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SOME MULTI–DIMENSIONAL HARDY TYPE INTEGRAL INEQUALITIES

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Abstract. In this paper we prove some new results concerning multi-dimensional Hardy type integral inequalities and also some corresponding limit Pólya–Knopp type inequalities.

1. Introduction

Let $1 \leqslant n < \infty$ be a natural number and $0 < p,q < \infty$. The n-dimensional integral Hardy operator H_n, defined for any non-negative function $f(\mathbf{y})$ on $\mathbb{R}^n_+ := \{\mathbf{y} = (y_1,\dots,y_n) : y_1,\dots,y_n \geqslant 0\}$, is given by

$$(H_nf)(\mathbf{x}) := \int_0^{x_1} \dots \int_0^{x_n} f(\mathbf{y})d\mathbf{y}, \qquad x_1,\dots,x_n > 0, \tag{1.1}$$

where $d\mathbf{y} := dy_1 \dots dy_n$. The problem to characterize weight functions w and v on $\mathbb{R}^n_+$ so that the inequality of the form

$$\left(\int_{\mathbb{R}^n_+} (H_nf)^q(\mathbf{x})w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n_+} f^p(\mathbf{y})v(\mathbf{y})d\mathbf{y}\right)^{\frac{1}{p}} \tag{1.2}$$

holds for all non-negative functions f on $\mathbb{R}^n_+$ is considered in several works (see [1], [3], [4], [7] and the references given there). The one-dimensional case is very well studied but even in two dimensions there is only one result for the inequality (1.2) to hold without any special restrictions on the weights, namely the following result of E. Sawyer for $1 < p \leqslant q < \infty$:

THEOREM 1.1. [7, Theorem 1] *Let $n = 2$ and $1 < p \leqslant q < \infty$. Then the inequality* (1.2) *holds for all non-negative functions f on $\mathbb{R}^2_+$ if and only if*

$$\sup_{t_1,t_2>0} W(t_1,t_2)^{\frac{1}{q}} V(t_1,t_2)^{\frac{1}{p'}} < \infty, \tag{1.3}$$

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$$\sup_{t_1,t_2>0} \frac{\left(\int_0^{t_1}\int_0^{t_2} V(x_1,x_2)^q w(x_1,x_2)dx_2dx_1\right)^{\frac{1}{q}}}{V(t_1,t_2)^{\frac{1}{p}}} < \infty, \tag{1.4}$$

and

$$\sup_{t_1,t_2>0} \frac{\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} W(x_1,x_2)^{p'} v(x_1,x_2)^{1-p'}dx_2dx_1\right)^{\frac{1}{p'}}}{W(t_1,t_2)^{\frac{1}{q'}}} < \infty, \tag{1.5}$$

where $p' := p/(p-1)$, $q' := q/(q-1)$, $W(t_1,t_2) := \int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1$ *and* $V(t_1,t_2) := \int_0^{t_1}\int_0^{t_2} v(y_1,y_2)^{1-p'}dy_2dy_1$.

Note that in the one-dimensional case the conditions corresponding to (1.3)–(1.5) are equivalent to each other (see [2]).

Moreover, it was recently discovered by A. Wedestig in her PhD thesis [8] (see also [9]) that if the weight on the right hand is of product type, then, in fact, (1.2) can be characterized by just one condition (or, more generally, just one of infinite possible conditions).

THEOREM 1.2. [9, Theorem 1.1] *Let* $n = 2$, $1 < p \leqslant q < \infty$, $s_1, s_2 \in (1,p)$ *and* $v(x_1,x_2) = v_1(x_1)v_2(x_2)$. *Then the inequality* (1.2) *holds for all measurable functions* f *if and only if* $A_W(s_1,s_2) < \infty$, *where*

$$\begin{aligned} A_W(s_1,s_2) &:= \sup_{t_1,t_2>0} V_1(t_1)^{\frac{s_1-1}{p}} V_2(t_2)^{\frac{s_2-1}{p}} \\ &\quad \times \left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^{\frac{q(p-s_1)}{p}} V_2(x_2)^{\frac{q(p-s_2)}{p}} dx_2dx_1\right)^{\frac{1}{q}} \end{aligned}$$

and

$$V_1(t_1) := \int_0^{t_1} v_1(x_1)^{1-p'}dx_1, \qquad V_2(t_2) := \int_0^{t_2} v_2(x_2)^{1-p'}dx_2.$$

Moreover, if C *is the best possible constant in* (1.2), *then*

$$\sup_{1<s_1,s_2<p} \left(\frac{\left(\frac{p}{p-s_1}\right)^p}{\left(\frac{p}{p-s_1}\right)^p + \frac{1}{s_1-1}}\right)^{\frac{1}{p}} \left(\frac{\left(\frac{p}{p-s_2}\right)^p}{\left(\frac{p}{p-s_2}\right)^p + \frac{1}{s_2-1}}\right)^{\frac{1}{p}} A_W(s_1,s_2)$$

$$\leqslant C \leqslant \inf_{1<s_1,s_2<p} A_W(s_1,s_2)\left(\frac{p-1}{p-s_1}\right)^{\frac{1}{p'}}\left(\frac{p-1}{p-s_2}\right)^{\frac{1}{p'}}.$$

Theorem 1.2 can be easily extended to n-dimensional case for any $n > 2$. Obviously, in such situation we will have n parameters $s_1,\ldots,s_n$ in the definition of the constant A_W. This fact was used in [8] (see also [9]) for a corresponding characterization of the weights v and w so that the following limit inequality of (1.2) holds:

$$\left(\int_{\mathbb{R}_+^n} (G_n f)^q(\mathbf{x})w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}_+^n} f^p(\mathbf{y})v(\mathbf{y})d\mathbf{y}\right)^{\frac{1}{p}} \tag{1.6}$$

with the multi-dimensional geometric mean operator G_n defined by

$$(G_nf)(\mathbf{x}) := \exp\left(\frac{1}{x_1 \dots x_n}\int_0^{x_1}\dots\int_0^{x_n}\log f(\mathbf{y})d\mathbf{y}\right), \quad x_1,\dots,x_n > 0. \tag{1.7}$$

THEOREM 1.3. [9, Theorem 3.1] *Let $n = 2$, $0 < p \leqslant q < \infty$ and $s_1, s_2 > 1$. The inequality (1.6) holds for all positive measurable functions f on $\mathbb{R}^2_+$ if and only if $D_W(s_1,s_2) < \infty$, where*

$$D_W(s_1,s_2) := \sup_{y_1,y_2>0} y_1^{\frac{s_1-1}{p}} y_2^{\frac{s_2-1}{p}}\left(\int_{y_1}^{\infty}\int_{y_2}^{\infty} x_1^{-\frac{s_1q}{p}} x_2^{-\frac{s_2q}{p}} u(x_1,x_2)dx_2dx_1\right)^{\frac{1}{q}}$$

and

$$u(x_1,x_2) := \left[\exp\left(\frac{1}{x_1x_2}\int_0^{x_1}\int_0^{x_2}\log\frac{1}{v(t_1,t_2)}dt_2dt_1\right)\right]^{\frac{q}{p}} w(x_1,x_2).$$

Moreover, the best possible constant C in (1.6) can be estimated in the following way:

$$\sup_{s_1,s_2>1}\left(\frac{e^{s_1}(s_1-1)}{e^{s_1}(s_1-1)+1}\right)^{\frac{1}{p}}\left(\frac{e^{s_2}(s_2-1)}{e^{s_2}(s_2-1)+1}\right)^{\frac{1}{p}} D_W(s_1,s_2)$$
$$\leqslant C \leqslant \inf_{s_1,s_2>1} e^{(s_1+s_2-2)/p} D_W(s_1,s_2).$$

Note especially that there is no restriction concerning product type of some of the weights in the conditions of Theorem 1.3 because of the special properties of the operator G_n (see, for instance, [5]). Also this result can be given in a natural n-dimensional setting.

