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ON THE GENERALIZED HYERS–ULAM STABILITY OF SWIATAK'S FUNCTIONAL EQUATION

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Abstract. In this paper we shall study the generalized Hyers-Ulam stability of Swiatak's functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y)+g(x)g(y),\quad x,y\in G,$$

where G is an abelian group and $f,g:G\longrightarrow\mathbb{C}$ are complex-valued functions satisfying the condition $g(e)\neq 0$.

1. Introduction

The following question concerning the stability of mappings has been raised by S. M. Ulam [27]:

Given a group G, a metric group (G',d) and a number $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\longrightarrow G'$ satisfies the inequality

$$d(f(xy),f(x)f(y))<\epsilon \text{ for all } x,y\in G,$$

then a homomorphism $h:G\longrightarrow G'$ exists such that

$$d(f(x),h(x))<\delta \text{ for all } x\in G?$$

The first affirmative partial answer to the question of Ulam was given in 1941 by D. H. Hyers [11] in the following result:

THEOREM 1.1. *If $f:V\longrightarrow X$ is a mapping satisfying*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\delta,$$

for all $x,y\in V$, where V and X are Banach spaces and δ is a given positive number, then there exists a unique additive mapping $T:V\longrightarrow X$ such that

$$\|f(x)-T(x)\|\leqslant\delta,$$

for all $x\in V$. Also, if the function $t\longrightarrow f(tx)$ from $\mathbb{R}$ to X is continuous for each fixed x in V, then T is linear.

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Hyers's result was extended and generalized in several directions. In [3] Bourgin treated the problem for additive mappings. In [17] Th. M. Rassias provided a generalization of the Hyers's Theorem by allowing the Cauchy difference to be unbounded as follows.

THEOREM 1.2. *Let $f : V \longrightarrow X$ be a mapping between Banach spaces and let $p < 1$ be fixed. If f satisfies the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|x\|^p)$$

for some $\theta \geqslant 0$ and for all $x, y \in V$ ($x, y \in V \setminus \{0\}$ if $p < 0$). Then there exists a unique additive mapping $T : V \longrightarrow X$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{|2-2^p|}\|x\|^p \tag{1.1}$$

for all $x \in V$ ($x \in V \setminus \{0\}$ if $p < 0$).

If, in addition, $f(tx)$ is continuous in t for each fixed x in V, then T is linear.

In [1] T. Aoki treated the problem for additive mappings. However T. Aoki had claimed the wrong result of the existence of a unique linear mapping. His claim is not true because he did not allow the mapping f to satisfy some continuity assumption. Thus it was proved for the first time by Th. M. Rassias [17] that there exists a unique linear mapping T satisfying (1.1).

Th. M. Rassias during the 27th International Symposium on functional equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. Z. Gajda [10] following the same approach as in [17], gave an affirmative solution to Rassias' question for $p > 1$. However, it was showed that a similar result for the case $p = 1$ does not hold. The reader can be refered to [12,13,14,18,19,23,24,25,26] for a comprehensive account of the Hyers-Ulam stability of functional equations. In [6,7] I. Fenyö established the stability of the Ulam problem for quadratic and other mappings. In [9] Z. Gajda and R. Ger showed that one can get analogous stability results for a subadditive multifunction. Some other stability results have been achieved also by the following authors: In [16] S.-M. Jung investigated the Hyers-Ulam-Rassias stability for more general mappings on restricted domains. In [21] F. Skof solved the Ulam problem on a restricted domain and she proved the stability problem for the quadratic equation

$$q(x+y) + q(x-y) = 2q(x) + 2q(y), \quad x, y \in G, \tag{1.2}$$

where $q : V \longrightarrow X$ and V, X are Banach spaces. In [4] P. W. Cholewa extended Skof's result to the case where V is an abelian group G. In fact he proved the following result

THEOREM 1.3. *Let $\eta > 0$ be a real number. If $f : G \longrightarrow X$ satisfies the inequality*

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \leqslant \eta, \quad x, y \in G, \tag{1.3}$$

then for every $x \in G$ the limit $q(x) = \lim_{n \to +\infty} \frac{f(2^n x)}{2^{2n}}$ exists and $q : G \longrightarrow X$ is the unique solution of (1.2) satisfying

$$|f(x) - q(x)| \leqslant \frac{\eta}{2}, \quad x \in G. \tag{1.4}$$

Later, I. Fenyö [7] improved the above bound obtained by F. Skof and P. W. Cholewa from $\frac{\eta}{2}$ to $\frac{\eta+\|f(0)\|}{2}$.

