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AN INEQUALITY FOR $_{r+1}\phi_r$ AND ITS APPLICATIONS

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Abstract. In this paper, we use the q-binomial formula to establish an inequality for the basic hypergeometric series $_{r+1}\phi_r$. As applications of the inequality, we derive a sufficient condition for convergence of a q-series and two other inequalities.

1. Introduction and main result

q-series, which are also called basic hypergeometric series, plays a very important role in many fields, such as affine root systems, Lie algebras and groups, number theory, orthogonal polynomials and physics, etc. Inequality technique is one of the useful tools in the study of special functions. In [1], the authors gave some inequalities for hypergeometric functions. In this paper, we derive an inequality for the basic hypergeometric series $_{r+1}\phi_r$. Some applications of the inequality are also given.

The main result of this paper is the following inequality.

THEOREM 1.1. *Suppose a_i, b_i and z be any real numbers such that $|z| < (\prod_{i=1}^{r+1} M_i)^{-1}$, $b_i < 1$ with $i = 1, 2, \cdots, r$. Then we have*

$$\left| {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, z \right) \right| \leqslant \frac{1}{(|z| \prod_{i=1}^{r+1} M_i; q)_\infty}, \tag{1.1}$$

where $b_{r+1} = 0$, $M_i = \max\left\{1, \frac{|1-a_i|}{1-b_i}\right\}$ for $i = 1, 2, \cdots, r+1$.

Before the proof of the theorem, we recall some definitions, notations and known results which will be used in this paper. Throughout the whole paper, it is supposed that $0 < q < 1$. The q-shifted factorials are defined as

$$(a;q)_0 = 1, \quad (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \quad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k). \tag{1.2}$$

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We also adopt the following compact notation for multiple q-shifted factorial:

$$(a_1, a_2, ..., a_m; q)_n = (a_1; q)_n (a_2; q)_n ... (a_m; q)_n, \tag{1.3}$$

where n is an integer or ∞.

The q-binomial theorem [2][3]

$$\sum_{k=0}^{\infty} \frac{(a;q)_k z^k}{(q;q)_k} = \frac{(az;q)_\infty}{(z;q)_\infty}, \quad |z| < 1. \tag{1.4}$$

When $a = 0$,

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}, \quad |z| < 1. \tag{1.5}$$

The q-Gauss sum [2][3]

$$\sum_{k=0}^{\infty} \frac{(a,b;q)_k}{(q,c;q)_k} \left(\frac{c}{ab}\right)^k = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad \left|\frac{c}{ab}\right| < 1. \tag{1.6}$$

Heine introduced the ${}_{r+1}\phi_r$ basic hypergeometric series, which is defined by[2][3]

$${}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, ..., a_{r+1} \\ b_1, b_2, ..., b_r \end{matrix} ; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_{r+1}; q)_n z^n}{(q, b_1, b_2, ..., b_r; q)_n}. \tag{1.7}$$

2. The proof of the theorem

In this section, we use the q-binomial formula to prove the theorem. In order to prove (1.1), we need the following lemma:

LEMMA 2.1. *Let a, b be given real numbers such that $b < 1$. Then for $0 \leqslant t \leqslant 1$, we have*

$$\left|\frac{1-at}{1-bt}\right| \leqslant \max\left\{1, \ \frac{|1-a|}{1-b}\right\} \tag{2.1}$$

Proof. Let

$$f(t) = \frac{1-at}{1-bt}, \quad 0 \leqslant t \leqslant 1,$$

then

$$f'(t) = \frac{b-a}{(1-bt)^2}, \quad 0 \leqslant t \leqslant 1.$$

So $f(t)$ is a monotone function with respect to $0 \leqslant t \leqslant 1$. Because of $f(0) = 1$ and $f(1) = \frac{1-a}{1-b}$, (2.1) holds. □

Now, we give the proof of theorem 1.1.