In Section 2. of this paper we prove some statements (see Lemmas 2.1–2.3), which are needed later on but also partially generalize the necessary part of Theorem 1.1 to n dimensions. Moreover, it is proved that also the natural end point condition in Theorem 1.2 can be used for a characterization of the weights so that (1.2) holds (see Theorem 2.1). This condition is natural since it obviously corresponds to the usual Muckenhoupt-Bradley condition in dimension 1. Moreover, a corresponding weight characterization is done by using a generalization of the Persson-Stepanov condition (see Theorem 2.2). It is also proved that some similar results can be obtained if we instead assume that the weight on the left hand side is of product type (see Theorems 2.4 and 2.5).

In Section 3. of this paper we prove some analogous multi-dimensional Hardy type inequalities for the case $0 < q < p < \infty$ when we alternately assume that the weight on the right (or left) hand side in (1.2) is of product type (see Theorems 3.1–3.4). Finally, in Section 4. we prove some natural corresponding limit inequalities (involving the geometric mean operator (1.7)) of Theorems 2.2 and 3.2. However, for the case $0 < q < p < \infty$ we have only obtained a sufficient condition.

Throughout this work an expression of the form $0 \cdot \infty$ is taken to be equal to zero. The notation $A \ll B$ means that $A \leqslant cB$ with some constant $c > 0$ depending at most on the dimension n and the parameters of summation p and q. Moreover, $A \approx B$ means that $A \ll B \ll A$. The inverse function of a function h is denoted by h^{-1}. We also use the symbols $:=$ and $=:$ for introducing new quantities or notations and the symbol $\Box$ to note the end of a proof.

2. Multi-dimensional Hardy type inequalities – the case $1 < p \leqslant q < \infty$

In this and the next sections we deal with the inequality (1.2), where one of the two weight functions v and w is of product type, that is where

$$v(\mathbf{y}) = v(y_1, \dots, y_n) = v_1(y_1)\dots v_n(y_n) \tag{2.1}$$

or

$$w(\mathbf{x}) = w(x_1, \dots, x_n) = w_1(x_1)\dots w_n(x_n). \tag{2.2}$$

Conditions (2.1) and (2.2) are satisfied, for instance, by a power function of n variables.

In this Section we obtain new necessary and sufficient conditions for the validity of (1.2) in the case $1 < p \leqslant q < \infty$ and when (2.1) is satisfied. The same problem is considered here with the assumption (2.2). Our estimates are n-dimensional analogies of well known Muckenhoupt-Mazya-Rozin-type and Persson-Stepanov-type criteria for the one-dimensional integral Hardy inequality (see [2], [4] and [6]).

In the next preliminary Lemmas we state some necessary conditions for the inequality (1.2) to hold in the case $1 < p \leqslant q < \infty$ without of any restrictions on the weight functions w and v. These Lemmas are useful in our proofs later on but also of independent interest because they indicate the problem to extend Theorem 1.1 to n-dimensional case.

LEMMA 2.1. *Let $1 < p \leqslant q < \infty$ and assume that the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with a finite constant C, which is independent on f. Then*

$$\sup_{\substack{t_i>0\\ i=1,\dots,n}} W(t_1, \dots, t_n)^{\frac{1}{q}} V(t_1, \dots, t_n)^{\frac{1}{p'}} < \infty, \tag{2.3}$$

where

$$W(t_1, \dots, t_n) := W(\mathbf{t}) = \int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} w(\mathbf{x})d\mathbf{x}$$

and

$$V(t_1, \dots, t_n) := V(\mathbf{t}) = \int_0^{t_1} \dots \int_0^{t_n} v(\mathbf{y})^{1-p'}d\mathbf{y}.$$

Proof. For $\mathbf{t} = (t_1, \dots, t_n)$ such that $t_i > 0,\ i = 1, \dots, n,$ we take a test function

$$f_{\mathbf{t}}(\mathbf{y}) := \chi_{[0,t_1]}(y_1)\dots\chi_{[0,t_n]}(y_n)v(\mathbf{y})^{1-p'} \tag{2.4}$$

and put it into the inequality (1.2). Then we have that

$$C \geqslant \frac{\left(\int_{\mathbb{R}^n_+} \left(\int_0^{x_1}\dots\int_0^{x_n} f_{\mathbf{t}}(\mathbf{y})d\mathbf{y}\right)^q w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}}}{\left(\int_{\mathbb{R}^n_+} f_{\mathbf{t}}^p(\mathbf{y})v(\mathbf{y})d\mathbf{y}\right)^{\frac{1}{p}}}$$

$$\geqslant \frac{\left(\int_{t_1}^{\infty}\dots\int_{t_n}^{\infty} w(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \left(\int_0^{t_1}\dots\int_0^{t_n} v(\mathbf{y})^{1-p'}d\mathbf{y}\right)}{\left(\int_0^{t_1}\dots\int_0^{t_n} v(\mathbf{y})^{1-p'}d\mathbf{y}\right)^{\frac{1}{p}}} = W(\mathbf{t})^{\frac{1}{q}} V(\mathbf{t})^{\frac{1}{p'}}.$$

Thus, (2.3) follows by taking the supremum over all $t_i > 0,\ i = 1, \dots, n.$ □

LEMMA 2.2. *Let* $1 < p \leqslant q < \infty$ *and suppose that the inequality* (1.2) *holds for all measurable functions* f *on* $\mathbb{R}^n_+$ *with some finite constant* C *independent on* f. *Then*

$$\sup_{\substack{t_i>0\\ i=1,\dots,n}} V(t_1,\dots,t_n)^{-\frac{1}{p}} \left(\int_0^{t_1}\dots\int_0^{t_n} w(\mathbf{x})V(\mathbf{x})^q d\mathbf{x}\right)^{\frac{1}{q}} < \infty. \tag{2.5}$$

Proof. This statement follows evidently by substituting into the inequality (1.2) the function $f_{\mathbf{t}}(\mathbf{y})$ (see (2.4)) for $\mathbf{t} = (t_1,\dots,t_n)$ such that $t_i > 0$, $i = 1,\dots,n$. □

LEMMA 2.3. *Let* $1 < p \leqslant q < \infty$ *and assume that the inequality* (1.2) *holds for all measurable functions* f *on* $\mathbb{R}^n_+$ *with some finite constant* C *independent on* f. *Then*

$$\sup_{\substack{t_i>0\\ i=1,\dots,n}} W(t_1,\dots,t_n)^{-\frac{1}{q'}} \left(\int_{t_1}^{\infty}\dots\int_{t_n}^{\infty} v(\mathbf{x})^{1-p'}W(\mathbf{x})^{p'} d\mathbf{x}\right)^{\frac{1}{p'}} < \infty. \tag{2.6}$$

Proof. By duality the inequality (1.2) is equivalent to the inequality

$$\left(\int_{\mathbb{R}^n_+} (H_n^* g)^{p'}(\mathbf{x})v^{1-p'}(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p'}} \leqslant C\left(\int_{\mathbb{R}^n_+} g^{q'}(\mathbf{y})w^{1-q'}(\mathbf{y})d\mathbf{y}\right)^{\frac{1}{q'}} \tag{2.7}$$

with the dual operator H_n^* defined by

$$(H_n^* g)(\mathbf{x}) := \int_{x_1}^{\infty}\dots\int_{x_n}^{\infty} g(\mathbf{y})d\mathbf{y}, \qquad x_1,\dots,x_n > 0. \tag{2.8}$$

Now (2.6) follows by substituting into the inequality (2.7) the function

$$g_{\mathbf{t}}(\mathbf{y}) := \chi_{[t_1,\infty)}(y_1)\dots\chi_{[t_n,\infty)}(y_n)w(\mathbf{y})$$

for $\mathbf{t} = (t_1,\dots,t_n)$ such that $t_i > 0$, $i = 1,\dots,n$, and taking supremum. □

REMARK 2.1. Note that for $n = 2$ the statements of Lemmas 2.1–2.3 follow from Theorem 1.1.

