Journal of
Mathematical
Inequalities
Volume 1, Number 3 (2007), 347–353

GENERAL FRAMEWORK FOR SENSITIVITY ANALYSIS TO A CLASS OF NONLINEAR RELAXED COCOERCIVE QUASIVARIATIONAL INCLUSIONS

RAM U. VERMA

(*communicated by G. Anastassiou*)

Abstract. Based on the generalized resolvent operator technique, sensitivity analysis results for relaxed cocoercive quasivariational inclusions are obtained, which generalize a broad range of sensitivity analysis results, including strongly monotone quasivariational inclusions. Generalized resolvent operator technique is constructed on the emergence of the new notion of A-monotonicity — a significant generalization to the notion of maximal monotonicity. The notion of A-monotonicity is also referred to as A-maximal monotonicity in literature. Furthermore, the relaxed cocoercivity is illustrated by some examples.

1. Introduction and Preliminaries

Resolvent operator techniques have been applied to studying nonlinear variational inequality/inclusion problems, including problems from model equilibria in economics, optimization and control theory, operations research, transportation network modeling, mathematical programming, and engineering sciences. Recently, Agarwal et al. [1] applied the resolvent operator technique to studying sensitivity analysis for qusivariational inclusions involving strongly monotone mappings, without any differentiability assumptions on solution variables with respect to perturbation parameters. The aim of this paper is to present the sensitivity analysis for the relaxed cocoercive quasivariational inclusions based on an application of the generalized resolvent operator technique. The framework of the generalized resolvent operator technique heavily relies on A-monotonicity — just recently introduced by the author [9] — is more general the existing general class of maximal monotone mappings, so it further empowers the resolvent operator technique. Furthermore, the class of A-monotone mappings generalizes recently introduced and studied notion of the H-monotone mappings by Fang and Huang [2] as well. For more details, we recommend [1-12].

Let X denote a real Hilbert space with the norm $\|.\|$ and inner product $\langle .,. \rangle$ on X.

Mathematics subject classification (2000): 49J40, 47H05.

Key words and phrases: Sensitivity analysis, relaxed cocoercive quasivariational inclusions, maximal monotone mapping, relaxed maximal monotone mapping, A-monotone mapping, H-monotone mappings, generalized resolvent operator technique.

.

Paper JMI-01-30

DEFINITION 1. [9] Let $A : X \to X$ be a nonlinear mapping on X and $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be A-monotone if M is (m)-relaxed monotone and $A + \rho M$ is maximal monotone for $\rho > 0$.

DEFINITION 2. [2] Let $H : X \to X$ be a nonlinear mapping on X and $M : X \to 2^X$ be a multivalued mapping on X. The map M is said to be H-monotone if $(H + \rho M)(X) = X$ for $\rho > 0$.

PROPOSITION 1. *Let $A : X \to X$ be an (r)-strongly monotone single valued mapping and let $M : X \to 2^X$ be an A-monotone mapping. Then M is maximal monotone.*

PROPOSITION 2. *Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-monotone mapping. Then the operator $(A + \rho M)^{-1}$ is single-valued.*

This leads to the generalized definition of the resolvent operator:

DEFINITION 3. [9] Let $A : X \to X$ be an (r)-strongly monotone mapping and let $M : X \to 2^X$ be an A-monotone mapping. Then the generalized resolvent operator $J^A_{M,\rho} : X \to X$ is defined by

$$J^A_{M,\rho}(u) = (A + \rho M)^{-1}(u).$$

Let $N : X \times L \to X$ be a nonlinear mapping and $M : X \times L \to 2^X$ be an A-monotone mapping with respect to first variable, where L is a nonempty open subset of X. Then the problem of finding an element $u \in X$ such that

$$0 \in N(u, \lambda) + M(u, \lambda), \tag{1}$$

where $\lambda \in L$ is the perturbation parameter, is called a class of generalized relaxed cocoercive variational inclusion (abbreviated RCVI) problems.

Next, a special case of the $RCVI\,(1)$ problem is: determine an element $u \in X$ such that

$$0 \in S(u, \lambda) + T(u, \lambda) + M(u, \lambda), \tag{2}$$

where $N(u, v, \lambda) = S(u, \lambda) + T(v, \lambda)$ for all $u, v \in X$, and $S, T : X \times L \to H$ are two nonlinear mappings. If $S = 0$ in (2), then (2) is equivalent to: find an element $u \in X$ such that

$$0 \in T(u, \lambda) + M(u, \lambda). \tag{3}$$

The solvability of the $RCVI$ problem (1) depends on the equivalence between (1) and the problem of finding the fixed point of the associated generalized resolvent operator.

