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NEW WEIGHTED SIMPSON TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we establish some weighted Simpson type inequalities and give several applications for Euler's Beta mapping and special means.

1. Introduction

The following inequality is well known in the literature as *Simpson's inequality:*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right]\right| \leqslant \frac{1}{2880}\left\|f^{(4)}\right\|_\infty (b-a)^5, \tag{1.1}$$

where the mapping $f : [a,b] \to R$ is assumed to be four times continuously differentiable on the interval and $f^{(4)}$ to be bounded on (a,b), that is,

$$\left\|f^{(4)}\right\|_\infty := \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty.$$

For some recent results which generalize, improve and extend the inequality (1.1), see the papers [2–7] and [9–14].

2. Notation

We introduce the following notation:

Generalized Partition: $I(t_1,t_2,t_3)$**:** $t_1,t_2,t_3 \in R, t_2 \in [a,b]$.

Canonical partition: $I^*(t_1,t_2,t_3)$**:** $t_1,t_2,t_3 \in R, a \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant b$.

Functions: (1) $g_i : R \to R$ is integrable and $k_i : R \to R$ is differentiable with $k_i' = g_i$ $(i = 1,\cdots,4)$.

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(2) Let $a \leqslant r_1 \leqslant r_2 \leqslant r_3 \leqslant b$. Define

$$s(x) := \begin{cases} k_1(x), & x \in [a, r_1), \\ k_2(x), & x \in [r_1, r_2), \\ k_3(x), & x \in [r_2, r_3), \\ k_4(x), & x \in [r_3, b]. \end{cases} \tag{2.1}$$

(3) Let $x, y, z \in R$ and $x < y$. Define

$$Q(x, y, z, \mu) := \begin{cases} (\int_x^y |t - z|^{\mu} \, dt)^{\frac{1}{\mu}} & \text{if } \mu \in [1, \infty), \\ \max\limits_{x \leqslant t \leqslant y} |t - z| & \text{if } \mu = \infty. \end{cases} \tag{2.2}$$

(4) Let $a \leqslant t_1 \leqslant t_2 \leqslant t_3 \leqslant b$, $\mu_i > 1$ $(i = 1, \cdots, 4)$ and $\mu \geqslant 1$. Define

$$h_1(\mu) := \left(\frac{(t_1 - a)^{\mu+1}}{\mu + 1} \right)^{\frac{1}{\mu}}; \quad h_2(\mu) := \left(\frac{(t_2 - t_1)^{\mu+1}}{\mu + 1} \right)^{\frac{1}{\mu}};$$

$$h_3(\mu) := \left(\frac{(t_3 - t_2)^{\mu+1}}{\mu + 1} \right)^{\frac{1}{\mu}}; \quad h_4(\mu) := \left(\frac{(b - t_3)^{\mu+1}}{\mu + 1} \right)^{\frac{1}{\mu}};$$

$$\overline{H}_{\mu} := (h_1(\mu), h_2(\mu), h_3(\mu), h_4(\mu)); \tag{2.3}$$

$$\overline{\mu} := (\mu_1, \mu_2, \mu_3, \mu_4);$$

$$\overline{H}_{\overline{\mu}} := (h_1(\mu_1), h_2(\mu_2), h_3(\mu_3), h_4(\mu_4)). \tag{2.4}$$

(5) Let $c \leqslant d$ in R. Define

$$\|f\|_{\mu,[c,d]} = \begin{cases} (\int_a^b |f(x)|^{\mu} \, dx)^{\frac{1}{\mu}} & \text{if } \mu \in [1, \infty), \\ \sup\limits_{t \in [c,d]} |f(t)| & \text{if } \mu = \infty. \end{cases} \tag{2.5}$$

Total variation: Let $c \leqslant d$ in R and let the function f be bounded variation on $[c, d]$. Define $V_c^d(f)$ to be the total variation of f on $[c, d]$.

3. Dragomir and Čuljak-Pečarić-Persson's Inequalities

In [6], Dragomir established the following two Simpson type inequalities:

THEOREM A. *Let $f : [a, b] \to R$ be a function with bounded variation on $[a, b]$. Then we have the inequality*

$$\left| \int_a^b f(x)dx - \frac{b - a}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a + b}{2}\right) \right] \right| \leqslant \frac{1}{3}(b - a)V_a^b(f). \tag{3.1}$$

The constant $\frac{1}{3}$ is the best possible in (2.1).

THEOREM B. *Let* $f : I \subset R \to R$ *be a differentiable function in* $Int(I)$ *and* $a,b \in Int(I)$ *with* $a < b$. *If* $f \in L_1[a,b]$, *then we have the inequality*

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right]\right| \leqslant \frac{1}{3}(b-a)\|f'\|_{1,[a,b]}. \tag{3.2}$$

In [3], Čuljak, Pečarić and Pessson established the following Simpson type inequalities:

THEOREM C. *Let* f *be a function with bounded variation on* $[a,b]$ *and* f *have no discontinuity in* t_2. *Then the inequality*

$$\left|\int_a^b f(x)dx - [(t_1-a)f(a) + (b-t_3)f(b) + (t_3-t_1)f(t_2)]\right| \leqslant Q(a,t_2,t_1,\infty)V_a^{t_2}(f) + Q(t_2,b,t_3,\infty)V_{t_2}^b(f) \tag{3.3}$$

holds in the generalized partition $I(t_1,t_2,t_3)$, *and the inequality*

$$\left|\int_a^b f(x)dx - [(t_1-a)f(a) + (b-t_3)f(b) + (t_3-t_1)f(t_2)]\right| \leqslant \overline{G}\cdot\overline{V} \tag{3.4}$$

holds in the canonical partition $I^*(t_1,t_2,t_3)$, *where* $\overline{G} := (t_1-a, t_2-t_1, t_3-t_2, b-t_3)$ *and* $\overline{V} := (V_a^{t_1}(f), V_{t_1}^{t_2}(f), V_{t_2}^{t_3}(f), V_{t_3}^b(f))$.

THEOREM D. *Soppose* $p_i, q_i \geqslant 1$ $(i = 1,\cdots,4)$ *with* $\frac{1}{p_i} + \frac{1}{q_i} = 1$ $(q_i = \infty$ *if* $p_i = 1)$. *Let* $f : [a,b] \to R$. *In the generalized partition* $I(t_1,t_2,t_3)$, *if* f *is differentiable with* $f' \in L_{p_1}[a,t_2] \cap L_{p_2}[t_2,b]$, *then we have the inequality*

$$\left|\int_a^b f(x)dx - [(t_1-a)f(a) + (b-t_3)f(b) + (t_3-t_1)f(t_2)]\right| \leqslant Q(a,t_2,t_1,q_1)\|f'\|_{p_1,[a,t_2]} + Q(t_2,b,t_3,q_2)\|f'\|_{p_2,[t_2,b]}. \tag{3.5}$$

In the canonical partition $I^*(t_1,t_2,t_3)$, *if* f *is differentiable with* $f' \in L_{p_1}[a,t_1] \cap L_{p_2}[t_1,t_2] \cap L_{p_3}[t_2,t_3] \cap L_{p_4}[t_3,b]$, *then we have the inequality*

$$\left|\int_a^b f(x)dx - [(t_1-a)f(a) + (b-t_3)f(b) + (t_3-t_1)f(t_2)]\right| \leqslant \overline{H}_{\overline{q}}\cdot\overline{I}_{\overline{p}}, \tag{3.6}$$

where $\overline{q} = (q_1,q_2,q_3,q_4)$ *and* $\overline{I}_{\overline{p}} := (\|f'\|_{p_1,[a,t_1]}, \|f'\|_{p_2,[t_1,t_2]}, \|f'\|_{p_3,[t_2,t_3]}, \|f'\|_{p_4,[t_3,b]})$.