Proof. From the lemma 2.1, we know

$$\left|\frac{(a_i;q)_n}{(b_i;q)_n}\right| = \left|\frac{1-a_i}{1-b_i}\right| \cdot \left|\frac{1-a_i q}{1-b_i q}\right| \cdot \cdots \cdot \left|\frac{1-a_i q^{n-1}}{1-b_i q^{n-1}}\right| \leqslant M_i^n, \tag{2.2}$$

where $b_{r+1}=0$, $M_i=\max\left\{1,\frac{|1-a_i|}{1-b_i}\right\}$ for $i=1,2,\cdots,r+1$.

Hence,

$$\left|\frac{(a_1,a_2,\cdots,a_{r+1};q)_n}{(b_1,b_2,\cdots,b_r;q)_n}\right|=\left|\frac{(a_1,a_2,\cdots,a_{r+1};q)_n}{(b_1,b_2,\cdots,b_{r+1};q)_n}\right|\leqslant\Big(\prod_{i=1}^{r+1}M_i\Big)^n. \tag{2.3}$$

It is obvious that

$$\frac{|z|^n}{(q;q)_n}>0,\quad n=1,2,\cdots.$$

Multiplying both sides of (2.3) by $\frac{|z|^n}{(q;q)_n}$ gives

$$\left|\frac{(a_1,a_2,\cdots,a_{r+1};q)_n z^n}{(b_1,b_2,\cdots,b_r;q)_n}\right|\leqslant\frac{(|z|\prod_{i=1}^{r+1}M_i)^n}{(q;q)_n}. \tag{2.4}$$

So, we have

$$\begin{aligned}&\left|{}_{r+1}\phi_r\left(\begin{matrix}a_1,a_2,\cdots,a_{r+1}\\ q,b_1,b_2,\cdots,b_r\end{matrix};q,z\right)\right|\\ &\qquad=\left|\sum_{n=0}^{\infty}\frac{(a_1,a_2,\cdots,a_{r+1};q)_n z^n}{(q,b_1,b_2,\cdots,b_r;q)_n}\right|\leqslant\sum_{n=0}^{\infty}\left|\frac{(a_1,a_2,\cdots,a_{r+1};q)_n z^n}{(q,b_1,b_2,\cdots,b_r;q)_n}\right|\\ &\qquad\leqslant\sum_{n=0}^{\infty}\frac{(|z|\prod_{i=1}^{r+1}M_i)^n}{(q;q)_n}.\end{aligned} \tag{2.5}$$

Using the q-binomial theorem (1.5) obtains

$$\sum_{n=0}^{\infty}\frac{(|z|\prod_{i=1}^{r+1}M_i)^n}{(q;q)_n}=\frac{1}{(|z|\prod_{i=1}^{r+1}M_i;q)_\infty}. \tag{2.6}$$

Substituting (2.6) into (2.5) gets (1.1). Thus, we complete the proof. □

3. Some applications of the inequality

Convergence is an important problem in the study of q-series. There are some results about it. For example, Ito used inequality technique to give a sufficient condition for convergence of a special q-series called Jackson integral [4]. In this section, we use the inequality obtained in the paper to give a sufficient condition for convergence of a q-series and two other inequalities.

THEOREM 3.1. *Suppose a_i, b_i, t be any real numbers such that $|t|<1$ and $b_i<1$ with $i=1,2,\cdots,r$. Let $\{c_n\}$ be any number series. If*

$$\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|=p<1,$$

then the q*-series*

$$\sum_{n=0}^{\infty} c_n \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; q, tq^n\right) \tag{3.1}$$

converges absolutely.

Proof. Since

$$\lim_{n\to\infty} q^n = 0, \tag{3.2}$$

there exists a integer N_0 such that, when $n > N_0$,

$$|t|q^n < (\prod_{i=1}^{r+1} M_i)^{-1}, \tag{3.3}$$

where $b_{r+1} = 0$, $M_i = \max\left\{1, \frac{|1-a_i|}{1-b_i}\right\}$ for $i = 1, 2, \cdots, r+1$.