The first main theorem in this Section reads:

THEOREM 2.1. *Let* $1 < p \leqslant q < \infty$ *and the weight function* v *be of product type* (2.1). *Then the inequality* (1.2) *holds for all measurable functions* f *on* $\mathbb{R}^n_+$ *with some finite constant* C, *which is independent on* f, *if and only if* $A_{M_n} < \infty$, *where*

$$A_{M_n} := \sup_{\substack{t_i>0\\ i=1,\dots,n}} W(t_1,\dots,t_n)^{\frac{1}{q}} V_1(t_1)^{\frac{1}{p'}}\dots V_n(t_n)^{\frac{1}{p'}} \tag{2.9}$$

and

$$V_i(t_i) := \int_0^{t_i} v_i(x_i)^{1-p'}dx_i, \qquad i = 1,\dots,n.$$

Moreover, $C \approx A_{M_n}$ *with constants of equivalence depending only on the parameters* p, q *and the dimension* n.

Proof. The necessary part of the proof follows from Lemma 2.1 while the sufficiency can be obtained from the n-dimensional extension of Theorem 1.2 and from the following Lemma 2.4. □

LEMMA 2.4. *Let*

$$A_{W_n} := A_{W_n}(s_1,\dots,s_n) := \sup_{\substack{t_i>0\\ i=1,\dots,n}} V_1(t_1)^{\frac{s_1-1}{p}}\dots V_n(t_n)^{\frac{s_n-1}{p}} \times \left(\int_{t_1}^{\infty}\dots\int_{t_n}^{\infty} w(\mathbf{x})V_1(x_1)^{\frac{q(p-s_1)}{p}}\dots V_n(x_n)^{\frac{q(p-s_n)}{p}}d\mathbf{x}\right)^{\frac{1}{q}},$$

where $s_i\in(1,p)$, $i=1,\dots,n$. *Then*

$$A_{W_n} \ll A_{M_n}. \tag{2.10}$$

Proof. Let $n=2$ and $s_1=s_2=\frac{1+p}{2}$. Then

$$A_{W_2} = \sup_{t_1,t_2>0} V_1(t_1)^{\frac{1}{2p'}}V_2(t_2)^{\frac{1}{2p'}}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^{\frac{q}{2p'}}V_2(x_2)^{\frac{q}{2p'}}dx_2dx_1\right)^{\frac{1}{q}}.$$

Since

$$V_i(x_i)^{\frac{q}{2p'}} = \frac{q}{2p'}\int_0^{x_i} v_i(y_i)^{1-p'}V_i(y_i)^{\frac{q}{2p'}-1}dy_i, \qquad i=1,2,$$

we have that

$$\begin{aligned}
V_1(x_1)^{\frac{q}{2p'}}\quad V_2(x_2)^{\frac{q}{2p'}} &\approx \left(\left[\int_0^{t_1}+\int_{t_1}^{x_1}\right] v_1(y_1)^{1-p'}V_1(y_1)^{\frac{q}{2p'}-1}dy_1\right)\\
&\quad\times\left(\left[\int_0^{t_2}+\int_{t_2}^{x_2}\right] v_2(y_2)^{1-p'}V_2(y_2)^{\frac{q}{2p'}-1}dy_2\right)\\
&= \int_0^{t_1} v_1(y_1)^{1-p'}V_1(y_1)^{\frac{q}{2p'}-1}dy_1\int_0^{t_2} v_2(y_2)^{1-p'}V_2(y_2)^{\frac{q}{2p'}-1}dy_2\\
&\quad+\int_{t_1}^{x_1} v_1(y_1)^{1-p'}V_1(y_1)^{\frac{q}{2p'}-1}dy_1\int_{t_2}^{x_2} v_2(y_2)^{1-p'}V_2(y_2)^{\frac{q}{2p'}-1}dy_2\\
&\quad+\int_0^{t_1} v_1(y_1)^{1-p'}V_1(y_1)^{\frac{q}{2p'}-1}dy_1\int_{t_2}^{x_2} v_2(y_2)^{1-p'}V_2(y_2)^{\frac{q}{2p'}-1}dy_2\\
&\quad+\int_{t_1}^{x_1} v_1(y_1)^{1-p'}V_1(y_1)^{\frac{q}{2p'}-1}dy_1\int_0^{t_2} v_2(y_2)^{1-p'}V_2(y_2)^{\frac{q}{2p'}-1}dy_2\\
&=: I_{11}+I_{22}+I_{12}+I_{21}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^{\frac{q}{2p'}}V_2(x_2)^{\frac{q}{2p'}}dx_2dx_1\\
&\qquad\approx \int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)\left[I_{11}+I_{22}+I_{12}+I_{21}\right]dx_2dx_1\\
&\qquad=: J_{11}+J_{22}+J_{12}+J_{21}.
\end{aligned}$$

Clearly that

$$V_1(t_1)^{\frac{1}{2p'}} V_2(t_2)^{\frac{1}{2p'}} [J_{11}]^{\frac{1}{q}} \ll A_{M_2}.$$

Further

$$\begin{aligned} J_{22} &= \int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)\left(\int_{t_1}^{x_1} v_1(y_1)^{1-p'} V_1(y_1)^{\frac{q}{2p'}-1} dy_1\right) \\ &\quad \times \left(\int_{t_2}^{x_2} v_2(y_2)^{1-p'} V_2(y_2)^{\frac{q}{2p'}-1} dy_2\right) dx_2 dx_1 \\ &= \int_{t_1}^{\infty}\int_{t_2}^{\infty} v_1(y_1)^{1-p'} V_1(y_1)^{\frac{q}{2p'}-1} v_2(y_2)^{1-p'} V_2(y_2)^{\frac{q}{2p'}-1} \\ &\quad \times \left(\int_{y_1}^{\infty}\int_{y_2}^{\infty} w(x_1,x_2) dx_2 dx_1\right) dy_2 dy_1 \\ &\leqslant A_{M_2}^q \int_{t_1}^{\infty} V_1(y_1)^{-\frac{q}{2p'}-1} v_1(y_1)^{1-p'} dy_1 \int_{t_2}^{\infty} V_2(y_2)^{-\frac{q}{2p'}-1} v_2(y_2)^{1-p'} dy_2 \\ &\ll A_{M_2}^q V_1(t_1)^{-\frac{q}{2p'}} V_2(t_2)^{-\frac{q}{2p'}}. \end{aligned}$$

Hence,

$$V_1(t_1)^{\frac{1}{2p'}} V_2(t_2)^{\frac{1}{2p'}} [J_{11}]^{\frac{1}{q}} \ll A_{M_2}.$$

The terms with J_{12} and J_{21} are estimated analogously. The method works for any $n > 2$ by induction. □

REMARK 2.2. The condition $A_{M_n} < \infty$ may be regarded as a natural end point of the conditions given in Theorem 1.2 and also as a natural generalization of the usual Muckenhoupt-Bradley condition in one dimension.