THEOREM E. *Let $f:[a,b]\to R$. In the generalized partition $I(t_1,t_2,t_3)$, if f is L_1-Lipschitzian on $[a,t_2]$ and L_2-Lipschitzian on $[t_2,b]$, then we have the inequality*

$$\left|\int_a^b f(x)dx-[(t_1-a)f(a)+(b-t_3)f(b)+(t_3-t_1)f(t_2)]\right| \leqslant Q(a,t_2,t_1,1)L_1+Q(t_2,b,t_3,1)L_2 \tag{3.7}$$

In the canonical partition $I^*(t_1,t_2,t_3)$, if f is L_1-Lipschitzian on $[a,t_1]$, L_2-Lipschitzian on $[t_1,t_2]$, L_3-Lipschitzian on $[t_2,t_3]$ and L_4-Lipschitzian on $[t_3,b]$, then we have the inequality

$$\left|\int_a^b f(x)dx-[(t_1-a)f(a)+(b-t_3)f(b)+(t_3-t_1)f(t_2)]\right| \leqslant \overline{H}_1\cdot\overline{L}, \tag{3.8}$$

where $\overline{L}:=(L_1,L_2,L_3,L_4)$.

4. Main Results

THEOREM 1. *Let $a\leqslant r_1\leqslant r_2\leqslant r_3\leqslant b$, the functions s, k_i, g_i $(i=1,\cdots,4)$ be defined as above, f be a function with bounded variation on $[a,b]$, and let f and s have no common point of discontinuity on $[a,b]$. Then we have the inequality*

$$|W_f|\leqslant\|k_1\|_{\infty,[a,r_1]}V_a^{r_1}(f)+\|k_2\|_{\infty,[r_1,r_2]}V_{r_1}^{r_2}(f) +\|k_3\|_{\infty,[r_2,r_3]}V_{r_2}^{r_3}(f)+\|k_4\|_{\infty,[r_3,b]}V_{r_3}^{b}(f) \tag{4.1}$$

where

$$\begin{aligned}W_f&=-k_1(a)f(a)+(k_1(r_1)-k_2(r_1))f(r_1)+(k_2(r_2)-k_3(r_2))f(r_2)\\&\quad+(k_3(r_3)-k_4(r_3))f(r_3)+k_4(b)f(b)-\int_a^{r_1}f(x)g_1(x)dx\\&\quad-\int_{r_1}^{r_2}f(x)g_2(x)dx-\int_{r_2}^{r_3}f(x)g_3(x)dx-\int_{r_3}^{b}f(x)g_4(x)dx.\end{aligned}$$

Proof. Using the integration by parts formula, we have the following identity

$$\begin{aligned}\int_a^b s(x)df(x)&=k_1(x)f(x)\Big|_a^{r_1}-\int_a^{r_1}f(x)g_1(x)dt+k_2(x)f(x)\Big|_{r_1}^{r_2}-\int_{r_1}^{r_2}f(x)g_2(x)dx\\&\quad+k_3(x)f(x)\Big|_{r_2}^{r_3}-\int_{r_2}^{r_3}f(x)g_3(x)dx+k_4(x)f(x)\Big|_{r_3}^{b}-\int_{r_3}^{b}f(x)g_4(x)dx\\&=-k_1(a)f(a)+(k_1(r_1)-k_2(r_1))f(r_1)+(k_2(r_2)-k_3(r_2))f(r_2)\\&\quad+(k_3(r_3)-k_4(r_3))f(r_3)+k_4(b)f(b)-\int_a^{r_1}f(x)g_1(x)dx\\&\quad-\int_{r_1}^{r_2}f(x)g_2(x)dx-\int_{r_2}^{r_3}f(x)g_3(x)dx-\int_{r_3}^{b}f(x)g_4(x)dx=W_f.\end{aligned} \tag{4.2}$$

It is well known [1, p. 159] that if $\mu, \nu : [c,d] \to R$ are such that μ is continuous on $[c,d]$ and ν is of bounded variation on $[c,d]$, then $\int_c^d \mu(t)d\nu(t)$ exists and [1, p. 177]

$$\left| \int_c^d \mu(x)d\nu(x) \right| \leqslant \|\mu\|_{\infty,[c,d]} V_c^d(\nu). \tag{4.3}$$

Using (4.2) and (4.3), we have

$$\begin{aligned} |W_f| &\leqslant \left| \int_a^{r_1} k_1(x)df(x) \right| + \left| \int_{r_1}^{r_2} k_2(x)df(x) \right| \\ &\quad + \left| \int_{r_2}^{r_3} k_3(x)df(x) \right| + \left| \int_{r_3}^{b} k_4(x)df(x) \right| \\ &\leqslant \|k_1\|_{\infty,[a,r_1]} V_a^{r_1}(f) + \|k_2\|_{\infty,[r_1,r_2]} V_{r_1}^{r_2}(f) \\ &\quad + \|k_3\|_{\infty,[r_2,r_3]} V_{r_2}^{r_3}(f) + \|k_4\|_{\infty,[r_3,b]} V_{r_3}^{b}(f) \end{aligned}$$

which is the inequality (4.1).

This completes the proof. □

Under the conditions of Theorem 1, we have the following corollary and remarks.

COROLLARY 1. *In Theorem 1, let* $k_1(x) = k_2(x)$ $(x \in R)$ *and* $k_3(x) = k_4(x)$ $(x \in R)$ *. Then we have the inequality*

$$\begin{aligned} |W_f| &\leqslant \|k_1\|_{\infty,[a,r_1]} V_a^{r_1}(f) + \|k_1\|_{\infty,[r_1,r_2]} V_{r_1}^{r_2}(f) \\ &\quad + \|k_3\|_{\infty,[r_2,r_3]} V_{r_2}^{r_3}(f) + \|k_3\|_{\infty,[r_3,b]} V_{r_3}^{b}(f), \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} W_f &= -k_1(a)f(a) + (k_1(r_2) - k_3(r_2))f(r_2) + k_3(b)f(b) \\ &\quad - \int_a^{r_2} f(x)g_1(x)dx - \int_{r_2}^{b} f(x)g_3(x)dx. \end{aligned}$$

REMARK 1. In Corollary 1, let $t_1, t_3 \in R$, $r_2 = t_2$, $k_1(x) = k_2(x) = x - t_1$ $(x \in R)$ and $k_3(x) = k_4(x) = x - t_3$ $(x \in R)$. Then we have the inequality

$$\begin{aligned} &\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (b - t_3)f(b) + (t_3 - t_1)f(t_2)] \right| \\ &\quad \leqslant Q(a, r_1, t_1, \infty)V_a^{r_1}(f) + Q(r_1, t_2, t_1, \infty)V_{r_1}^{t_2}(f) \\ &\qquad + Q(t_2, r_3, t_3, \infty)V_{t_2}^{r_3}(f) + Q(r_3, b, t_3, \infty)V_{r_3}^{b}(f) \\ &\quad \leqslant Q(a, t_2, t_1, \infty)V_a^{t_2}(f) + Q(t_2, b, t_3, \infty)V_{t_2}^{b}(f) \end{aligned}$$

which refines the inequality (3.3).