When $n > N_0$, letting $z = tq^n$ in (1.1) gives

$$\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; q, tq^n\right)\right| \leqslant \frac{1}{(|t|q^n \prod_{i=1}^{r+1} M_i; q)_\infty}. \tag{3.4}$$

Multiplying both sides of (3.4) by $|c_n|$ gets

$$\left|c_n \cdot {}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; q, tq^n\right)\right| \leqslant \frac{|c_n|}{(|t|q^n \prod_{i=1}^{r+1} M_i; q)_\infty}. \tag{3.5}$$

The ratio test shows that the series

$$\sum_{n=0}^{\infty} \frac{c_n}{(|t|q^n \prod_{i=1}^{r+1} M_i; q)_\infty}$$

is absolutely convergent. From (3.5), it is sufficient to establish that (3.1) is absolutely convergent. □

THEOREM 3.2. *Suppose* a_i, b_i, t *be any real numbers such that* $0 < t < 1$ *and* $b_i < 1$ *with* $i = 1, 2, \cdots, r$. *Then for* $q < (\prod_{i=1}^{r+1} M_i)^{-1}$, *we have*

$$\sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n}\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; q, q^n\right)\right| \tag{3.6}$$

$$\leqslant \frac{(t\prod_{i=1}^{r+1} M_i; q)_\infty}{(t, \prod_{i=1}^{r+1} M_i; q)_\infty} - \frac{1}{(\prod_{i=1}^{r+1} M_i; q)_\infty}.$$

Proof. Using the theorem 3.1, we can easily know

$$\sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n}\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix}; q, q^n\right)\right|$$

converges absolutely.

Letting $z = q^n$ in (1.1) gives

$$\left| {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right| \leqslant \frac{1}{(q^n \prod_{i=1}^{r+1} M_i; q)_\infty}, \tag{3.7}$$

where $n = 1, 2, \cdots$. Multiplying both sides of (3.7) by $\frac{t^n}{(q;q)_n}$ gets

$$\frac{t^n}{(q;q)_n} \left| {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right| \leqslant \frac{t^n}{(q;q)_n (q^n \prod_{i=1}^{r+1} M_i; q)_\infty}. \tag{3.8}$$

Hence,

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n} \left| {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right| \\ &\qquad \leqslant \sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n (q^n \prod_{i=1}^{r+1} M_i; q)_\infty} \\ &\qquad = \sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n (q^n \prod_{i=1}^{r+1} M_i; q)_\infty} - \frac{1}{(\prod_{i=1}^{r+1} M_i; q)_\infty}. \end{aligned} \tag{3.9}$$

Employing the q-binomial theorem (1.4) obtains

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n (q^n \prod_{i=1}^{r+1} M_i; q)_\infty} \\ &\qquad = \frac{1}{(\prod_{i=1}^{r+1} M_i; q)_\infty} \sum_{n=0}^{\infty} \frac{(\prod_{i=1}^{r+1} M_i; q)_n t^n}{(q;q)_n} = \frac{(t \prod_{i=1}^{r+1} M_i; q)_\infty}{(t, \prod_{i=1}^{r+1} M_i; q)_\infty}. \end{aligned} \tag{3.10}$$

Substituting (3.10) into (3.9), we get (3.6). □

COROLLARY 1. *Suppose a_i, b_i, t be any real numbers such that $|t| < 1$ and $b_i < 1$ with $i = 1, 2, \cdots, r$. Then for $q < (\prod_{i=1}^{r+1} M_i)^{-1}$, we have*

$$\begin{aligned} &\left| \sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n} {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right| \\ &\qquad \leqslant \frac{(t \prod_{i=1}^{r+1} M_i; q)_\infty}{(t, \prod_{i=1}^{r+1} M_i; q)_\infty} - \frac{1}{(\prod_{i=1}^{r+1} M_i; q)_\infty}. \end{aligned} \tag{3.11}$$