The alternative criterion for the Hardy inequality (1.2) to hold with product type weight v satisfying (2.1) in the case $1 < p \leqslant q < \infty$ is stated by the following

THEOREM 2.2. *Let $1 < p \leqslant q < \infty$ and the weight function v be of product type* (2.1). *Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}_+^n$ with some finite constant C, which is independent on f, if and only if $A_{PS_n} < \infty$, where*

$$A_{PS_n} := \sup_{\substack{t_i>0 \\ i=1,\dots,n}} V_1(t_1)^{-\frac{1}{p}} \dots V_n(t_n)^{-\frac{1}{p}} \times \left(\int_0^{t_1}\dots\int_0^{t_n} w(\mathbf{x}) V_1(x_1)^q \dots V_n(x_n)^q d\mathbf{x}\right)^{\frac{1}{q}}. \tag{2.11}$$

Moreover, $C \approx A_{PS_n}$ with constants of equivalence depending only on the parameters p, q and n.

Proof. The necessary part follows from Lemma 2.2. The proof of the sufficiency can be obtained from Theorem 2.1 and the following Lemma 2.5. □

LEMMA 2.5. *We have*

$$A_{M_n} \ll A_{PS_n}. \tag{2.12}$$

Proof. Let $n=2$. Suppose first that $V_1(\infty)=V_2(\infty)=\infty$. Then

$$\begin{aligned}
\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1 &= \int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^qV_1(x_1)^{-q}dx_1dx_2\\
&= q\int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^q\left(\int_{x_1}^{\infty} V_1(y_1)^{-q-1}dV_1(y_1)\right)dx_1dx_2\\
&= q\int_{t_2}^{\infty}\int_{t_1}^{\infty} V_1(y_1)^{-q-1}\left(\int_{t_1}^{y_1} w(x_1,x_2)V_1(x_1)^qdx_1\right)dV_1(y_1)dx_2\\
&\leqslant q\int_{t_1}^{\infty} V_1(y_1)^{-q-1}\int_{t_2}^{\infty} V_2(x_2)^qV_2(x_2)^{-q}\left(\int_0^{y_1} w(x_1,x_2)V_1(x_1)^qdx_1\right)dx_2dV_1(y_1)\\
&\leqslant q^2\int_{t_1}^{\infty}\int_{t_2}^{\infty} V_1(y_1)^{-q-1}V_2(y_2)^{-q-1}\\
&\quad\times\left(\int_0^{y_2}\int_0^{y_1} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^qdx_1dx_2\right)dV_2(y_2)dV_1(y_1)\\
&\leqslant q^2A_{PS_2}^q\int_{t_1}^{\infty}\int_{t_2}^{\infty} V_1(y_1)^{-\frac{q}{p'}-1}V_2(y_2)^{-\frac{q}{p'}-1}dV_2(y_2)dV_1(y_1)\\
&= (p')^2A_{PS_2}^qV_1(t_1)^{-\frac{q}{p'}}V_2(t_2)^{-\frac{q}{p'}}.
\end{aligned}$$

Thus, we get that

$$V_1(t_1)^{\frac{1}{p'}}V_2(t_2)^{\frac{1}{p'}}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1\right)^{\frac{1}{q}} \ll A_{PS_2}.$$

Further, suppose that $V_i(\infty)<\infty$ for all $i=1,2$. Note that

$$V_i(x_i)^{-q} = V_i(\infty)^{-q} + q\int_{x_i}^{\infty} V_i(y_i)^{-q-1}dV_i(y_i), \qquad i=1,2. \tag{2.13}$$

Therefore,

$$\begin{aligned}
&\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1\\
&= \int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^qV_1(x_1)^{-q}V_2(x_2)^{-q}dx_1dx_2\\
&= V_1(\infty)^{-q}V_2(\infty)^{-q}\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^qdx_2dx_1\\
&\quad + qV_1(\infty)^{-q}\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q\left(\int_{x_2}^{\infty} V_2(y_2)^{-q-1}dV_2(y_2)\right)dx_2dx_1\\
&\quad + qV_2(\infty)^{-q}\int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q\left(\int_{x_1}^{\infty} V_1(y_1)^{-q-1}dV_1(y_1)\right)dx_1dx_2
\end{aligned}$$

$$
\begin{aligned}
&+q^2\int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q\\
&\times\left(\int_{x_1}^{\infty}\int_{x_2}^{\infty} V_1(y_1)^{-q-1}V_2(y_2)^{-q-1}dV_2(y_2)dV_1(y_1)\right)dx_1dx_2\\
=&: J_{11}+J_{12}+J_{21}+J_{22}.
\end{aligned}
$$

Obviously that

$$
\begin{aligned}
J_{11} &\leqslant V_1(\infty)^{-q}V_2(\infty)^{-q}\int_0^{\infty}\int_0^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q dx_2dx_1\\
&\leqslant A_{PS_2}^q V_1(\infty)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}.
\end{aligned}
$$

By changing the order of integration we have that

$$
\begin{aligned}
J_{12} &\leqslant qV_1(\infty)^{-q}\int_0^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q\\
&\quad\times\left(\int_{x_2}^{\infty}V_2(y_2)^{-q-1}dV_2(y_2)\right)dx_2dx_1\\
&\leqslant qV_1(\infty)^{-q}\int_0^{\infty}\int_{t_2}^{\infty}V_2(y_2)^{-q-1}\\
&\quad\times\left(\int_0^{y_2} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q dx_2\right)dV_2(y_2)dx_1\\
&= qV_1(\infty)^{-q}\int_{t_2}^{\infty}V_2(y_2)^{-q-1}\\
&\quad\times\left(\int_0^{y_2}\int_0^{\infty} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q dx_2dx_1\right)dV_2(y_2)\\
&\leqslant qA_{PS_2}^q V_1(\infty)^{-\frac{q}{p'}}\int_{t_2}^{\infty}V_2(y_2)^{-\frac{q}{p'}-1}dV_2(y_2)\\
&= p'A_{PS_2}^q\left[V_1(\infty)^{-\frac{q}{p'}}V_2(t_2)^{-\frac{q}{p'}}-V_1(\infty)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}\right].
\end{aligned}
$$

Analogously,

$$
J_{21}\leqslant p'A_{PS_2}^q\left[V_1(t_1)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}-V_1(\infty)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}\right].
$$

By changing the order of integration we have for J_{22} that

$$
\begin{aligned}
J_{22} &\leqslant q^2\int_{t_1}^{\infty}\int_{t_2}^{\infty}\left(\int_0^{y_1}\int_0^{y_2} w(x_1,x_2)V_1(x_1)^qV_2(x_2)^q dx_2dx_1\right)\\
&\quad\times V_1(y_1)^{-q-1}V_2(y_2)^{-q-1}dV_2(y_2)dV_1(y_1)\\
&\leqslant (p')^2A_{PS_2}^q\left[V_1(t_1)^{-\frac{q}{p'}}V_2(t_2)^{-\frac{q}{p'}}-V_1(\infty)^{-\frac{q}{p'}}V_2(t_2)^{-\frac{q}{p'}}\right.\\
&\quad\left.-V_1(t_1)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}+V_1(\infty)^{-\frac{q}{p'}}V_2(\infty)^{-\frac{q}{p'}}\right].
\end{aligned}
$$

Therefore, it follows that

$$V_1(t_1)^{\frac{1}{p'}}V_2(t_2)^{\frac{1}{p'}}\left[J_{11}+J_{12}+J_{21}+J_{22}\right]^{\frac{1}{q}} \leqslant (p')^{\frac{2}{q}}A_{PS_2}.$$

