-α demipseudomonotonicity as well as two classes of vector variational-like inequalities with relaxed η-α peseudomonotone and η-α demipseudomonotone mappings in Banach spaces, respectively. By using KKM technique, we obtain some existence theorems of solutions for vector variational-like inequalities with relaxed η-α pseudomonotone mappings in reflexive Banach spaces. We also show the solvability of vector variational-like inequalities with relaxed η-α demipseudomonotone mappings in reflexive Banach spaces by means of Kakutani-Fan-Glicksberg fixed-point theorem. The results presented in this paper extend and improve some corresponding results of [1, 2, 5, 7, 8].

Let X be a real Banach space and $P \subset X$ be a pointed closed convex cone with $\operatorname{int} P \neq \emptyset$, where $\operatorname{int} P$ is the interior of P. With P, we define relations " $\geqslant$ ", " $\not\geqslant$ ", " $<$ ", and " $\not<$ " as follows: for any $x, y \in X$,

$x \geqslant y \Longleftrightarrow x - y \in P$;

$x \not\geqslant y \Longleftrightarrow x - y \notin P$;

$x < y \Longleftrightarrow y - x \in \operatorname{int} P$;

$x \not< y \Longleftrightarrow y - x \notin \operatorname{int} P$.

In what follows, unless otherwise specified, we always suppose that K is a nonempty closed convex subset of a real Banach space X and $(Y, \leqslant)$ is an ordered Banach space induced by the pointed closed convex cone P with $\operatorname{int} P \neq \emptyset$. Denote by $L(X, Y)$ the space of all the continuous linear mappings from $X \to Y$.

DEFINITION 1.1. A mapping $T : K \to L(X, Y)$ is said to be relaxed η-α pseudomonotone if there exist $\eta : K \times K \to X$ and $\alpha : X \to Y$ with $\alpha(tz) = t^p \alpha(z)$ for all $t > 0$ and $z \in X$ such that

$$\langle Ty, \eta(x, y)\rangle \not< 0 \Longrightarrow \langle Tx, \eta(x, y)\rangle \geqslant \alpha(x - y), \quad \forall x, y \in K, \tag{1.1}$$

where $p > 1$ is a constant.

Some special cases:

(1) If $\eta(x, y) = x - y$ for all $x, y \in K$, then (1.1) reduces to that for any $x, y \in K$,

$$\langle Ty, x - y)\rangle \not< 0 \Longrightarrow \langle Tx, x - y)\rangle \geqslant \alpha(x - y),$$

where $p > 1$ is a constant;

(2) If $Y = R$, then (1.1) reduces to that for any $x, y \in K$,

$$\langle Ty, \eta(x, y))\rangle \geqslant 0 \Longrightarrow \langle Tx, \eta(x, y)\rangle \geqslant \alpha(x - y), \tag{1.2}$$

and T is said to be relaxed η-α pseudomonotone (see [1]);

(3) If $Y = R$, $\eta(x,y) = x - y$ for all $x, y \in K$ and $\alpha(x-y) \equiv 0$, then (1.2) reduces to that for any $x, y \in K$,

$$\langle Ty, x - y)\rangle \geqslant 0 \Longrightarrow \langle Tx, x - y)\rangle \geqslant 0,$$

and T is said to be pseudomonotone (see [6, 16]).

EXAMPLE 1.1. Let $X = K = (-\infty, +\infty)$, $Y = R^2$, $P = R^2_+$, $\eta(x,y) = e^x - e^y$ for all $x, y \in X$, and

$$T(x) = \begin{cases} \begin{pmatrix} \frac{3}{2}x \\ \frac{3}{2}x \end{pmatrix}, & x \geqslant 0, \\ \begin{pmatrix} -\frac{1}{2}x \\ -\frac{1}{2}x \end{pmatrix}, & x < 0; \end{cases} \qquad \alpha(x) = \begin{pmatrix} -\frac{1}{2}x^2 \\ -\frac{1}{2}x^2 \end{pmatrix}, \quad \forall x \in X.$$

Then it is easy to see that T is relaxed η-α pseudomonotone.