Proof. Because of

$$\begin{aligned} &\left| \sum_{n=1}^{\infty} \frac{t^n}{(q;q)_n} {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right| \\ &\qquad \leqslant \sum_{n=1}^{\infty} \frac{|t|^n}{(q;q)_n} \left| {}_{r+1}\phi_r \left(\begin{matrix} a_1, a_2, \cdots, a_{r+1} \\ b_1, b_2, \cdots, b_r \end{matrix} ; q, q^n \right) \right|, \end{aligned} \tag{3.12}$$

the result follows immediately from (3.6). □

THEOREM 3.3. *Suppose a_i, b_i, a, b be any real numbers such that $b_i < 1$ with $i = 1,2,\cdots,r$ and $0 < b < a\prod_{i=1}^{r+1} M_i < 1$. Then for $q < (\prod_{i=1}^{r+1} M_i)^{-1}$, we have*

$$\sum_{n=1}^{\infty} \frac{(a;q)_n}{(q,b;q)_n}\Big(\frac{b}{a\prod_{i=1}^{r+1} M_i}\Big)^n \left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,q^n\right)\right| \leqslant \frac{(b/a,b/a\prod_{i=1}^{r+1} M_i;q)_\infty}{(\prod_{i=1}^{r+1} M_i,b,b/a\prod_{i=1}^{r+1} M_i;q)_\infty} - \frac{1}{(\prod_{i=1}^{r+1} M_i;q)_\infty}. \tag{3.13}$$

Proof. Using the theorem 3.1, we can easily know

$$\sum_{n=1}^{\infty} \frac{(a;q)_n}{(q,b;q)_n}\Big(\frac{b}{a\prod_{i=1}^{r+1} M_i}\Big)^n \left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,q^n\right)\right|$$

converges absolutely.

Letting $z = q^n$ in (1.1) gives

$$\left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,q^n\right)\right| \leqslant \frac{1}{(q^n\prod_{i=1}^{r+1} M_i;q)_\infty}, \tag{3.14}$$

where $n = 1,2,\cdots$.

It is obvious that, under the conditions of the theorem,

$$\frac{(a;q)_n}{(q,b;q)_n}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n > 0, \quad n = 1,2,\cdots.$$

Multiplying both sides of (3.14) by

$$\frac{(a;q)_n}{(q,b;q)_n}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n$$

gets

$$\frac{(a;q)_n}{(q,b;q)_n}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n \left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,q^n\right)\right| \leqslant \frac{(a;q)_n}{(q,b;q)_n(q^n\prod_{i=1}^{r+1} M_i;q)_\infty}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n. \tag{3.15}$$

Hence,

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(a;q)_n}{(q,b;q)_n}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n \left|{}_{r+1}\phi_r\left(\begin{matrix} a_1,a_2,\cdots,a_{r+1} \\ b_1,b_2,\cdots,b_r \end{matrix};q,q^n\right)\right| \\ &\leqslant \sum_{n=1}^{\infty} \frac{(a;q)_n}{(q,b;q)_n(q^n\prod_{i=1}^{r+1} M_i;q)_\infty}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n \\ &= \sum_{n=0}^{\infty} \frac{(a;q)_n}{(q,b;q)_n(q^n\prod_{i=1}^{r+1} M_i;q)_\infty}\left(\frac{b}{a\prod_{i=1}^{r+1} M_i}\right)^n - \frac{1}{(\prod_{i=1}^{r+1} M_i;q)_\infty}.\end{aligned} \tag{3.16}$$

Employing the q-Gauss sum (1.6) obtains

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(a;q)_n}{(q,b;q)_n(q^n\prod_{i=1}^{r+1}M_i;q)_\infty}\left(\frac{b}{a\prod_{i=1}^{r+1}M_i}\right)^n\\ &=\frac{1}{(\prod_{i=1}^{r+1}M_i;q)_\infty}\sum_{n=0}^{\infty}\frac{(a,\prod_{i=1}^{r+1}M_i;q)_n}{(q,b;q)_n}\left(\frac{b}{a\prod_{i=1}^{r+1}M_i}\right)^n\\ &=\frac{(b,b/\prod_{i=1}^{r+1}M_i;q)_\infty}{(\prod_{i=1}^{r+1}M_i,b,b/a\prod_{i=1}^{r+1}M_i;q)_\infty}. \qquad (3.17)\end{aligned}$$

Substituting (3.17) into (3.16), we get (3.13). □

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