REMARK 2. In Corollary 1, let $r_j = t_j$ $(j = 1, 2, 3)$, $k_1(x) = k_2(x) = x - t_1 (x \in R)$, $k_3(x) = k_4(x) = x - t_3$ $(x \in R)$ and $g_i(x) \equiv 1 (i = 1, \cdots, 4;\ x \in R)$. Then the inequality (4.4) reduces to the inequality (3.4).

THEOREM 2. *Let $a \leqslant r_1 \leqslant r_2 \leqslant r_3 \leqslant b$, and let the functions f, s, k_i, g_i $(i=1,\cdots,4)$ be defined as in Theorem 1. Further, let $g_i(x) > 0$ $(x \in R; i=1,\cdots,4)$ and $k_1(r_1)=k_2(r_1)= k_3(r_3)=k_4(r_3)=0$. Then we have the inequality*

$$\begin{aligned}\left|\overline{W}_f\right| &\leqslant (-k_1(a))V_a^{r_1}(f)+k_2(r_2)V_{r_1}^{r_2}(f)\\ &\qquad +(-k_3(r_2))V_{r_2}^{r_3}(f)+k_4(b)V_{r_3}^{b}(f)-[M_1+M_2]\\ &\leqslant (-k_1(a))V_a^{r_1}(f)+k_2(r_2)V_{r_1}^{r_2}(f)+(-k_3(r_2))V_{r_2}^{r_3}(f)+k_4(b)V_{r_3}^{b}(f),\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}\overline{W}_f = &-k_1(a)f(a)+(k_2(r_2)-k_3(r_2))f(r_2)+k_4(b)f(b)\\ &-\int_a^{r_1} f(x)g_1(x)dx-\int_{r_1}^{r_2} f(x)g_2(x)dx\\ &-\int_{r_2}^{r_3} f(x)g_3(x)dx-\int_{r_3}^{b} f(x)g_4(x)dx;\end{aligned}$$

$$M_1=\begin{cases}[k_1(a)+k_2(r_2)]\,V_{r_1}^{c_1}(f), & \text{as } \begin{array}{l}-k_1(a)\leqslant k_2(r_2) \text{ and}\\ c_1=(k_2)^{-1}(-k_1(a))\end{array}\\ [-k_1(a)-k_2(r_2)]\,V_{c_2}^{r_1}(f), & \text{as } \begin{array}{l}k_2(r_2)\leqslant -k_1(a) \text{ and}\\ c_2=(k_1)^{-1}(-k_2(r_2))\end{array}\end{cases}$$

and

$$M_2=\begin{cases}[k_3(r_2)+k_4(b)]\,V_{r_3}^{d_1}(f), & \text{as } \begin{array}{l}-k_3(r_2)\leqslant k_4(b) \text{ and}\\ d_1=(k_4)^{-1}(-k_3(r_2))\end{array}\\ [-k_3(r_2)-k_4(b)]\,V_{d_2}^{r_3}(f), & \text{as } \begin{array}{l}k_4(b)\leqslant -k_3(r_2) \text{ and}\\ d_2=(k_3)^{-1}(-k_4(b))\end{array}\end{cases}.$$

Proof. Using the assumption $k_1(r_1)=k_2(r_1)= k_3(r_3)=k_4(r_3)=0$ in (4.2), we have the following identity

$$\int_a^b s(x)df(x)=\overline{W}_f. \tag{4.6}$$

Next, we shall discuss the following four cases.

Case 1. Let $-k_1(a)\leqslant k_2(r_2)$ and $-k_3(r_2)\leqslant k_4(b)$ with $c_1=(k_2)^{-1}(-k_1(a))$ and $d_1=(k_4)^{-1}(-k_3(r_2))$.

Using (4.3) and (4.6), we have

$$\begin{aligned}|\overline{W}_f| &\leqslant \left|\int_a^{r_1} s(x)df(x)\right|+\left|\int_{r_1}^{c_1} s(x)df(x)\right|+\left|\int_{c_1}^{r_2} s(x)df(x)\right|\\ &\qquad +\left|\int_{r_2}^{r_3} s(x)df(x)\right|+\left|\int_{r_3}^{d_1} s(x)df(x)\right|+\left|\int_{d_1}^{b} s(x)df(x)\right|\\ &=\left|\int_a^{r_1} k_1(x)df(x)\right|+\left|\int_{r_1}^{c_1} k_2(x)df(x)\right|+\left|\int_{c_1}^{r_2} k_2(x)df(x)\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\int_{r_2}^{r_3} k_3(x)df(x)\right| + \left|\int_{r_3}^{d_1} k_4(x)df(x)\right| + \left|\int_{d_1}^{b} k_4(x)df(x)\right| \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + (-k_1(a))V_{r_1}^{c_1}(f) + k_2(r_2)V_{c_1}^{r_2}(f) \\
&\quad + (-k_3(r_2))V_{r_2}^{r_3}(f) + (-k_3(r_2))V_{r_3}^{d_1}(f) + k_4(b)V_{d_1}^{b}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) - [k_1(a)+k_2(r_2)]\, V_{r_1}^{c_1}(f) \\
&\quad + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [k_3(r_2)+k_4(b)]\, V_{r_3}^{d_1}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [M_1+M_2] \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f)
\end{aligned}
$$

which is the inequality (4.5).

Case 2. Let $-k_1(a) \leqslant k_2(r_2)$ and $k_4(b) \leqslant -k_3(r_2)$ with $c_1 = (k_2)^{-1}(-k_1(a))$ and $d_2 = (k_3)^{-1}(-k_4(b))$.

Using (4.3) and (4.6), we have

$$
\begin{aligned}
|\overline{W}_f| &\leqslant \left|\int_a^{r_1} s(x)df(x)\right| + \left|\int_{r_1}^{c_1} s(x)df(x)\right| + \left|\int_{c_1}^{r_2} s(x)df(x)\right| \\
&\quad + \left|\int_{r_2}^{d_2} s(x)df(x)\right| + \left|\int_{d_2}^{r_3} s(x)df(x)\right| + \left|\int_{r_3}^{b} s(x)df(x)\right| \\
&= \left|\int_a^{r_1} k_1(x)df(x)\right| + \left|\int_{r_1}^{c_1} k_2(x)df(x)\right| + \left|\int_{c_1}^{r_2} k_2(x)df(x)\right| \\
&\quad + \left|\int_{r_2}^{d_2} k_3(x)df(x)\right| + \left|\int_{d_2}^{r_3} k_3(x)df(x)\right| + \left|\int_{r_3}^{b} k_4(x)df(x)\right| \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + (-k_1(a))V_{r_1}^{c_1}(f) + k_2(r_2)V_{c_1}^{r_2}(f) \\
&\quad + (-k_3(r_2))V_{r_2}^{d_2}(f) + k_4(b)V_{d_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) - [k_1(a)+k_2(r_2)]\, V_{r_1}^{c_1}(f) \\
&\quad + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [-k_3(r_2)-k_4(b)]\, V_{d_2}^{r_3}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [M_1+M_2] \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f)
\end{aligned}
$$

which is the inequality (4.5).