DEFINITION 1.2. Let $T : K \to L(X,Y)$ and $\eta : K \times K \to X$ be two mappings. T is said to be η-hemicontinuous if, for any $x, y \in K$, the mapping

$$t \mapsto \langle T(x + t(y - x)), \eta(y,x)\rangle$$

is continuous at 0^+.

REMARK 1.1. If $Y = R$, then Definition 1.2 reduces to the definition of η-hemicontinuous in the sense of [2, 22].

DEFINITION 1.3. A mapping $T : K \to Y$ is said to be completely continuous if, for any net $\{x_\gamma\} \in K$, $x_\gamma \to x_0 \in K$ weakly, then $Tx_\gamma \to Tx_0$.

DEFINITION 1.4. A mapping $G : M \subset X \to 2^X$ is said to be a KKM mapping if, for any finite set $A \subset M$, $\operatorname{co} A \subset \bigcup_{x \in A} G(x)$, where 2^X denotes the family of all the nonempty subsets of X and $\operatorname{co} A$ is the convex hull of A.

LEMMA 1.1. ([24]) *Let M be a nonempty subset of a Hausdorff topological vector space X and $G : M \subset X \to 2^X$ be a KKM mapping. If $G(x)$ is closed in X for every $x \in M$ and compact for some $x \in M$, then $\bigcap_{x \in M} G(x) \neq \emptyset$.*

LEMMA 1.2. ([4, 3]) *Let $(Y, \leqslant)$ be an ordered Banach space induced by the pointed closed convex cone P with $\operatorname{int} P \neq \emptyset$. For any $a, b, c \in Y$, the following implications hold:*

$c \nleq a \geqslant b \Longrightarrow b \ngeq c$;
$c \ngeq a \leqslant b \Longrightarrow b \nleq c$.

LEMMA 1.3. *Let $T : K \to L(X,Y)$ be η-hemicontinuous and η-α pseudomonotone. Assume that*

(i) $\eta(x,x) = 0$, $\forall x \in K$;
(ii) for any fixed $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex.

Then the following problems are equivalent:

(a) $x \in K$, $\langle Tx, \eta(y,x)\rangle \nleq 0$, $\forall y \in K$;
(b) $x \in K$, $\langle Ty, \eta(y,x)\rangle \geqslant \alpha(y-x)$, $\forall y \in K$.

Proof. $(a) \Longrightarrow (b)$. The result directly follows from Definition 1.1.

$(b) \Longrightarrow (a)$. Let $x \in K$ be a solution of problem (b). For any given $y \in K$, let $y_t = x + t(y-x)$ for all $t \in (0,1)$. It follows from (b) that

$$\begin{aligned} \langle Ty_t, \eta(y_t, x)\rangle &\geqslant \alpha(y_t - x) \\ &= \alpha(t(y-x)) \\ &= t^p \alpha(y-x). \end{aligned} \tag{1.3}$$

Now condition (ii) implies that

$$\begin{aligned} \langle Ty_t, \eta(y_t, x)\rangle &= \langle Ty_t, \eta((1-t)x+ty), x)\rangle \\ &\leqslant (1-t)\langle Ty_t, \eta(x,x)\rangle + t\langle Ty_t, \eta(y,x)\rangle \\ &= t\langle T(x+t(y-x)), \eta(y,x)\rangle . \end{aligned} \tag{1.4}$$

It follows from (1.3)-(1.4) that

$$\langle T(x+t(y-x)), \eta(y,x)\rangle \geqslant t^{p-1}\alpha(y-x) \tag{1.5}$$

for all $y \in K$. Since T is η-hemicontinuous and $p > 1$, letting $t \to 0$ in (1.5), we get

$$\langle Tx, \eta(y,x)\rangle \geqslant 0$$

and so

$$\langle Tx, \eta(y,x)\rangle \not< 0, \quad \forall y \in K.$$

This completes the proof. □

2. Vector Variational-Like Inequalities with Relaxed η-α Monotone Mappings

In this section, we always suppose that K is a nonempty closed convex subset of a real reflexive Banach space X and $(Y, \leqslant)$ is an ordered Banach space induced by the pointed closed convex cone P with $\operatorname{int} P \neq \emptyset$. Denote by $L(X,Y)$ the space of all the continuous linear mappings from X to Y. We will discuss the existence of solutions for the following vector variational-like inequality with relaxed η-α pseudomonotone mappings: find $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle \not< 0, \quad \forall y \in K. \tag{2.1}$$