Consider now one of the mixed cases when $V_1(\infty)=\infty$ and $V_2(\infty)<\infty$. Write

$$\begin{aligned}\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1 &= \int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^qV_1(x_1)^{-q}dx_1dx_2\\ &= q\int_{t_2}^{\infty}\int_{t_1}^{\infty} w(x_1,x_2)V_1(x_1)^q\left(\int_{x_1}^{\infty}V_1(y_1)^{-q-1}dV_1(y_1)\right)dx_1dx_2\\ &\leqslant q\int_{t_2}^{\infty}\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\left(\int_0^{y_1}w(x_1,x_2)V_1(x_1)^qdx_1\right)dV_1(y_1)dx_2.\end{aligned}$$

Further, by using (2.13) with $i=2$ we get that

$$\begin{aligned}&\int_{t_1}^{\infty}\int_{t_2}^{\infty} w(x_1,x_2)dx_2dx_1\\ &\leqslant q\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\int_{t_2}^{\infty}\left(\int_0^{y_1}w(x_1,x_2)V_1(x_1)^qdx_1\right)V_2(x_2)^qV_2(x_2)^{-q}dx_2dV_1(y_1)\\ &= qV_2(\infty)^{-q}\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\int_{t_2}^{\infty}\left(\int_0^{y_1}w(x_1,x_2)V_1(x_1)^qdx_1\right)V_2(x_2)^qdx_2dV_1(y_1)\\ &\quad+q^2\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\int_{t_2}^{\infty}\left(\int_0^{y_1}w(x_1,x_2)V_1(x_1)^qdx_1\right)V_2(x_2)^q\\ &\quad\times\left(\int_{x_2}^{\infty}V_2(y_2)^{-q-1}dV_2(y_2)\right)dx_2dV_1(y_1)\\ &\leqslant qV_2(\infty)^{-q}\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\left(\int_0^{\infty}\int_0^{y_1}w(x_1,x_2)V_1(x_1)^qV_2(x_2)^qdx_1dx_2\right)dV_1(y_1)\\ &\quad+q^2\int_{t_1}^{\infty}V_1(y_1)^{-q-1}\int_{t_2}^{\infty}V_2(y_2)^{-q-1}\\ &\quad\times\left(\int_0^{y_1}\int_0^{y_2}w(x_1,x_2)V_1(x_1)^qV_2(x_2)^qdx_2dx_1\right)dV_2(y_2)dV_1(y_1)\\ &\leqslant (p')^2A_{PS_2}^qV_1(t_1)^{-\frac{q}{p'}}V_2(t_2)^{-\frac{q}{p'}}.\end{aligned}$$

Hence, it yields that

$$V_1(t_1)^{\frac{1}{p'}}V_2(t_2)^{\frac{1}{p'}}\left(\int_{t_1}^{\infty}\int_{t_2}^{\infty}w(x_1,x_2)dx_2dx_1\right)^{\frac{1}{q}} \leqslant (p')^{\frac{2}{q}}A_{PS_2}.$$

The case $V_1(\infty)<\infty$, $V_2(\infty)=\infty$ can be proved analogously. The proof for $n=2$ is complete. For any $n>2$ the statement of Lemma follows by induction. □

Further we discuss the inequality (1.2) with the left hand side weight function of product type. In particular, in Theorems 2.4 and 2.5 we state a Muckenhoupt-type and Persson-Stepanov-type criteria for the inequality (1.2) to hold in the case

$1 < p \leqslant q < \infty$ with the left hand side weight w to be of product type (2.2). The proofs of these results are analogous to the proofs of Theorems 2.1, 2.2 and based on some statements formulated below. The first of them is dual to n-dimensional extension of Theorem 1.2 and reads:

THEOREM 2.3. *Let* $1 < q' \leqslant p' < \infty$, $s_i \in (1, q')$, $i = 1, \dots, n$, *and the weight function* w *be of product type* (2.2) *Then the inequality* (2.7) *holds for all measurable functions* g *if and only if* $A^*_{W_n} < \infty$, *where*

$$A^*_{W_n} := A^*_{W_n}(s_1, \dots, s_n) := \sup_{\substack{t_i>0 \\ i=1,\dots,n}} W_1(t_1)^{\frac{s_1-1}{q'}} \dots W_n(t_n)^{\frac{s_n-1}{q'}}$$
$$\times \left(\int_0^{t_1} \dots \int_0^{t_n} v(\mathbf{x})^{1-p'} W_1(x_1)^{\frac{p'(q'-s_1)}{q'}} \dots W_n(x_n)^{\frac{p'(q'-s_n)}{q'}} d\mathbf{x} \right)^{\frac{1}{p'}}$$

and

$$W(t_i) := \int_{t_i}^{\infty} w_i(x_i) dx_i, \qquad i = 1, \dots, n.$$

Moreover, $C \approx A^*_{W_n}$ *with constants of equivalence depending only on the parameters* p, q *and* n.

The following two auxiliary statements are similar to Lemmas 2.4 and 2.5, respectively.

LEMMA 2.6. *Let*

$$A^*_{M_n} := \sup_{\substack{t_i>0 \\ i=1,\dots,n}} W_1(t_1)^{\frac{1}{q}} \dots W_n(t_n)^{\frac{1}{q}} V(t_1, \dots, t_n)^{\frac{1}{p'}}.$$

Then

$$A^*_{W_n} \ll A^*_{M_n}. \tag{2.14}$$

LEMMA 2.7. *Let*

$$A^*_{PS_n} := \sup_{\substack{t_i>0 \\ i=1,\dots,n}} W_1(t_1)^{-\frac{1}{q'}} \dots W_n(t_n)^{-\frac{1}{q'}}$$
$$\times \left(\int_{t_1}^{\infty} \dots \int_{t_n}^{\infty} v(\mathbf{x})^{1-p'} W_1(x_1)^{p'} \dots W_n(x_n)^{p'} d\mathbf{x} \right)^{\frac{1}{p'}}.$$

Then

$$A^*_{M_n} \ll A^*_{PS_n}. \tag{2.15}$$

Now by passing to the dual inequality (2.7) of (1.2) we can get a Muckenhoupt-type and Persson-Stepanov-type criteria for (1.2) with the left hand side weight w of product type (2.2). The necessity in the proofs of these results follow from Lemmas 2.1 and 2.3, while the sufficient parts can be proved in the similar ways as in Theorems 2.1 and 2.2 but by using Theorem 2.3, Lemmas 2.6 and 2.7 instead of Theorem 1.2, Lemmas 2.4 and 2.5, respectively.

THEOREM 2.4. *Let $1 < p \leqslant q < \infty$ and the weight function w be of product type* (2.2)*. Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $A^*_{M_n} < \infty$. Moreover, $C \approx A^*_{M_n}$ with constants of equivalence depending only on the parameters p, q and n.*

THEOREM 2.5. *Let $1 < p \leqslant q < \infty$ and the weight function w be of product type. Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $A^*_{PS_n} < \infty$. Moreover, $C \approx A^*_{PS_n}$ with constants of equivalence depending only on the parameters p, q and n.*

3. Multi-dimensional Hardy type inequalities – the case $1 < q < p < \infty$

In this Section we will prove the similar results as in the previous Section but in the case $1 < q < p < \infty$. Let us introduce the following n-dimensional versions of the Mazya-Rozin and Persson-Stepanov conditions in this case:

$$B_{MR_n} := \left(\int_{\mathbb{R}^n_+} W(\mathbf{t})^{\frac{r}{q}} V_1(t_1)^{\frac{r}{q'}} \dots V_n(t_n)^{\frac{r}{q'}} dV_1(t_1) \dots dV_n(t_n) \right)^{\frac{1}{r}},$$

$$\begin{aligned} B_{PS_n} := & \left(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \dots \int_0^{t_n} w(\mathbf{x}) V_1(x_1)^q \dots V_n(x_n)^q d\mathbf{x} \right)^{\frac{r}{q}} \right. \\ & \left. \times V_1(t_1)^{-\frac{r}{q}} \dots V_n(t_n)^{-\frac{r}{q}} dV_1(t_1) \dots dV_n(t_n) \right)^{\frac{1}{r}}. \end{aligned}$$

The following comparison between these constants is useful later on but also of independent interest.