In [15] K. W. Jun and Y.-H. Lee obtained the Hyers-Ulam-Rassias stability of the Pexider equations $f(x+y)+f(x-y) = 2g(x)+2g(y),\ x,y \in G$ and $f(x+y)+g(x-y) = 2h(x)+2k(y),\ x,y \in G$.

Recently B. Bouikhalene, E. Elqorachi and A. Redouani [2] proved the Hyers-Ulam-Rassias stability of O'Connor's and Gajda's type functional equations $f(x-y) = a(x)\overline{a}(y)$ and $f(x+y)+f(x-y) = 2a(x)\overline{a}(y),\ x,y \in G$. Following this investigation we deal with the stability of the functional equation

$$f(x+y)+f(x-y) = 2f(x)+2f(y)+g(x)g(y),\ x,y \in G, \tag{1.5}$$

where G is an abelian group and $f,g : G \longrightarrow \mathbb{C}$. This equation was introduced by Swiatak [22] as a generalization of the equation (1.2) (parallelogram law for which $g = 0$). In the case where $G = K = \mathbb{R}$, it is a characteristic of even polynomial of order 2. Swiatak found the general solution of (1.5), where $f,g : G \longrightarrow K$ and K is a commutative ring without zero divisors, under the additional hypothesis that $g(e) \neq 0$. Later the functional equation (1.5) was completely solved by J. K. Chung , B. R. Ebank, C. T. Ng and P. K. Sahoo [5].

In this paper we study the stability of this equation under the assumption of Swiatak, namely $g(e) \neq 0$ and $K = \mathbb{C}$.

2. Main results

In the following theorems we investigate the generalized Hyers-Ulam stability of the functional equation (1.5).

THEOREM 2.1. *Let $\delta > 0$ and $f,g : G \longrightarrow \mathbb{C}$ be mappings which satisfy the inequality*

$$|f(x+y)+f(x-y)-2f(x)-2f(y)-g(x)g(y)| \leqslant \delta, \tag{2.1}$$

for all $x,y \in G$ and $g(e) \neq 0$. Then there exists a unique mapping $q : G \longrightarrow \mathbb{C}$, given by $q(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^{2n}}$, which satisfies the quadratic functional equation

$$q(x+y)+q(x-y) = 2q(x)+2q(y),\ x,y \in G,$$

such that

$$|g(x)-g(e)| \leqslant \frac{2\delta}{|g(e)|},\ x \in G, \tag{2.2}$$

and

$$|f(x)-q(x)-f(e)| \leqslant \frac{2\delta^2}{|g(e)|^2}+2\delta,\ x \in G. \tag{2.3}$$

Proof. Let $\{f,g\}$ be a solution of inequality (2.1). Suppose

$$\theta(x,y) = g(x)g(y)+2f(x)+2f(y)-f(x+y)-f(x-y), \tag{2.4}$$

for all $x,y \in G$.

We note that for all $x, y \in G$ one has

$$|\theta(x,y)| \leqslant \delta.$$

Setting $x = y = e$ in (2.4) yields

$$\theta(e,e) = g(e)^2 + 2f(e). \tag{2.5}$$

By taking $y = e$ we get from (2.4) that

$$\theta(x,e) = g(x)g(e) + 2f(e),\ x \in G. \tag{2.6}$$

From (2.5) and (2.6) we deduce that

$$\theta(x,e) - \theta(e,e) = g(x)g(e) - g(e)^2 \tag{2.7}$$

$$= g(e)(g(x) - g(e)). \tag{2.8}$$

Since $g(e) \neq 0$ it follows that

$$|g(x) - g(e)| \leqslant \frac{2\delta}{|g(e)|},\ x \in G, \tag{2.9}$$

which establishes (2.2).

Now, equality (2.4) implies that

$$\begin{aligned}(f - f(e))(x+y) + (f - f(e))(x-y) &= 2(f - f(e))(x) + 2(f - f(e))(y) \\ &\quad + g(x)g(y) + 2f(e) - \theta(x,y).\end{aligned}$$

We get

$$\begin{aligned}(f - f(e))(x+y) + (f - f(e))(x-y) &= 2(f - f(e))(x) + 2(f - f(e))(y) \\ &\quad + g(x)g(y) - g(x)g(e) + \theta(x,e) - \theta(x,y) \\ &= 2(f - f(e))(x) + 2(f - f(e))(y) \\ &\quad + g(x)(g(y) - g(e)) + \theta(x,e) - \theta(x,y) \\ &= 2(f - f(e))(x) + 2(f - f(e))(y) \\ &\quad + g(x)\frac{\theta(y,e) - \theta(e,e)}{g(e)} + \theta(x,e) - \theta(x,y).\end{aligned}$$