THEOREM 2.1. *Let K be a nonempty bounded closed convex subset of a real reflexive Banach space X and $(Y, \leqslant)$ be an order Banach space induced by the point closed convex cone P with $\operatorname{int} P \neq \emptyset$. Let $T : K \to L(X,Y)$ be η-hemicontinuous and relaxed η-α pseudomonotone. Assume that*

(i) $\eta(x,x) = 0$, $\forall x \in K$;

(ii) *for any given points $y, z \in K$, the mapping $x \mapsto \langle Tz, \eta(x,y)\rangle$ is convex and the mapping $x \mapsto \langle Tz, \eta(y,x)\rangle$ is completely continuous;*

(iii) $\alpha : X \to Y$ *is completely continuous.*

Then problem (2.1) is solvable.

Proof. Define two set-valued mappings $F, G : K \to 2^X$ as follows:

$$F(y) = \{x \in K : \langle Tx, \eta(y,x)\rangle \not< 0\}, \quad \forall y \in K,$$

and

$$G(y) = \{x \in K : \langle Ty, \eta(y,x)\rangle \geqslant \alpha(y-x)\}, \quad \forall y \in K.$$

We claim that F is a KKM mapping. In fact, if F is not a KKM mapping, then there exist $\{y_1, \cdots, y_n\} \subset K$ and $t_i > 0$ $(i = 1, 2, \cdots, n)$ with $\sum_{i=1}^n t_i = 1$ such that

$$y = \sum_{i=1}^n t_i y_i \notin \bigcup_{i=1}^n F(y_i).$$

By the definition of F, we have

$$\langle Ty, \eta(y_i, y)\rangle < 0, \quad i = 1, 2, \cdots, n. \tag{2.2}$$

It follows from (2.2) and condition (ii) that

$$\begin{aligned} 0 &= \langle Ty, \eta(y,y)\rangle = \langle Ty, \eta(\sum_{i=1}^n t_i y_i, y)\rangle \\ &\leqslant \sum_{i=1}^n t_i \langle Ty, \eta(y_i, y)\rangle < 0, \end{aligned}$$

which is a contradiction. This implies that F is a KKM mapping. Since T is relaxed η-α pseudomonotone, we have

$$F(y) \subset G(y), \quad \forall y \in K.$$

So G is also a KKM mapping. Since K is bounded, closed and convex, we know that K is weakly compact. From the assumptions, we know that $G(y)$ is weakly closed for all $y \in K$. In fact, since $x \mapsto \langle Tz, \eta(y,x)\rangle$ and α are completely continuous, we know that $G(y)$ is weakly closed for all $y \in K$ and so $G(y)$ is weakly compact in K for all $y \in K$. It follows from Lemmas 1.1 and 1.3 that

$$\bigcap_{y \in K} F(y) = \bigcap_{y \in K} G(y) \neq \emptyset.$$

Hence, there exists $x \in K$ such that

$$\langle Tx, \eta(y,x)\rangle \not< 0, \quad \forall y \in K.$$

This completes the proof. $\square$

THEOREM 2.2. *Let K be a nonempty unbounded closed convex subset of a real reflexive Banach space X and $(Y, \leqslant)$ be an order Banach space induced by the point closed convex cone P with $\operatorname{int} P \neq \emptyset$. Let $T : K \to L(X,Y)$ be η-hemicontinuous and relaxed η-α pseudomonotone. Assume that*

(i) *there exist a constant $r > 0$ and $y_0 \in K$ with $\|y_0\| \leqslant r$ such that*

$$\langle Tz, \eta(z, y_0)\rangle > 0, \quad \forall z \in K \text{ with } \|z\| = r; \tag{2.3}$$

(ii) $\eta(x,y)+\eta(y,x)=0$, $\forall x,y\in K$;
(iii) *for any given points* $y,z\in K$, *the mapping* $x\mapsto \langle Tz,\eta(x,y)\rangle$ *is convex and completely continuous;*
(iv) $\alpha : X\to Y$ *is completely continuous.*
Then problem (2.1) is solvable.