Case 3. Let $k_2(r_2) \leqslant -k_1(a)$ and $-k_3(r_2) \leqslant k_4(b)$ with $c_2 = (k_1)^{-1}(-k_2(r_2))$ and $d_1 = (k_4)^{-1}(-k_3(r_2))$.

Using (4.3) and (4.6), we have

$$
\begin{aligned}
|\overline{W}_f| &\leqslant \left|\int_a^{c_2} s(x)df(x)\right| + \left|\int_{c_2}^{r_1} s(x)df(x)\right| + \left|\int_{r_1}^{r_2} s(x)df(x)\right| \\
&\quad + \left|\int_{r_2}^{r_3} s(x)df(x)\right| + \left|\int_{r_3}^{d_1} s(x)df(x)\right| + \left|\int_{d_1}^{b} s(x)df(x)\right|
\end{aligned}
$$

$$
\begin{aligned}
&= \left|\int_a^{c_2} k_1(x)df(x)\right| + \left|\int_{c_2}^{r_1} k_1(x)df(x)\right| + \left|\int_{r_1}^{r_2} k_2(x)df(x)\right| \\
&\qquad + \left|\int_{r_2}^{r_3} k_3(x)df(x)\right| + \left|\int_{r_3}^{d_1} k_4(x)df(x)\right| + \left|\int_{d_1}^{b} k_4(x)df(x)\right| \\
&\leqslant (-k_1(a))V_a^{c_2}(f) + k_2(r_2)V_{c_2}^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) \\
&\qquad + (-k_3(r_2))V_{r_2}^{r_3}(f) + (-k_3(r_2))V_{r_3}^{d_1}(f) + k_4(b)V_{d_1}^{b}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) - [-k_1(a) - k_2(r_2)]\, V_{c_2}^{r_1}(f) \\
&\qquad + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [k_3(r_2) + k_4(b)]\, V_{r_3}^{d_1}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [M_1+M_2] \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f)
\end{aligned}
$$

which is the inequality (4.5).

Case 4. Let $k_2(r_2) \leqslant -k_1(a)$ and $k_4(b) \leqslant -k_3(r_2)$ with $c_2 = (k_1)^{-1}(-k_2(r_2))$ and $d_2 = (k_3)^{-1}(-k_4(b))$.

Using (4.3) and (4.6), we have

$$
\begin{aligned}
|\overline{W}_f| &\leqslant \left|\int_a^{c_2} s(x)df(x)\right| + \left|\int_{c_2}^{r_1} s(x)df(x)\right| + \left|\int_{r_1}^{r_2} s(x)df(x)\right| \\
&\qquad + \left|\int_{r_2}^{d_2} s(x)df(x)\right| + \left|\int_{d_2}^{r_3} s(x)df(x)\right| + \left|\int_{r_3}^{b} s(x)df(x)\right| \\
&= \left|\int_a^{c_2} k_1(x)df(x)\right| + \left|\int_{c_2}^{r_1} k_1(x)df(x)\right| + \left|\int_{r_1}^{r_2} k_2(x)df(x)\right| \\
&\qquad + \left|\int_{r_2}^{d_2} k_3(x)df(x)\right| + \left|\int_{d_2}^{r_3} k_3(x)df(x)\right| + \left|\int_{r_3}^{b} k_4(x)df(x)\right| \\
&\leqslant (-k_1(a))V_a^{c_2}(f) + k_2(r_2)V_{c_2}^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) \\
&\qquad + (-k_3(r_2))V_{r_2}^{d_2}(f) + k_4(b)V_{d_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) - [-k_1(a) - k_2(r_2)]\, V_{c_2}^{r_1}(f) \\
&\qquad + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [-k_3(r_2) - k_4(b)]\, V_{d_2}^{r_3}(f) \\
&= (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f) - [M_1+M_2] \\
&\leqslant (-k_1(a))V_a^{r_1}(f) + k_2(r_2)V_{r_1}^{r_2}(f) + (-k_3(r_2))V_{r_2}^{r_3}(f) + k_4(b)V_{r_3}^{b}(f)
\end{aligned}
$$

which is the inequality (4.5).

This completes the proof. □

Under the conditions of Theorem 2, we have the following corollaries and remarks.

COROLLARY 2. *In Theorem 2, let* $t_j,\ r_j,\ \ k_i,\ g_i\ \ (j = 1, 2, 3;\ i = 1, \cdots, 4)$ *be* *defined as in Remark 2, then we have the inequality*

$$\left|\int_a^b f(x)dx - [(t_1-a)f(a)+(b-t_3)f(b)+(t_3-t_1)f(t_2)]\right| \leqslant \overline{G}\cdot\overline{V} - [M_1+M_2] \leqslant \overline{G}\cdot\overline{V}, \tag{4.7}$$

where

$$M_1 = \begin{cases} (a-2t_1+t_2)V_{t_1}^{c_1}(f), & as \quad t_1 \leqslant \frac{a+t_2}{2} \text{ and } c_1 = 2t_1 - a \\ (2t_1-a-t_2)V_{c_2}^{t_1}(f), & as \quad \frac{a+t_2}{2} \leqslant t_1 \text{ and } c_2 = 2t_1 - t_2 \end{cases}$$

$$M_2 = \begin{cases} (t_2-2t_3+b)V_{t_3}^{d_1}(f), & as \quad t_3 \leqslant \frac{t_2+b}{2} \text{ and } d_1 = 2t_3 - t_2 \\ (2t_3-t_2-b)V_{d_2}^{t_3}(f), & as \quad \frac{t_2+b}{2} \leqslant t_3 \text{ and } d_2 = 2t_3 - b \end{cases}$$

and $\overline{G}$, $\overline{V}$ *are defined as in Theorem C.*

REMARK 3. In Corollary 2, the inequality (4.7) refines the inequality (3.4).