By using the inequality (2.9) we get

$$|g(x)| \leqslant |g(e)| + \frac{2\delta}{|g(e)|}.$$

Hence,

$$|g(x)\frac{\theta(y,e) - \theta(e,e)}{g(e)}| \leqslant 2\delta + \frac{4\delta^2}{|g(e)|^2}$$

and consequently, we obtain

$$\begin{aligned}&|(f - f(e))(x+y) + (f - f(e))(x-y) - 2(f - f(e))(x) - 2(f - f(e))(y)| \\ &\qquad \leqslant 4\delta\left(\frac{\delta}{|g(e)|^2} + 1\right)\end{aligned} \tag{2.10}$$

for all $x,y \in G$. From Theorem 1.3 there exists a unique mapping $q : G \longrightarrow \mathbb{C}$ given by $q(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^{2n}}$ satisfying the functional equation (1.2) and

$$|f(x) - q(x) - f(e)| \leqslant 2\delta \left(\frac{\delta}{|g(e)|^2} + 1 \right)$$

for all $x \in G$, which establishes (2.3). This completes the proof of the theorem. □

THEOREM 2.2. *Let $f,g : G \longrightarrow \mathbb{C}$ and $\varphi : G \times G \longrightarrow [0,+\infty[$ be mappings which satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y) - g(x)g(y)| \leqslant \varphi(x,y), \tag{2.11}$$

for all $x,y \in G$ and $g(e) \neq 0$. Suppose also that φ satisfies the following inequalities

i)

$$\sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i y) < +\infty, \ x,y \in G.$$

ii)

$$\sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, e)^2 < +\infty, \ x \in G.$$

Then

$$|g(x) - g(e)| \leqslant \frac{\varphi(e,e) + \varphi(x,e)}{|g(e)|}, \tag{2.12}$$

and the function $q : G \longrightarrow \mathbb{C}$, given by $q(x) = \lim_{n\to+\infty} \frac{f(2^n x)}{2^{2n}}$ defines a unique quadratic function such that

$$\begin{aligned} \left| f(x) - q(x) - \frac{1}{3} f(e) \right| \ \leqslant \ & \frac{1}{|g(e)|^2} \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} (\varphi(2^i x, e) + 2 \mid f(e) \mid)^2 \\ & + \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i x), \quad x \in G. \end{aligned} \tag{2.13}$$

Proof. Substituting x in place of y in (2.11), we easily obtain

$$|f(2x) + f(e) - 4f(x) - g(x)^2| \leqslant \varphi(x,x), \quad x \in G. \tag{2.14}$$

By setting $y = e$, respectively $x = y = e$ in (2.11), we obtain

$$|-2f(e) - g(x)g(e)| \leqslant \varphi(x,e), \quad x \in G, \tag{2.15}$$

and

$$|-2f(e) - g(e)^2| \leqslant \varphi(e,e), \quad x \in G. \tag{2.16}$$

Adding (2.15) and (2.16) by the triangle inequality yields for all $x \in G$

$$|g(x) - g(e)| \leqslant \frac{\varphi(e,e) + \varphi(x,e)}{|g(e)|},$$

which establishes (2.12).

From (2.15) and (2.14), we get for all $x \in G$ that

$$|g(x)| \leqslant \frac{\varphi(x,e)+2|f(e)|}{|g(e)|} \tag{2.17}$$

and

$$\left|f(x)-\frac{1}{4}f(2x)-\frac{1}{3}\left(1-\frac{1}{4}\right)f(e)\right| \leqslant \frac{1}{4}\varphi(x,x)+\frac{(\varphi(x,e)+2|f(e)|)^2}{4|g(e)|^2},\ x \in G. \tag{2.18}$$

Making the inductive assumption we obtain

$$\begin{aligned}\left|f(x)-\frac{1}{4^n}f(2^n x)-\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(e)\right| \ &\leqslant\ \frac{1}{|g(e)|^2}\sum_{i=0}^{n-1}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2 \\ &+\sum_{i=0}^{n-1}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x),\ x\in G,\end{aligned} \tag{2.19}$$

for any positive integer n. Clearly (2.19) is true for the case $n=1$, since setting $n=1$ in (2.19) implies (2.18). We now assume that the inductive assumption is true for $n-1$ i.e. that

$$\begin{aligned}&\left|f(x)-\frac{1}{4^{n-1}}f(2^{n-1}x)-\frac{1}{3}(1-\frac{1}{4^{n-1}})f(e)\right| \\ &\qquad \leqslant \frac{1}{|g(e)|^2}\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x).\end{aligned}$$