LEMMA 3.1. *We have*

$$B_{PS_n} \ll B_{MR_n}. \tag{3.1}$$

Proof. It yields that

$$\begin{aligned} & \int_0^{t_1} \dots \int_0^{t_n} w(\mathbf{x}) V_1(x_1)^q \dots V_n(x_n)^q d\mathbf{x} \\ & \quad = q^n \int_0^{t_1} \dots \int_0^{t_n} w(\mathbf{x}) \\ & \quad \times \left(\int_0^{x_1} \dots \int_0^{x_n} V_1(y_1)^{q-1} \dots V_n(y_n)^{q-1} dV_n(y_n) \dots dV_1(y_1) \right) d\mathbf{x} \\ & \quad \leqslant q^n \int_0^{t_1} \dots \int_0^{t_n} W(\mathbf{y}) V_1(y_1)^{q-1} \dots V_n(y_n)^{q-1} dV_n(y_n) \dots dV_1(y_1) \end{aligned}$$

[applying Hölder's inequality with the exponents r/q and p/q]

$$= q^n \int_0^{t_1} \cdots \int_0^{t_n} \left\{ W(\mathbf{y}) V_1(y_1)^{(q-1)+\frac{q}{2p}} \dots V_n(y_n)^{(q-1)+\frac{q}{2p}} \right\}$$
$$\times V_1(y_1)^{-\frac{q}{2p}} \dots V_n(y_n)^{-\frac{q}{2p}} dV_n(y_n) \dots dV_1(y_1)$$
$$\leqslant q^n \left(\int_0^{t_1} \cdots \int_0^{t_n} W(\mathbf{y})^{\frac{r}{q}} V_1(y_1)^{(q-1+\frac{q}{2p})\frac{r}{q}} \dots V_n(y_n)^{(q-1+\frac{q}{2p})\frac{r}{q}} dV_n(y_n) \dots \right.$$
$$\left. \times V_1(y_1)\right)^{\frac{q}{r}} \left(\int_0^{t_1} \cdots \int_0^{t_n} V_1(y_1)^{-\frac{1}{2}} \dots V_n(y_n)^{-\frac{1}{2}} dV_n(y_n) \dots V_1(y_1) \right)^{\frac{q}{p}}$$
$$= q^n 2^{\frac{qn}{p}} \left(\int_0^{t_1} \cdots \int_0^{t_n} W(\mathbf{y})^{\frac{r}{q}} V_1(y_1)^{\frac{r}{q'}+\frac{r}{2p}} \dots \right.$$
$$\left. \times V_n(y_n)^{\frac{r}{q'}+\frac{r}{2p}} dV_n(y_n) \dots dV_1(y_1) \right)^{\frac{q}{r}} V_1(t_1)^{\frac{q}{2p}} \dots V_n(t_n)^{\frac{q}{2p}}.$$

Hence, we obtain that

$$B_{PS_n}^r \leqslant q^{\frac{rn}{q}} 2^{\frac{rn}{p}} \int_{\mathbb{R}_+^n} \left(\int_0^{t_1} \cdots \int_0^{t_n} W(\mathbf{y})^{\frac{r}{q}} V_1(y_1)^{\frac{r}{q'}+\frac{r}{2p}} \dots \right.$$
$$\left. \times V_n(y_n)^{\frac{r}{q'}+\frac{r}{2p}} dV_n(y_n) \dots dV_1(y_1) \right)$$
$$\times V_1(t_1)^{\frac{r}{2p}-\frac{r}{q}} \dots V_n(t_n)^{\frac{r}{2p}-\frac{r}{q}} dV_1(t_1) \dots dV_n(t_n).$$

Therefore, by changing the order of integration, we get that

$$B_{PS_n}^r \leqslant q^{\frac{rn}{q}} 2^{\frac{rn}{p}} \int_{\mathbb{R}_+^n} W(\mathbf{y})^{\frac{r}{q}} V_1(y_1)^{\frac{r}{q'}+\frac{r}{2p}} \dots V_n(y_n)^{\frac{r}{q'}+\frac{r}{2p}}$$
$$\times \left(\int_{y_1}^{\infty} \cdots \int_{y_n}^{\infty} V_1(t_1)^{\frac{r}{2p}-\frac{r}{q}} \dots V_n(t_n)^{\frac{r}{2p}-\frac{r}{q}} dV_n(t_n) \dots dV_1(t_1) \right)$$
$$\times dV_1(y_1) \dots dV_n(y_n) \leqslant q^{\frac{rn}{q}} 2^{\frac{rn}{p}} \left(\frac{2p}{r} \right)^n B_{MR_n}^r$$

and the required estimate (3.1) is proved. □

Next we will state a similar comparison between the following dual versions of the constants B_{MR_n} and B_{PS_n} :

$$B_{MR_n}^* := \left(\int_{\mathbb{R}_+^n} V(\mathbf{t})^{\frac{r}{p'}} W_1(t_1)^{\frac{r}{p}} \dots W_n(t_n)^{\frac{r}{p}} d\left[-W_1(t_1)\right] \dots d\left[-W_n(t_n)\right] \right)^{\frac{1}{r}},$$
$$B_{PS_n}^* := \left(\int_{\mathbb{R}_+^n} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} v(\mathbf{x})^{1-p'} W_1(x_1)^{p'} \dots W_n(x_n)^{p'} d\mathbf{x} \right)^{\frac{r}{p'}} \right.$$
$$\left. \times W_1(t_1)^{-\frac{r}{p'}} \dots W_n(t_n)^{-\frac{r}{p'}} d\left[-W_1(t_1)\right] \dots d\left[-W_n(t_n)\right] \right)^{\frac{1}{r}}.$$

LEMMA 3.2. *It yields that*

$$B^*_{PS_n} \ll B^*_{MR_n}. \tag{3.2}$$

Proof. The proof is similar to that of Lemma 3.1 so we omit the details. □

The following Theorems state necessary and sufficient conditions for the validity of (1.2) in the case $1 < q < p < \infty$ with weights satisfying the following additional conditions:

$$V_1(\infty) = \ldots = V_n(\infty) = \infty \tag{3.3}$$

or

$$W_1(0) = \ldots = W_n(0) = \infty \tag{3.4}$$

THEOREM 3.1. *Let $1 < q < p < \infty$ and $1/r = 1/q - 1/p$. Suppose that the weight function v satisfies the conditions (2.1) and (3.3). Then the inequality (1.2) holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $B_{MR_n} < \infty$. Moreover, $C \approx B_{MR_n}$ with constants of equivalence depending only on the parameters p, q and the dimension n.*

Proof. Necessity. Suppose that the inequality (1.2) holds with $C < \infty$ and put

$$f(\mathbf{y}) = W(\mathbf{y})^{\frac{r}{pq}} V_1(y_1)^{\frac{r}{pq'}} v_1(y_1)^{1-p'} \ldots V_n(y_n)^{\frac{r}{pq'}} v_n(y_n)^{1-p'}.$$