COROLLARY 3. *Suppose* $g : R \to R$ *is positive and continuous and* $h : R \to R$ *is differentiable with* $h'(x) = g(x)$ *on* R. *In Theorem 2, let* $r_1 = h^{-1}(\frac{5h(a)+h(b)}{6})$, $r_2 = h^{-1}(\frac{h(a)+h(b)}{2})$, $r_3 = h^{-1}(\frac{h(a)+5h(b)}{6})$, $k_1(x) = k_2(x) = h(x) - \frac{5h(a)+h(b)}{6}$ $(x \in R)$, $k_3(x) = k_4(x) = h(x) - \frac{h(a)+5h(b)}{6}$ $(x \in R)$ *and* $g_i(x) = g(x)$ $(x \in R,\ i = 1,\cdots,4)$. *Then we have* $c_1 = h^{-1}(\frac{2h(a)+h(b)}{3})$, $d_2 = h^{-1}(\frac{h(a)+2h(b)}{3})$ *and the inequality*

$$\begin{aligned} &\left|\int_a^b f(x)g(x)dx - \frac{\int_a^b g(x)dx}{3}\left[\frac{f(a)+f(b)}{2} + 2f(r_2)\right]\right| \\ &\quad\leqslant \frac{h(b)-h(a)}{6}V_a^{r_1}(f) + \frac{h(b)-h(a)}{3}V_{r_1}^{r_2}(f) + \frac{h(b)-h(a)}{3}V_{r_2}^{r_3}(f) \\ &\qquad + \frac{h(b)-h(a)}{6}V_{r_3}^{b}(f) - \left[\frac{h(b)-h(a)}{6}V_{r_1}^{c_1}(f) + \frac{h(b)-h(a)}{6}V_{d_2}^{r_3}(f)\right] \\ &\quad= \frac{h(b)-h(a)}{3}V_a^b(f) - \frac{h(b)-h(a)}{6}\left[V_a^{c_1}(f) + V_{d_2}^b(f)\right] \\ &\quad= \frac{\int_a^b g(x)dx}{3}V_a^b(f) - \frac{\int_a^b g(x)dx}{6}\left[V_a^{c_1}(f) + V_{d_2}^b(f)\right] \\ &\quad\leqslant \frac{\int_a^b g(x)dx}{3}V_a^b(f) \end{aligned} \tag{4.8}$$

which is the weighted Simpson type inequality for functions of bounded variation.

REMARK 4. In Corollary 3, let $g(x) \equiv 1$ and $h(x) = x$ $(x \in R)$. Then we have the inequality

$$\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})\right]\right| \leqslant \frac{1}{3}(b-a)V_a^b(f) - \frac{1}{6}(b-a)\left[V_a^{\frac{2a+b}{3}}(f) + V_{\frac{a+2b}{3}}^b(f)\right] \leqslant \frac{1}{3}(b-a)V_a^b(f) \tag{4.9}$$

which refines the inequality (3.1).

THEOREM 3. *Soppose $\mu_i, \nu_i \geqslant 1$ $(i = 1, \cdots, 4)$ with $\frac{1}{\mu_i} + \frac{1}{\nu_i} = 1$ ($\nu_i = \infty$ if $\mu_i = 1$). Let $a \leqslant r_1 \leqslant r_2 \leqslant r_3 \leqslant b$ and let the functions s, k_i, g_i $(i = 1, \cdots, 4)$ be defined as in Theorem 1. If $f : [a,b] \to R$ is defferentiable with $f' \in L_{\mu_1}[a, r_1] \cap L_{\mu_2}[r_1, r_2] \cap L_{\mu_3}[r_2, r_3] \cap L_{\mu_4}[r_3, b]$, then we have the inequality*

$$|W_f| \leqslant \|k_1\|_{\nu_1,[a,r_1]} \|f'\|_{\mu_1,[a,r_1]} + \|k_2\|_{\nu_2,[r_1,r_2]} \|f'\|_{\mu_2,[r_1,r_2]} + \|k_3\|_{\nu_3,[r_2,r_3]} \|f'\|_{\mu_3,[r_2,r_3]} + \|k_4\|_{\nu_4,[r_3,b]} \|f'\|_{\mu_4,[r_3,b]}, \tag{4.10}$$

where W_f is defined as in Theorem 1.

Proof. Using the integration by parts formula, we have the following identity

$$\begin{aligned}\int_a^b s(x)f'(x)dx &= k_1(x)f(x)\Big|_a^{r_1} - \int_a^{r_1} f(x)g_1(x)dt + k_2(x)f(x)\Big|_{r_1}^{r_2} - \int_{r_1}^{r_2} f(x)g_2(x)dx \\ &+ k_3(x)f(x)\Big|_{r_2}^{r_3} - \int_{r_2}^{r_3} f(x)g_3(x)dx + k_4(x)f(x)\Big|_{r_3}^{b} - \int_{r_3}^{b} f(x)g_4(x)dx \\ &= -k_1(a)f(a) + (k_1(r_1) - k_2(r_1))f(r_1) + (k_2(r_2) - k_3(r_2))f(r_2) \\ &+ (k_3(r_3) - k_4(r_3))f(r_3) + k_4(b)f(b) - \int_a^{r_1} f(x)g_1(x)dx \\ &- \int_{r_1}^{r_2} f(x)g_2(x)dx - \int_{r_2}^{r_3} f(x)g_3(x)dx - \int_{r_3}^{b} f(x)g_4(x)dx = W_f.\end{aligned} \tag{4.11}$$

Using the Hölder's inequality and (4.11), we have

$$\begin{aligned}|W_f| &\leqslant \left|\int_a^{r_1} k_1(x)f'(x)dx\right| + \left|\int_{r_1}^{r_2} k_2(x)f'(x)dx\right| \\ &+ \left|\int_{r_2}^{r_3} k_3(x)f'(x)dx\right| + \left|\int_{r_3}^{b} k_4(x)f'(x)dx\right| \\ &\leqslant \|k_1\|_{\nu_1,[a,r_1]} \|f'\|_{\mu_1,[a,r_1]} + \|k_2\|_{\nu_2,[r_1,r_2]} \|f'\|_{\mu_2,[r_1,r_2]} \\ &+ \|k_3\|_{\nu_3,[r_2,r_3]} \|f'\|_{\mu_3,[r_2,r_3]} + \|k_4\|_{\nu_4,[r_3,b]} \|f'\|_{\mu_4,[r_3,b]}\end{aligned}$$

which is the inequality (4.10).

This completes the proof. □

Under the conditions of Theorem 3, we have the following corollaries and remarks.

COROLLARY 4. *In Theorem 3, let* $\mu_1 = \mu_2$, $\nu_1 = \nu_2$, $\mu_3 = \mu_4$, $\nu_3 = \nu_4$, $k_1(x) = k_2(x)$ $(x \in R)$ *and* $k_3(x) = k_4(x)$ $(x \in R)$. *Futher, let* $f' \in L_{\mu_1}[a, r_2] \cap L_{\mu_3}[r_2, b]$. *Using the Hölder's inequality*

$$\sum_{i=1}^{2} \alpha_i^{\frac{1}{\mu_j}} \beta_i^{\frac{1}{\nu_j}} \leqslant (\sum_{i=1}^{2} \alpha_i)^{\frac{1}{\mu_j}} (\sum_{i=1}^{2} \beta_i)^{\frac{1}{\nu_j}}$$