Then for any positive integer n we get by using the triangle inequality that

$$\begin{aligned}&\left|f(x)-\frac{1}{4^n}f(2^n x)-\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(e)\right| \\ &\quad \leqslant \left|f(x)-\frac{1}{4^{n-1}}f(2^{n-1}x)-\frac{1}{3}(1-\frac{1}{4^{n-1}})f(e)\right| \\ &\qquad +\left|\frac{1}{4^{n-1}}f(2^{n-1}x)+\frac{1}{3}(1-\frac{1}{4^{n-1}})f(e)-\frac{1}{4^n}f(2^n x)-\frac{1}{3}\left(1-\frac{1}{4^n}\right)f(e)\right| \\ &\quad \leqslant \frac{1}{|g(e)|^2}\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x) \\ &\qquad +\frac{1}{4^{n-1}}\left|f(2^{n-1}x)-\frac{1}{4}f(22^{n-1}x)-\frac{1}{4}f(e)\right| \\ &\quad \leqslant \frac{1}{|g(e)|^2}\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=0}^{n-2}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x) \\ &\qquad +\frac{1}{4^{n-1}}\left[\frac{1}{4}\varphi(2^{n-1}x,2^{n-1}x)+\frac{(\varphi(2^{n-1}x,e)+2|f(e)|)^2}{4|g(e)|^2}\right]\end{aligned}$$

$$= \frac{1}{|g(e)|^2}\sum_{i=0}^{n-1}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=0}^{n-1}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x),\ x\in G.$$

Thus the inductive assumption (2.19) applies for all positive integer values of n. We now wish to construct a Cauchy sequence of functional values. If $m>n>0$, then $m-n\in\mathbb{N}$. Replacing n by $m-n$ in (2.19) one has

$$\begin{aligned}\left|f(x)-\frac{1}{4^{m-n}}f(2^{m-n}x)\right| &\leqslant \frac{1}{3}\left(1-\frac{1}{4^{m-n}}\right)|f(e)|\\ &+\frac{1}{|g(e)|^2}\sum_{i=0}^{m-n-1}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=0}^{m-n-1}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x).\end{aligned}$$

If we replace x by $2^n x$, we obtain

$$\begin{aligned}\left|f(2^n x)-\frac{1}{4^{m-n}}f(2^m x)\right| &\leqslant \frac{1}{3}\left(1-\frac{1}{4^{m-n}}\right)|f(e)|\\ &+\frac{1}{|g(e)|^2}\sum_{i=0}^{m-n-1}\frac{1}{4^{i+1}}(\varphi(2^{i+n}x,e)+2|f(e)|)^2\\ &+\sum_{i=0}^{m-n-1}\frac{1}{4^{i+1}}\varphi(2^{i+n}x,2^{i+n}x).\end{aligned}\tag{2.20}$$

Dividing (2.20) by 2^{2n} yields

$$\begin{aligned}\left|\frac{1}{4^n}f(2^n x)-\frac{1}{4^m}f(2^m x)\right| &\leqslant \frac{1}{3}\left(\frac{1}{4^n}-\frac{1}{4^m}\right)|f(e)|\\ &+\frac{1}{|g(e)|^2}\sum_{i=n}^{m-1}\frac{1}{4^{i+1}}(\varphi(2^i x,e)+2|f(e)|)^2+\sum_{i=n}^{m-1}\frac{1}{4^{i+1}}\varphi(2^i x,2^i x).\end{aligned}$$

The last inequality and the assumptions of the theorem show that the sequence $\{\frac{f(2^n x)}{2^{2n}}\}$ is a Cauchy sequence for each fixed $x\in G$. We call this limit $q(x)=\lim_{n\to+\infty}\frac{f(2^n x)}{2^{2n}}$. Next, we will show that $q(x)$ is a quadratic function. From (2.11) and (2.17), we get

$$\begin{aligned}&\left|\frac{f(2^n(x+y))+f(2^n(x-y))-2f(2^n x)-2f(2^n y)}{4^n}\right|\\ &\qquad\leqslant \frac{(\varphi(2^n x,e)+2|f(e)|)(\varphi(2^n y,e)+2|f(e)|)}{4^n|g(e)|^2}+\frac{\varphi(2^n x,2^n y)}{4^n},\end{aligned}\tag{2.21}$$

for all $x,y\in G$ and for all $n\in\mathbb{N}$. By the assumptions of the theorem, the right-hand side of (2.21) converges to zero as n tends to infinity, so that

$$q(x+y)+q(x-y)-2q(x)-2q(y)=0,\tag{2.22}$$

for all $x,y\in G$. Hence q is indeed a quadratic function.