Proof. Define $K_r=\{z\in K:\|z\|\leqslant r\}$. By Theorem 2.1, there exists $x\in K_r$ such that

$$\langle Tx,\eta(y,x)\rangle \not< 0,\quad \forall y\in K_r. \tag{2.4}$$

Letting $y=y_0$ in (2.4), we have

$$\langle Tx,\eta(y_0,x)\rangle \not< 0.$$

By condition (ii),

$$\langle Tx,\eta(x,y_0)\rangle \not> 0. \tag{2.5}$$

Combining (2.3) and (2.5), we know that $\|x\|<r$. For any given $y\in K$, choose $t>0$ small enough such that $t\in(0,1)$ and

$$x+t(y-x)\in K_r.$$

It follows from (2.4) and condition (iii) that

$$(1-t)\langle Tx,\eta(x,x)\rangle + t\langle Tx,\eta(y,x)\rangle \geqslant \langle Tx,\eta(x+t(y-x),x)\rangle \not< 0. \tag{2.6}$$

Now (2.6) and Lemma 1.2 imply that

$$\langle Tx,\eta(y,x)\rangle \not< 0,\quad \forall y\in K.$$

This completes the proof. □

REMARK 2.1. Theorems 2.1 and 2.2 improve and generalize the corresponding results of [1] and [7] when $f\equiv 0$.

3. Vector Variational-Like Inequalities with Relaxed η-α Demipseudomonotone Mappings

Throughout this section, let X be a real reflexive Banach space, $K\subset X$ be a nonempty closed convex set, and $(Y,\leqslant)$ be an ordered Banach space induced by the pointed closed convex cone P with $\operatorname{int}P\neq\emptyset$. Denote by $L(X,Y)$ the space of all the continuous linear mappings from X into Y. We will discuss the existence of solutions for the following vector variational-like inequality with relaxed η-α demipseudomonotone mapping: find $x\in K$ such that

$$\langle T(x,x),\eta(y,x)\rangle \not< 0,\quad \forall y\in K.$$

DEFINITION 3.1. A mapping $A : K \times K \to L(X, Y)$ is said to be relaxed η-α demipseudomonotone if the following conditions hold:

(i) for any $u \in K$, $A(u, \cdot)$ is relaxed η-α pseudomonotone;

(ii) for any $v \in K$ and $\omega \in X$, $\langle A(\cdot, v), \omega \rangle$ is completely continuous.

THEOREM 3.1. *Let $K \subset X$ be a nonempty bounded and closed convex subset and $A : K \times K \to L(X, Y)$ be a nonlinear mapping. Suppose that*

(a) *A is relaxed η-α demipseudomonotone;*

(b) *for each $x \in K$, $A(x, \cdot) : K \to L(X, Y)$ is finite-dimensional continuous; i.e., for any finite-dimensional subspace $D \subset X$, $A(x, \cdot) : K \cap D \to L(X, Y)$ is continuous;*

(c) $\eta(x, y) + \eta(y, x) = 0$, $\forall x, y \in K$;

(d) *for any given points $\omega, y, z \in K$, the mapping $x \mapsto \langle T(\omega, z), \eta(x, y) \rangle$ is convex and the mapping $x \mapsto \eta(x, y)$ is completely continuous;*

(e) *$\alpha : X \to Y$ is convex and completely continuous.*

Then there exists $u \in K$ such that

$$\langle A(u, u), \eta(v, u) \rangle \not< 0, \quad \forall v \in K. \tag{3.1}$$