and (4.10), *we have the inequality*

$$\begin{aligned}
|W_f| &\leqslant \|k_1\|_{\nu_1,[a,r_1]} \|f'\|_{\mu_1,[a,r_1]} + \|k_1\|_{\nu_1,[r_1,r_2]} \|f'\|_{\mu_1,[r_1,r_2]} \\
&\qquad + \|k_3\|_{\nu_3,[r_2,r_3]} \|f'\|_{\mu_3,[r_2,r_3]} + \|k_3\|_{\nu_3,[r_3,b]} \|f'\|_{\mu_3,[r_3,b]} \\
&\leqslant \left[(\|k_1\|_{\nu_1,[a,r_1]})^{\nu_1} + (\|k_1\|_{\nu_1,[r_1,r_2]})^{\nu_1} \right]^{\frac{1}{\nu_1}} \times \\
&\qquad \times \left[(\|f'\|_{\mu_1,[a,r_1]})^{\mu_1} + (\|f'\|_{\mu_1,[r_1,r_2]})^{\mu_1} \right]^{\frac{1}{\mu_1}} \\
&+ \left[(\|k_3\|_{\nu_3,[r_2,r_3]})^{\nu_3} + (\|k_3\|_{\nu_3,[r_3,b]})^{\nu_3} \right]^{\frac{1}{\nu_3}} \times \\
&\qquad \times \left[(\|f'\|_{\mu_3,[r_2,r_3]})^{\mu_3} + (\|f'\|_{\mu_3,[r_3,b]})^{\mu_3} \right]^{\frac{1}{\mu_3}} \\
&= \|k_1\|_{\nu_1,[a,r_2]} \|f'\|_{\mu_1,[a,r_2]} + \|k_3\|_{\nu_3,[r_2,b]} \|f'\|_{\mu_3,[r_2,b]}, \qquad (4.12)
\end{aligned}$$

where W_f *is defined as in Corollary 1.*

REMARK 5. In Corollary 4, let $\mu_1 = p_1$, $\nu_1 = q_1$, $\mu_3 = p_2$, $\nu_3 = q_2$, and let t_j, r_j, k_i, g_i $(j = 1, 2, 3;\ i = 1, \cdots, 4)$ be defined as in Remark 1. Then we have the inequality

$$\begin{aligned}
&\left| \int_a^b f(x)dx - [(t_1 - a)f(a) + (b - t_3)f(b) + (t_3 - t_1)f(t_2)] \right| \\
&\quad \leqslant Q(a, r_1, t_1, q_1) \|f'\|_{p_1,[a,r_1]} + Q(r_1, t_2, t_1, q_1) \|f'\|_{p_1,[r_1,t_2]} \\
&\qquad + Q(t_2, r_3, t_3, q_2) \|f'\|_{p_2,[t_2,r_3]} + Q(r_3, b, t_3, q_2) \|f'\|_{p_2,[r_3,b]} \\
&\quad \leqslant Q(a, t_2, t_1, q_1) \|f'\|_{p_1,[a,t_2]} + Q(t_2, b, t_3, q_2) \|f'\|_{p_2,[t_2,b]} \qquad (4.13)
\end{aligned}$$

which refines the inequality (3.5).

REMARK 6. In Theorem 3, let t_j, r_j, k_i, g_i $(j = 1, 2, 3;\ i = 1, \cdots, 4)$ be defined as in Remark 2, then the inequality (4.10) reduces to the inequality (3.6).

THEOREM 4. *Let* r_j, s, k_i, g_i $(j = 1, 2, 3;\ i = 1, \cdots, 4)$ *be defined as in Theorem 1 and let* $f : [a, b] \to R$ *be defferentiable with* $f' \in L_1[a, r_1] \cap L_1[r_1, r_2] \cap L_1[r_2, r_3] \cap L_1[r_3, b]$. *Further, let* $g_i(x) > 0$ $(x \in R; i = 1, \cdots, 4)$ *and* $k_1(r_1) = k_2(r_1) =$

$k_3(r_3) = k_4(r_3) = 0$. *Then we have the inequality*

$$\begin{aligned}\left|\overline{W_f}\right| &\leqslant (-k_1(a)) \|f'\|_{1,[a,r_1]} + k_2(r_2) \|f'\|_{1,[r_1,r_2]} \\ &\quad + (-k_3(r_2)) \|f'\|_{1,[r_2,r_3]} + k_4(b) \|f'\|_{1,[r_3,b]} - [M_3 + M_4] \\ &\leqslant (-k_1(a)) \|f'\|_{1,[a,r_1]} + k_2(r_2) \|f'\|_{1,[r_1,r_2]} \\ &\quad + (-k_3(r_2)) \|f'\|_{1,[r_2,r_3]} + k_4(b) \|f'\|_{1,[r_3,b]}, \end{aligned} \tag{4.14}$$

where $\overline{W}_f$ *is defined as in Theorem 2,*

$$M_3 = \begin{cases} [k_1(a) + k_2(r_2)] \|f'\|_{1,[r_1,c_1]}, & \text{as } \begin{array}{l} -k_1(a) \leqslant k_2(r_2) \text{ and} \\ c_1 = (k_2)^{-1}(-k_1(a)) \end{array} \\ [-k_1(a) - k_2(r_2)] \|f'\|_{1,[c_2,r_1]}, & \text{as } \begin{array}{l} k_2(r_2) \leqslant -k_1(a) \text{ and} \\ c_2 = (k_1)^{-1}(-k_2(r_2)) \end{array} \end{cases}$$

and

$$M_4 = \begin{cases} [k_3(r_2) + k_4(b)] \|f'\|_{1,[r_3,d_1]}, & \text{as } \begin{array}{l} -k_3(r_2) \leqslant k_4(b) \text{ and} \\ d_1 = (k_4)^{-1}(-k_3(r_2)) \end{array} \\ [-k_3(r_2) - k_4(b)] \|f'\|_{1,[d_2,r_3]}, & \text{as } \begin{array}{l} k_4(b) \leqslant -k_3(r_2) \text{ and} \\ d_2 = (k_3)^{-1}(-k_4(b)) \end{array} \end{cases}.$$

Proof. The proof is similar to that of Theorem 2.

Under the conditions of Theorem 4, we have the following corollaries and remark. □

COROLLARY 5. *Let* t_j, r_j, k_i, g_i $(j = 1,2,3,\ i = 1,\cdots,4)$ *be defined as in Corollary 2 . In Theorem 4, . we have the inequality*

$$\begin{aligned}&\left|\int_a^b f(x)dx - [(t_1 - a)f(a) + (b - t_3)f(b) + (t_3 - t_1)f(t_2)]\right| \\ &\qquad \leqslant \overline{G} \cdot \overline{V'} - [M_3 + M_4] \leqslant \overline{G} \cdot \overline{V'}, \end{aligned} \tag{4.15}$$

where

$$M_3 = \begin{cases} (a - 2t_1 + t_2) \|f'\|_{1,[t_1,c_1]}, & \text{as} \quad t_1 \leqslant \frac{a+t_2}{2} \text{ and } c_1 = 2t_1 - a \\ (2t_1 - a - t_2) \|f'\|_{1,[c_2,t_1]}, & \text{as} \quad \frac{a+t_2}{2} \leqslant t_1 \text{ and } c_2 = 2t_1 - t_2 \end{cases},$$

$$M_4 = \begin{cases} (t_2 - 2t_3 + b) \|f'\|_{1,[t_3,d_1]}, & \text{as} \quad t_3 \leqslant \frac{t_2+b}{2} \text{ and } d_1 = 2t_3 - t_2 \\ (2t_3 - t_2 - b) \|f'\|_{1,[d_2,t_3]}, & \text{as} \quad \frac{t_2+b}{2} \leqslant t_3 \text{ and } d_2 = 2t_3 - b \end{cases},$$

$\overline{V'} := \left(\|f'\|_{1,[a,t_1]}, \|f'\|_{1,[t_1,t_2]}, \|f'\|_{1,[t_2,t_3]}, \|f'\|_{1,[t_3,b]}\right)$ *and* $\overline{G}$ *is defined as in Theorem C.*

REMARK 7. In Corollary 5, the inequality (4.15) refines the inequality (3.6) in the case $p_i = 1$ and $q_i = \infty$ $(i = 1,\cdots,4)$.