From (2.19) we get

$$\left|f(x) - q(x) - \frac{1}{3}f(e)\right| \leqslant \frac{1}{|g(e)|^2} \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} (\varphi(2^i x, e) + 2|f(e)|)^2 + \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i x), \quad x \in G.$$

To prove the uniqueness, suppose that there exists another quadratic function B such that

$$\left|f(x) - B(x) - \frac{1}{3}f(e)\right| \leqslant \frac{1}{|g(e)|^2} \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} (\varphi(2^i x, e) + 2|f(e)|)^2 + \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i x), \quad x \in G.$$

For all $x \in G$ and for all $n \in \mathbb{N}$, we have

$$|q(x) - B(x)| = \frac{1}{4^n}|q(2^n x) - B(2^n x)| \leqslant \frac{1}{4^n}[|f(2^n x) - q(2^n x)| + |f(2^n x) - B(2^n x)|]$$

$$\leqslant \frac{2}{4^n}\left\{\frac{1}{3}|f(e)| + \frac{1}{|g(e)|^2} \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} (\varphi(2^i x, e) + 2|f(e)|)^2 + \sum_{i=0}^{\infty} \frac{1}{4^{i+1}} \varphi(2^i x, 2^i x)\right\} \tag{2.23}$$

by the triangle inequality. Since the right-hand side of (2.23) converges to zero as n tends to infinity, one gets that $q(x) = B(x)$ for all $x \in G$, so that q is unique. This completes the proof. □

THEOREM 2.3. *Let $f, g : G \longrightarrow \mathbb{C}$ and $\varphi : G \times G \longrightarrow [0, +\infty[$ be mappings which satisfy the inequality*

$$|f(x+y) + f(x-y) - 2f(x) - 2f(y) - g(x)g(y)| \leqslant \varphi(x, y), \tag{2.24}$$

for all $x, y \in G$ and $g(e) \neq 0$. Suppose also that φ satisfies the following inequalities

i)

$$\sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^{i+1}}, \frac{y}{2^{i+1}}\right) < +\infty, \ x, y \in G.$$

ii)

$$\sum_{i=0}^{\infty} 4^i \varphi\left(\frac{x}{2^{i+1}}, e\right)^2 < +\infty, \ x \in G.$$

Then

$$|g(x) - g(e)| \leqslant \frac{\varphi(e, e) + \varphi(x, e)}{|g(e)|}, \tag{2.25}$$

and the function $q : G \longrightarrow \mathbb{C}$*, given by* $q(x) = \lim_{n\to+\infty} 2^{2n}[f(2^n x) + \frac{1}{3}(1-2^{2n})f(e)]$ *defines a unique quadratic function such that*

$$\begin{aligned} |f(x) - q(x)| \ &\leqslant \ \frac{1}{|g(e)|^2}\sum_{i=0}^{\infty} 4^i\left(\varphi\left(\frac{x}{2^{i+1}}, e\right) + 2\,|\,f(e)\,|\right)^2 \\ &\quad + \sum_{i=0}^{\infty} 4^i\varphi\left(\frac{x}{2^{i+1}}, \frac{x}{2^{i+1}}\right), \quad x \in G. \end{aligned} \tag{2.26}$$

In the next corollary one has the Hyers-Ulam-Rassias stability of the Swiatak's functional equation (1.5) on a normed space G.

COROLLARY 2.4. *Let* $k < 1$*,* $\theta \geqslant 0$ *be real numbers. Suppose that the functions* $p, q : G \longrightarrow \mathbb{C}$ *are such that*

$$|p(x+y) + p(x-y) - 2p(x) - 2p(y) - q(x)q(y)| \leqslant \theta(|x|^k + |y|^k), \tag{2.27}$$

for all $x, y \in \mathbb{R}$ *and* $q(0) \neq 0$*. Then*

$$|q(x) - q(0)| \leqslant \frac{\theta|x|^k}{|q(0)|}, \tag{2.28}$$

and the function $B : G \longrightarrow \mathbb{C}$*, given by* $B(x) = \lim_{n\to+\infty} \frac{p(2^n x)}{2^{2n}}$ *defines a unique quadratic function such that*

$$\begin{aligned} |p(x) - B(x) - \frac{1}{3}p(0)| \ &\leqslant \ \frac{1}{|q(0)|^2}\sum_{i=0}^{\infty}\frac{1}{4^{i+1}}(\theta\mid 2^i x\mid^k + 2|p(0)|)^2 \\ &\quad + \frac{\theta|x|^k}{2}\frac{1}{1-2^{k-2}}, \quad x \in \mathbb{R}. \end{aligned} \tag{2.29}$$

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