Proof. Let $D \subset X$ be a finite-dimensional subspace with $K_D = D \cap K \neq \emptyset$. For each $\omega \in K$, consider the following problem: find $u_0 \in K_D$ such that

$$\langle A(\omega, u_0), \eta(v, u_0) \rangle \not< 0, \quad \forall v \in K_D. \tag{3.2}$$

Since $K_D \subset D$ is bounded and closed convex, $A(\omega, \cdot)$ is continuous on K_D and relaxed η-α pseudomonotone for each fixed $\omega \in K$, it follows from Theorem 2.1 that problem (3.2) has a solution $u_0 \in K_D$.

Define a set-valued mapping $T : K_D \to 2^{K_D}$ as follows:

$$T\omega = \{u \in K_D : \langle A(\omega, u), \eta(v, u) \rangle \not< 0, \forall v \in K_D\}, \quad \forall \omega \in K_D.$$

By Lemma 1.3, for each given $\omega \in K_D$,

$$\begin{aligned} &\{u \in K_D : \langle A(\omega, u), \eta(v, u) \rangle \not< 0, \forall v \in K_D\} \\ &= \{u \in K_D : \langle A(\omega, v), \eta(v, u) \rangle \geqslant \alpha(v - u), \forall v \in K_D\}. \end{aligned}$$

From conditions (c)–(e), it is easy to see that $T : K_D \to 2^{K_D}$ has nonempty bounded and closed convex values.

Now we prove that $T : K_D \to 2^{K_D}$ is upper semicontinuous. In fact, we only need to prove that $T : K_D \to 2^{K_D}$ has closed graph. Let $(\omega_\gamma, u_\gamma) \in \mathrm{Graph}(T)$ and $(\omega_\gamma, u_\gamma) \to (\omega_0, u_0)$ weakly. Then

$$\begin{aligned} \langle A(\omega_\gamma, v), \eta(v, u_0) \rangle + \langle A(\omega_\gamma, v), \eta(v, u_\gamma) - \eta(v, u_0) \rangle &= \langle A(\omega_\gamma, v), \eta(v, u_\gamma) \rangle \\ &\geqslant \alpha(v - u_\gamma), \quad \forall v \in K_D. \end{aligned} \tag{3.3}$$

Since A is relaxed η-α demipseudomonotone,

$$\langle A(\omega_\gamma, v), \eta(v, u_0) \rangle \to \langle A(\omega_0, v), \eta(v, u_0) \rangle, \quad \forall v \in K_D \tag{3.4}$$

and

$$\langle A(\omega_\gamma, v), x\rangle \to \langle A(\omega_0, v), x\rangle, \quad \forall x \in X.$$

By the principle of uniform boundedness (see [19]), we know that $\{A(\omega_\gamma, v)\}$ is uniform boundedness. Again from conditions (c)-(e), (3.3) and (3.4), we get

$$\langle A(\omega_0, v), \eta(v, u_0)\rangle \geqslant \alpha(v - u_0), \quad \forall v \in K_D, \tag{3.5}$$

i.e., $(\omega_0, u_0) \in \operatorname{Graph} T$. This implies that $\operatorname{Graph} T$ is weakly closed and so closed. Therefore, $T : K_D \to 2^{K_D}$ is upper semicontinuous.

By the Kakutani-Fan-Glicksberg fixed point theorem (see [24]), T has a fixed point $\omega_0 \in K_D$, i.e.,

$$\langle A(\omega_0, \omega_0), \eta(v, \omega_0)\rangle \not< 0, \quad \forall v \in K_D. \tag{3.6}$$

Let

$$\Sigma = \{D \subset X : D \text{ is a finite-dimensional subspace with } D \cap K \neq \emptyset\}$$

and

$$W_D = \{u \in K : \langle A(u, v), \eta(v, u)\rangle \geqslant \alpha(v - u), \forall v \in K_D\}, \quad \forall D \in \Sigma.$$

It follows from (3.6) and Lemma 1.3 that W_D is nonempty and bounded. Denote by $\overline{W}_D$ the weak closure of W_D in X. Then $\overline{W}_D$ is weakly compact in X.