COROLLARY 6. *Let* $g,\ h,\ r_j,\ \ k_i,\ g_i\ \ (j=1,2,3,\ i=1,\cdots,4)$ *be defined as in Corollary 3. In Theorem 4, we have the inequality*

$$\begin{aligned}
&\left|\int_a^b f(x)g(x)dx - \frac{\int_a^b g(x)dx}{3}\left[\frac{f(a)+f(b)}{2}+2f(r_2)\right]\right| \\
&\leqslant \frac{h(b)-h(a)}{6}\|f'\|_{1,[a,r_1]} + \frac{h(b)-h(a)}{3}\|f'\|_{1,[r_1,r_2]} + \frac{h(b)-h(a)}{3}\|f'\|_{1,[r_2,r_3]} \\
&+ \frac{h(b)-h(a)}{6}\|f'\|_{1,[r_3,b]} - \left[\frac{h(b)-h(a)}{6}\|f'\|_{1,[r_1,c_1]} + \frac{h(b)-h(a)}{6}\|f'\|_{1,[d_2,r_3]}\right] \\
&= \frac{h(b)-h(a)}{3}\|f'\|_{1,[a,b]} - \frac{h(b)-h(a)}{6}\left[\|f'\|_{1,[a,c_1]} + \|f'\|_{1,[d_2,b]}\right] \\
&= \frac{\int_a^b g(x)dx}{3}\|f'\|_{1,[a,b]} - \frac{\int_a^b g(x)dx}{6}\left[\|f'\|_{1,[a,c_1]} + \|f'\|_{1,[d_2,b]}\right] \\
&\leqslant \frac{\int_a^b g(x)dx}{3}\|f'\|_{1,[a,b]} \qquad (4.16)
\end{aligned}$$

which is the weighted Simpson type inequality in L_1 *norm.*

REMARK 8. In Corollary 6, let $g(x)\equiv 1\ \ (x\in R)$ and $h(x)=x\ \ (x\in R)$. Then we have the inequality

$$\begin{aligned}
&\left|\int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2}+2f(\frac{a+b}{2})\right]\right| \\
&\leqslant \frac{1}{3}(b-a)\|f'\|_{1,[a,b]} - \frac{1}{6}(b-a)\left[\|f'\|_{1,[a,\frac{2a+b}{3}]} + \|f'\|_{1,[\frac{a+2b}{3},b]}\right] \\
&\leqslant \frac{1}{3}(b-a)\|f'\|_{1,[a,b]} \qquad (4.17)
\end{aligned}$$

which refines the inequality (3.2).

THEOREM 5. *Let* $W_f,\ s,\ r_i,\ k_i,\ g_i\ \ (i=1,\cdots,4)$ *be defined as in Theorem 1 and let* f *be* L_1*-Lipschitzian on* $[a,r_1]$, L_2*-Lipschitzian on* $[r_1,r_2]$, L_3*-Lipschitzian on* $[r_2,r_3]$ *and* L_4*-Lipschitzian on* $[t_3,b]$. *Then we have the inequality*

$$|W_f| \leqslant L_1\|k_1\|_{1,[a,r_1]} + L_2\|k_2\|_{1,[r_1,r_2]} + L_3\|k_3\|_{1,[r_2,r_3]} + L_4\|k_4\|_{1,[r_3,b]}. \qquad (4.18)$$

Proof. Using (4.2), we have

$$\begin{aligned}
|W_f| = \left|\int_a^b s(x)df(x)\right| &\leqslant \left|\int_a^{r_1} k_1(x)df(x)\right| + \left|\int_{r_1}^{r_2} k_2(x)df(x)\right| \\
&+ \left|\int_{r_2}^{r_3} k_3(x)df(x)\right| + \left|\int_{r_3}^{b} k_4(x)df(x)\right| \\
&\leqslant L_1\|k_1\|_{1,[a,r_1]} + L_2\|k_2\|_{1,[r_1,r_2]} + L_3\|k_3\|_{1,[r_2,r_3]} + L_4\|k_4\|_{1,[r_3,b]}
\end{aligned}$$

which is the inequality (4.18). This completes the proof. $\square$

REMARK 9. In Theorem 2, let $t_j,\ r_j,\ k_i,\ g_i\ (j=1,2,3;\ i=1,\cdots,4)$ be defined as in Remark 2, then the inequality (4.18) reduces to the inequality (3.8).

5. Applications for Euler's Beta function

Consider the Euler's Beta function for real numbers

$$B(p,q) := \int_0^1 x^{p-1}(1-x)^{q-1}dx, \quad p,q>0$$

and the function

$$e_{p,q}(x) := x^{p-1}(1-x)^{q-1}, x\in[0,1].$$

In [8], Dragomir gets the following results:

We have for $p,q>1$ that

$$e'_{p,q}(x) = e_{p-1,q-1}(x)\left[p-1-(p+q-2)x\right] \tag{5.1}$$

and as

$$|p-1-(p+q-2)x| \leqslant \max\{p-1,q-1\} \tag{5.2}$$

for all $x\in[0,1]$, then

$$\begin{aligned}\left\|e'_{p,q}\right\|_{1,[0,1]} &\leqslant \max\{p-1,q-1\}\left\|e_{p-2,q-2}\right\|_{1,[0,1]}\\ &= \max\{p-1,q-1\}B(p-1,q-1).\end{aligned} \tag{5.3}$$

Using Remark 8 and (5.1) – (5.3), we have the following corollary:

COROLLARY 7. *Let $p,q>1$. Then we have the inequality*

$$\begin{aligned}\left|B(p,q)-\frac{2^{3-p-q}}{3}\right| &\leqslant \frac{1}{3}\max\{p-1,q-1\}B(p-1,q-1)\\ &\quad -\frac{1}{6}\left[\left\|e'_{p,q}\right\|_{1,\left[0,\frac{1}{3}\right]}+\left\|e'_{p,q}\right\|_{1,\left[\frac{2}{3},1\right]}\right]\\ &\leqslant \frac{1}{3}\max\{p-1,q-1\}B(p-1,q-1).\end{aligned}$$

6. Applications for the Special Means

Let us recall the following means of the two nonnegative number a and b:

1. The arithematic mean

$$A = A(a,b) := \frac{a+b}{2}, \ \ a,b\geqslant 0;$$

2. The geometric mean

$$G = G(a,b) := \sqrt{ab}, \ \ a,b > 0;$$

3. The harmonic mean

$$H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \ \ a,b > 0;$$

4. The logarithmic mean

$$L = L(a,b) = \begin{cases} \frac{b-a}{\ln b - \ln a} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \ a,b > 0;$$

5. The identric mean

$$I = I(a,b) := \begin{cases} \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, \ a,b > 0;$$

6.The p-logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b \\ a & \text{if } a = b \end{cases}, p \in R \setminus \{-1,0\}, \ a,b > 0.$$

It is well known that L_p is monotonically increasing in $p \in R$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality

$$H \leqslant G \leqslant L \leqslant I \leqslant A.$$

In what follows, by the use of Remark 8, we point out some inequalities for the above means.