It is easy to see that $\left(\int_{\mathbb{R}^n_+} f^p(\mathbf{x})v(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} = B^{\frac{r}{p}}_{MR_n}$. On the left hand side we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n_+} (H_n f)^q(\mathbf{x}) w(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n_+} \left(\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{t}) d\mathbf{t}\right) \left(\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{y}) d\mathbf{y}\right)^{q-1} w(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n_+} f(\mathbf{t}) \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} \left(\int_0^{x_1} \cdots \int_0^{x_n} f(\mathbf{y}) d\mathbf{y}\right)^{q-1} w(\mathbf{x}) d\mathbf{x}\right) d\mathbf{t}\right)^{\frac{1}{q}} \\
&= \left(\int_{\mathbb{R}^n_+} W(\mathbf{t})^{\frac{r}{pq}} V_1(t_1)^{\frac{r}{pq'}} \ldots V_n(t_n)^{\frac{r}{pq'}} \left(\int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} \left(\int_0^{x_1} \cdots \int_0^{x_n} W(\mathbf{y})^{\frac{r}{pq}}\right.\right.\right. \\
&\quad \times \left.\left.\left. V_1(y_1)^{\frac{r}{pq'}} \ldots V_n(y_n)^{\frac{r}{pq'}} dV_n(y_n) \ldots dV_1(y_1)\right)^{q-1} w(\mathbf{x}) d\mathbf{x}\right) dV_1(t_1) \ldots dV_n(t_n)\right)^{\frac{1}{q}} \\
&\geqslant \left(\int_{\mathbb{R}^n_+} W(\mathbf{t})^{\frac{r}{pq}+1} V_1(t_1)^{\frac{r}{pq'}} \ldots V_n(t_n)^{\frac{r}{pq'}} \left(\int_0^{t_1} \cdots \int_0^{t_n} W(\mathbf{y})^{\frac{r}{pq}}\right.\right. \\
&\quad \times \left.\left. V_1(y_1)^{\frac{r}{pq'}} \ldots V_n(y_n)^{\frac{r}{pq'}} dV_n(y_n) \ldots dV_1(y_1)\right)^{q-1} dV_1(t_1) \ldots dV_n(t_n)\right)^{\frac{1}{q}}
\end{aligned}$$

[since the function W is non-increasing and $r/pq' + 1 = r/p'q$]

$$\geqslant \left(\int_{\mathbb{R}_+^n}\left(\int_0^{t_1}\dots\int_0^{t_n} V_1(y_1)^{\frac{r}{pq'}}\dots V_n(y_n)^{\frac{r}{pq'}}dV_n(y_n)\dots dV_1(y_1)\right)^{q-1}\right.$$
$$\left.\times\, W(\mathbf{t})^{\frac{r}{q}}V_1(t_1)^{\frac{r}{pq'}}\dots V_n(t_n)^{\frac{r}{pq'}}dV_1(t_1)\dots dV_n(t_n)\right)^{\frac{1}{q}} = \left(\frac{p'q}{r}\right)^{\frac{n}{q'}} B_{MR_n}^{\frac{r}{q}}$$

and the estimate $B_{MR_n}^{\frac{r}{q}} \ll CB_{MR_n}^{\frac{r}{p}}$ follows. Therefore, $B_{MR_n} \ll C < \infty$.

Sufficiency. Suppose that $B_{MR_n} < \infty$. On the strength of (3.3) we find that

$$\int_{\mathbb{R}_+^n}(H_nf)^q(\mathbf{x})w(\mathbf{x})d\mathbf{x}$$
$$= \int_{\mathbb{R}_+^n}(H_nf)^q(\mathbf{x})V_1(x_1)^qV_1(x_1)^{-q}\dots V_n(x_n)^qV_n(x_n)^{-q}w(\mathbf{x})d\mathbf{x}$$
$$= q^n\int_{\mathbb{R}_+^n}(H_nf)^q(\mathbf{x})V_1(x_1)^q\dots V_n(x_n)^q$$
$$\times\left(\int_{x_1}^{\infty}\dots\int_{x_n}^{\infty}V_1(y_1)^{-q-1}\dots V_n(y_n)^{-q-1}dV_n(y_n)\dots dV_1(y_1)\right)w(\mathbf{x})d\mathbf{x}$$
$$\leqslant q^n\int_{\mathbb{R}_+^n}(H_nf)^q(\mathbf{y})V_1(y_1)^{-q-1}\dots V_n(y_n)^{-q-1}$$
$$\times\left(\int_0^{y_1}\dots\int_0^{y_n}w(\mathbf{x})V_1(x_1)^q\dots V_n(x_n)^qd\mathbf{x}\right)dV_n(y_n)\dots dV_1(y_1)$$
$$= q^n\int_{\mathbb{R}_+^n}\left\{(H_nf)^q(\mathbf{y})V_1(y_1)^{-q}\dots V_n(y_n)^{-q}\right\}\left\{V_1(y_1)^{-1}\dots V_n(y_n)^{-1}\right.$$
$$\left.\times\left(\int_0^{y_1}\dots\int_0^{y_n}w(\mathbf{x})V_1(x_1)^q\dots V_n(x_n)^qd\mathbf{x}\right)\right\}dV_n(y_n)\dots dV_1(y_1)$$

[by using Hölder's inequality with exponents p/q and r/q]

$$\ll B_{PS_n}^q\left(\int_{\mathbb{R}_+^n}(H_nf)^p(\mathbf{y})V_1(y_1)^{-p}\dots V_n(y_n)^{-p}dV_n(y_n)\dots dV_1(y_1)\right)^{\frac{q}{p}}.$$

Moreover, according to Theorem 2.2,

$$\left(\int_{\mathbb{R}_+^n}(H_nf)^p(\mathbf{y})V_1(y_1)^{-p}\dots V_n(y_n)^{-p}dV_n(y_n)\dots dV_1(y_1)\right)^{\frac{q}{p}}$$
$$\ll\left(\int_{\mathbb{R}_+^n}f^p(\mathbf{x})v_1(x_1)\dots v_n(x_n)d\mathbf{x}\right)^{\frac{q}{p}}.$$

By combining these inequalities we have that

$$\int_{\mathbb{R}_+^n}(H_nf)^q(\mathbf{x})w(\mathbf{x})d\mathbf{x}\ll B_{PS_n}^q\left(\int_{\mathbb{R}_+^n}f^p(\mathbf{x})v_1(x_1)\dots v_n(x_n)d\mathbf{x}\right)^{\frac{q}{p}}. \tag{3.5}$$

Therefore, in view of Lemma 3.1, the inequality (1.2) holds and the proof is complete. □

The corresponding result with the constant B_{PS_n} involved reads:

THEOREM 3.2. *Let $1 < q < p < \infty$ and $1/r = 1/q - 1/p$. Suppose that the weight function v satisfies the conditions* (2.1) *and* (3.3). *Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $B_{PS_n} < \infty$. Moreover, $C \approx B_{PS_n}$ with constants of equivalence depending only on the parameters p, q and the dimension n.*

Proof. The necessity follows from Lemma 3.1 and Theorem 3.1. The sufficiency is proved by (3.5). □

REMARK 3.1. Note that the sufficient parts of Theorems 3.1 and 3.2 in fact hold for all $0 < q < p < \infty$. Moreover, the necessary parts of these Theorems are correct even without assuming that the condition (3.3) is satisfied.

By passing to the dual inequality (2.7) of (1.2) we can in a similar way as above (but now using Lemma 3.2 instead of Lemma 3.1) get the following results for the case $1 < q < p < \infty$ with the left hand side weight w of product type (2.2).