For any $D_i \in \Sigma$, $i = 1, 2, \cdots, N$, we know that $W_{\cup_i D_i} \subset \cap W_{D_i}$ and so $\{\overline{W}_D : D \in \Sigma\}$ has the finite intersection property. It follows that

$$\bigcap_{D \in \Sigma} \overline{W}_D \neq \emptyset.$$

Let $u \in \bigcap_{D \in \Sigma} \overline{W}_D$. We claim that

$$\langle A(u, u), \eta(v, u)\rangle \not< 0, \quad \forall v \in K.$$

In fact, for each $v \in K$, let $D \in \Sigma$ be such that $v \in K_D$ and $u \in K_D$. Since $\overline{W}_D$ is weakly closed, there exists a net $\{u_\gamma\} \subset W_D$ such that u_γ converges to u with respect to the weak topology of X. It follows that

$$\langle A(u_\gamma, v), \eta(v, u_\gamma)\rangle \geqslant \alpha(v - u_\gamma).$$

Similar to the argument of (3.5), conditions (a), (c), (d) and (e) imply that

$$\langle A(u, v), \eta(v, u)\rangle \geqslant \alpha(v - u), \quad \forall v \in K.$$

By Lemma 1.3,

$$\langle A(u, u), \eta(v, u)\rangle \not< 0, \quad \forall v \in K.$$

This completes the proof. $\square$

THEOREM 3.2. *Let $K \subset X$ be a nonempty unbounded and closed convex subset and $A : K \times K \to L(X, Y)$ be a nonlinear mapping. Suppose that*

(a) *A is relaxed η-α demipseudomonotone;*

(b) *for each $x \in K$, $A(x, \cdot) : K \to L(X, Y)$ is finite-dimensional continuous;*

(c) *$\eta(x,y)+\eta(y,x)=0$, $\forall x,y \in K$;*
(d) *for any given points $\omega, y, z \in K$, the mapping $x \mapsto \langle T(\omega,z), \eta(x,y)\rangle$ is convex and $x \mapsto \eta(x,y)$ completely continuous;*
(e) *$\alpha : X \to Y$ is convex and completely continuous;*
(f) *there exist a constant $r > 0$ and $y_0 \in K$ with $\|y_0\| \leqslant r$ such that*

$$\langle T(z,z), \eta(z,y_0)\rangle > 0, \quad \forall z \in K \textit{ with } \|z\| = r. \tag{3.7}$$

Then there exists $u \in K$ such that

$$\langle A(u,u), \eta(v,u)\rangle \nless 0, \quad \forall v \in K.$$

Proof. Define $K_r = \{z \in K : \|z\| \leqslant r\}$. By Theorem 3.1, there exists $x \in K_r$ such that

$$\langle T(x,x), \eta(y,x)\rangle \nless 0, \quad \forall y \in K_r. \tag{3.8}$$

Letting $y = y_0$ in (3.8), we have

$$\langle T(x,x), \eta(y_0,x)\rangle \nless 0.$$

It follows from condition (c) that

$$\langle T(x,x), \eta(x,y_0)\rangle \ngtr 0. \tag{3.9}$$

Combining (3.7) and (3.9), we know that $\|x\| < r$. For any given $y \in K$, choose $t > 0$ small enough such that $t \in (0,1)$ and

$$x + t(y-x) \in K_r.$$

It follows from (3.8) and condition (d) that

$$(1-t)\langle T(x,x), \eta(x,x)\rangle + t\langle T(x,x), \eta(y,x)\rangle \geqslant \langle T(x,x), \eta(x+t(y-x),x)\rangle \nless 0.$$

By condition (c) and Lemma 1.2,

$$\langle T(x,x), \eta(y,x)\rangle \nless 0, \quad \forall y \in K.$$

This completes the proof. □

REMARK 3.1. Theorems 3.1 and 3.2 improve and generalize the corresponding results of [2], [7] and [8] when $f \equiv 0$.

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