Case 1. Let $f : [a,b] \to R \ \ (0 < a < b)$, $f(x) = x^p \ \ (p \in R \setminus \{-1,0,1\})$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L_p^p(a,b), \qquad \frac{f(a)+f(b)}{2} = A(a^p,b^p),$$

$$f\left(\frac{a+b}{2}\right) = A^p(a,b), \qquad \|f'\|_{1,[a,b]} = |p|\, L_{p-1}^{p-1}(a,b)(b-a)$$

$$\|f'\|_{1,\left[a,\frac{2a+b}{3}\right]} = |p|\, L_{p-1}^{p-1}\left(a, \frac{2a+b}{3}\right)\frac{b-a}{3},$$

$$\|f'\|_{1,\left[\frac{a+2b}{3},b\right]} = |p|\, L_{p-1}^{p-1}\left(\frac{a+2b}{3}, b\right)\frac{b-a}{3}$$

Using the inequality (4.17), we get

$$\begin{aligned} &\left| L_p^p(a,b) - \frac{1}{3}A(a^p,b^p) - \frac{2}{3}A^p(a,b) \right| \\ &\leqslant |p|\, L_{p-1}^{p-1}\frac{b-a}{3} - |p| \left[L_{p-1}^{p-1}(a, \frac{2a+b}{3}) + L_{p-1}^{p-1}(\frac{a+2b}{3}, b) \right] \frac{b-a}{18} \\ &\leqslant |p|\, L_{p-1}^{p-1}\frac{b-a}{3}. \end{aligned} \tag{6.1}$$

Case 2. Let $f : [a,b] \to R \ \ (0 < a < b), f(x) = \frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L^{-1}(a,b), \quad \frac{f(a)+f(b)}{2} = H^{-1}(a,b),$$

$$f\left(\frac{a+b}{2}\right) = A^{-1}(a,b), \quad \|f'\|_{1,[a,b]} = \frac{b-a}{G^2(a,b)}$$

$$\|f'\|_{1,\left[a,\frac{2a+b}{3}\right]} = \frac{b-a}{3G^2(a,\frac{2a+b}{3})}, \quad \|f'\|_{1,\left[\frac{a+2b}{3},b\right]} = \frac{b-a}{3G^2(\frac{a+2b}{3},b)}.$$

Using the inequality (4.17), we get

$$\begin{aligned} |3AH - AL - 2HL| &\leqslant \left[\frac{b-a}{G^2} - \frac{b-a}{6G^2(a,\frac{2a+b}{3})} - \frac{b-a}{6G^2(\frac{a+2b}{3},b)}\right] AHL \\ &\leqslant \frac{b-a}{G^2}AHL. \end{aligned} \tag{6.2}$$

Case 3. Let $f : [a,b] \to R \ \ (0 < a < b), f(x) = \ln x$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = \ln I(a,b), \quad \frac{f(a)+f(b)}{2} = \ln G(a,b),$$

$$f\left(\frac{a+b}{2}\right) = \ln A(a,b), \quad \|f'\|_{1,[a,b]} = \frac{b-a}{L(a,b)}$$

$$\|f'\|_{1,\left[a,\frac{2a+b}{3}\right]} = \frac{b-a}{3L(a,\frac{2a+b}{3})},\text{q} \ \|f'\|_{1,\left[\frac{a+2b}{3},b\right]} = \frac{b-a}{3L(\frac{a+2b}{3},b)}.$$

Using the inequality (4.17), we get

$$\begin{aligned} \left|\ln\left[\frac{I}{G^{\frac{1}{3}}A^{\frac{2}{3}}}\right]\right| &\leqslant \frac{b-a}{3L} - \frac{b-a}{18L(a,\frac{2a+b}{3})} - \frac{b-a}{18L(\frac{a+2b}{3},b)} \\ &\leqslant \frac{b-a}{3L}. \end{aligned} \tag{6.3}$$

REMARK 10. The inequalities (6.1)–(6.3) are improvements of the inequalities (3.2)–(3.4) in [6].

REFERENCES

[1] T. M. APOSTOL, *Mathematical Analysis*, Second Edition, Addision-Wesley Publishing Company , 1975.

[2] D. CRUZ-URIBLE AND C. J. NEUGEBAUER, *Sharp error bounds for the trapezoidal rule and Simpson's rule*, JIPAM. J. Inequal. Pure Appl. Math. 3 (2002), no. 4, Article 49, 22 pp.

[3] V. ČULJAK, J. PEČARIĆ AND L. E. PERSSON, *A note on Simpson's type numerical integration*, Soochow J. Math. 29 (2003), no. 2, 191–200.

[4] S. S. DRAGOMIR, *On Simpson's quadrature formula and applications*, Mathematica 43 (66) (2001), no. 2, 185–194 (2003).

[5] S. S. DRAGOMIR, *On Simpson's quadrature formula for Lipschitzian mappings and applications*, Soochow J. Math. 25 (1999), no. 2, 175–180.

[6] S. S. DRAGOMIR, *On Simpson's quadrature formula for mappings of bounded variation and applications*, Tamkang J. of Math. 30 (1999), no. 1, 53–58.
[7] S. S. DRAGOMIR, R. P. AGARWAL AND P. CERONE, *On Simpson's inequality and applications*, J. Inequal. Appl. 5 (2000), no. 6, 533–579.
[8] S. S. DRAGOMIR, *On the Ostrowski's integral inequality for mappings with bounded variation and applications*, Math. Inequal. Appl. no. 1, 59–66.
[9] J. PEČARIĆ AND S. VAROŠANEC, *A note on Simpson's inequality for Lipschitzian functions*, Soochow J. Math. 27 (2001), no. 1, 53–57.
[10] J. PEČARIĆ AND S. VAROŠANEC, *Simpson's formula for functions whose derivatives belong to L_p spaces*, Appl. Math. Lett. 14 (2001), no. 2, 131–135.
[11] J. PEČARIĆ AND S. VAROŠANEC, *A note on Simpson's inequality for functions of bounded variation*, Tamkang J. of Math. 31 (2000), no. 3, 239–242.
[12] K. L. TSENG, G. S. YANG AND S. S. DRAGOMIR, *On Weighted Simpson Type Inequalities and Applications*, RGMIA Res. Rep. Coll., 7 (1) (2004), Art. 17.
[13] N. UJEVIĆ, *New bounds for Simpson's inequality*, Tamkang J. of Math. 33 (2002), no. 2, 129–138.
[14] G. S. YANG AND H. F. CHU, *A note on Simpson's inequality for function of bounded variation*, Tamsui Oxf. J. Math. Sci. 16 (2000), no. 2, 229–240.

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