THEOREM 3.3. *Let $1 < q < p < \infty$ and $1/r = 1/q - 1/p$. Assume that the weight function w satisfies the conditions* (2.2) *and* (3.4). *Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $B^*_{MR_n} < \infty$. Moreover, $C \approx B^*_{MR_n}$ with constants of equivalence depending only on the parameters p, q and the dimension n.*

THEOREM 3.4. *Let $1 < q < p < \infty$ and $1/r = 1/q - 1/p$. Suppose that the weight function w satisfies the conditions* (2.2) *and* (3.4). *Then the inequality* (1.2) *holds for all measurable functions f on $\mathbb{R}^n_+$ with some finite constant C, which is independent on f, if and only if $B^*_{PS_n} < \infty$. Moreover, $C \approx B^*_{PS_n}$ with constants of equivalence depending only on the parameters p, q and the dimension n.*

4. Multi-dimensional limit Pólya–Knopp type inequalities

In this Section we will apply the results of Theorems 2.2 and 3.2 for the corresponding investigation of the inequality (1.6) with the geometric mean operator given by (1.7). Namely, we will characterize the inequality (1.6) in the case $0 < p \leqslant q < \infty$ and give a sufficient condition for (1.6) to hold in the case $0 < q < p < \infty$.

According to Jensen's inequality it holds for any $\mathbf{x} \in \mathbb{R}^n_+$ that

$$(G_n f)(\mathbf{x}) \leqslant \frac{1}{x_1 \ldots x_n}(H_n f)(\mathbf{x}). \tag{4.1}$$

This fact allows us to find a upper estimate for the best constant of (1.6) via the inequality (1.2) for the Hardy operator H_n, which is considered in a previous Section for $p, q > 1$

and with a product type weight on one side. It is useful to rewrite (1.6) in the following way

$$\left(\int_{\mathbb{R}^n_+} (G_n g)^q(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{q}} \leqslant C\left(\int_{\mathbb{R}^n_+} g^p(\mathbf{x})d\mathbf{x}\right)^{\frac{1}{p}} \tag{4.2}$$

with $g(\mathbf{x}) = f(\mathbf{x})v(\mathbf{x})^{1/p}$ and

$$u(\mathbf{x}) := (G_n v)(\mathbf{x})^{-\frac{q}{p}} w(\mathbf{x}). \tag{4.3}$$

Further, for any $0 < s < q$ we put $\widetilde{p} := p/s$, $\widetilde{q} := q/s$ and after a new substitution $g(\mathbf{x}) = h(\mathbf{x})^{1/s}$ the inequality (4.2) gets the form

$$\left(\int_{\mathbb{R}^n_+} (G_n h)^{\widetilde{q}}(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{1/\widetilde{q}} \leqslant \widetilde{C}\left(\int_{\mathbb{R}^n_+} h^{\widetilde{p}}(\mathbf{x})d\mathbf{x}\right)^{1/\widetilde{p}}, \tag{4.4}$$

where $\widetilde{C} = C^s$. Therefore, in view of (4.1) we have that the inequality corresponding to (4.4) for the operator

$$(\widetilde{H}_n h)(\mathbf{x}) := \frac{1}{x_1 \ldots x_n}(H_n h)(\mathbf{x}) \tag{4.5}$$

has the form

$$\left(\int_{\mathbb{R}^n_+} \left(\widetilde{H}_n h\right)^{\widetilde{q}}(\mathbf{x})u(\mathbf{x})d\mathbf{x}\right)^{1/\widetilde{q}} \leqslant \bar{C}\left(\int_{\mathbb{R}^n_+} h^{\widetilde{p}}(\mathbf{x})d\mathbf{x}\right)^{1/\widetilde{p}}. \tag{4.6}$$

This is an inequality for the Hardy operator H_n with $1 < \widetilde{p}, \widetilde{q} < \infty$, $w(\mathbf{x}) = (x_1 \ldots x_n)^{-\widetilde{q}}u(\mathbf{x})$ and with the product weight $v(\mathbf{x}) \equiv 1$. Now we are ready to state and prove our results for the inequality (1.6). Our main result for the case $0 < p \leqslant q < \infty$ reads:

THEOREM 4.1. *Let $0 < p \leqslant q < \infty$. Then the inequality* (1.6) *holds for all positive measurable functions f on $\mathbb{R}^n_+$ if and only if $A_{G_n} < \infty$, where*

$$A_{G_n} := \sup_{\substack{t_i > 0 \\ i=1,\ldots,n}} t_1^{-1/p} \ldots t_n^{-1/p} \left(\int_0^{t_1} \ldots \int_0^{t_n} u(\mathbf{x})d\mathbf{x}\right)^{1/q} \tag{4.7}$$

with $u(x)$ defined by (4.3). *Moreover, $C \approx A_{G_n}$ with constants of equivalence depending only on the parameters p, q and the dimension n.*

Proof. Sufficiency. On the strength of (4.1) and Theorem 2.2 for $1 < \widetilde{p} \leqslant \widetilde{q} < \infty$ the inequality (4.6) holds if

$$\bar{A}_{G_n} := \sup_{\substack{t_i > 0 \\ i=1,\ldots,n}} t_1^{-1/\widetilde{p}} \ldots t_n^{-1/\widetilde{p}} \left(\int_0^{t_1} \ldots \int_0^{t_n} u(\mathbf{x})d\mathbf{x}\right)^{1/\widetilde{q}} < \infty.$$

Note that because of definitions of $\widetilde{p}$ and $\widetilde{q}$ it yields that

$$(\bar{A}_{G_n})^{\frac{1}{s}} = A_{G_n}.$$

Therefore, according to the fact that $C = \widetilde{C}^{1/s}$ it follows that $A_{G_n} < \infty$ is a sufficient condition for the validity of the inequality (1.6) in the case $0 < p \leqslant q < \infty$.

Necessity. Suppose that (1.6) and, thus, (4.2) holds with $C < \infty$. Take a test function

$$g_{\mathbf{t}}(\mathbf{y}) = \chi_{[0,t_1]}(y_1)t_1^{-\frac{1}{p}} \dots \chi_{[0,t_n]}(y_n)t_n^{-\frac{1}{p}}$$

and put it into the inequality (4.2). The function $g_{\mathbf{t}}(\mathbf{y})$ is such that the right hand side of (4.2) is equal to 1. Therefore,

$$C \geqslant \left(\int_{\mathbb{R}^n_+} (G_n g_{\mathbf{t}})^q (\mathbf{x}) u(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}} \geqslant t_1^{-\frac{1}{p}} \dots t_n^{-\frac{1}{p}} \left(\int_0^{t_1} \dots \int_0^{t_n} u(\mathbf{x}) d\mathbf{x}\right)^{\frac{1}{q}}.$$

Hence, by taking supremum over all $t_i,\ i = 1, \dots, n,$ we have that $A_{G_n} < \infty$ and the proof is complete. □

REMARK 4.1. Our proof above shows that Theorem 4.1 may be regarded as a limit case of the result in Theorem 2.2.

Moreover, the inequality (4.1) and Theorem 3.2 allow us to obtain a sufficient condition for (1.6) to hold in the case $0 < q < p < \infty$. We state this result in the following form:

THEOREM 4.2. *Let* $0 < q < p < \infty$. *Then the inequality* (1.6) *holds if* $B_{G_n} < \infty$, *where*

$$B_{G_n} := \left(\int_{\mathbb{R}^n_+} \left(\int_0^{t_1} \dots \int_0^{t_n} u(\mathbf{x}) d\mathbf{x}\right)^{\frac{r}{q}} t_1^{-\frac{r}{q}} \dots t_n^{-\frac{r}{q}} dt_1 \dots dt_n\right)^{\frac{1}{r}}.$$

Proof. The statement follows from Theorem 3.2 by using the same arguments as for the proof of a sufficiency part of Theorem 4.1. □

REMARK 4.2. Note that the condition $B_{G_n} < \infty$ is also necessary for (1.6) to hold in the case $0 < q < p < \infty$ with the additional assumption that the weight function u is of product type. In this case we also have that $C \approx B_{G_n}$, where C is the best constant in (1.6).

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