Note that if M is A-monotone, then the generalized resolvent operator $J^M_{\rho,A}$ in first argument is defined by

$$J^{M(.,y)}_{\rho,A}(u) = (A + \rho M(.,y))^{-1}(u)\, \forall u \in X, \tag{4}$$

where $\rho > 0$, and A is an (r)-strongly monotone mapping.

LEMMA 1. *An element $u \in X$ is a solution to (1) if and only if there is an $u \in X$ such that*

$$u = G(u,\lambda) := J_{\rho,A}^{M(.,\lambda)}(A(u) - \rho N(u,\lambda)), \tag{5}$$

where $J_{\rho,A}^{M(.,\lambda)} = (A + \rho M(.,\lambda))^{-1}$ *and* $\rho > 0$.

Proof. The proof follows from the definition of the generalized resolvent operator. If $u \in X$ such that

$$u = J_{\rho,A}^{M(.,\lambda)}(A(u) - \rho N(u,\lambda)),$$

then we have

$$(A(u) - \rho N(u,\lambda)) \in (A + \rho M(.,\lambda))(u),$$

so,

$$0 \in N(u,\lambda) + M(u,\lambda),$$

that is, u is a solution to (1). □

DEFINITION 4. A mapping $T : X \times L \to X$ is said to be (m)-relaxed monotone in the first argument if there exists a positive constant m such that

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geqslant (-m)\|x - y\|^2 \, \forall (x,y,\lambda) \in X \times L.$$

DEFINITION 5. A mapping $T : X \times L \to X$ is said to be $(s) - cocoercive$ in the first argument if there exists a positive constant s such that

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geqslant (s)\|T(x) - T(y)\|^2 \, \forall (x,y,\lambda) \in H \times L.$$

DEFINITION 6. A mapping $T : X \times L \to X$ is said to be (m)-relaxed cocoercive in the first argument if there exists a positive constant m such that

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geqslant (-m)\|T(x) - T(y)\|^2 \, \forall (x,\lambda) \in X \times L.$$

DEFINITION 7. A mapping $T : X \times L \to X$ is said to be $(\gamma,m)-relaxed$ cocoercive in the first argument if there exist positive constants γ and m such that

$$\langle T(x,\lambda) - T(y,\lambda), x - y\rangle \geqslant (-m)\|T(x) - T(y)\|^2 + \gamma\|x - y\|^2 \, \forall (x,y,u,\lambda) \in H \times L.$$

EXAMPLE 1. Consider a nonexpansive mapping $T : X \to X$ on X. If we set $A = I - T$, then A is $(\frac{1}{2})$-cocoercive, where I is the identity.

EXAMPLE 2. Consider a projection $P : X \to C$, where C is a nonempty closed convex subset of X. Then P is (1)-cocoercive since P is nonexpansive.

EXAMPLE 3. Consider an (r)-strongly monotone (and hence (r)-expanding)) mapping $T : X \to X$ on X. Then T is $(1, r + r^2)$-relaxed cocoercive. For all $u, v \in X$, we have

$$\langle T(x) - T(y), x - y\rangle \geqslant (-1)\|T(x) - T(y)\|^2 + (r + r^2)\|x - y\|^2.$$

Clearly, every (m)-cocoercive mapping is (m)-relaxed cocoercive, while each (r)-strongly monotone mapping is $(1, r + r^2)$-relaxed cocoercive.

DEFINITION 8. A mapping $T : X \times L \to X$ is said to be (μ)-Lipschitz continuous in the first argument if there exists a positive constant μ such that

$$\|T(x,\lambda) - T(y,\lambda)\| \leqslant \mu\|x - y\| \; \forall (x,y,\lambda) \in X \times X \times L.$$

2. Quasivariational Inclusions

This section deals with the main results on the sensitivity analysis and its specializations in literature.

LEMMA 2. *[9] Let $A : X \to X$ be (r)-strongly monotone and $M : X \times L \to 2^X$ be A-monotone. Then the generalized resolvent operator $J_\rho^{M(.,\lambda)} : X \times L \to X$ is $(\frac{1}{r-\rho m})$-Lipschitz continuous for $0 < \rho < \frac{r}{m}$.*

THEOREM 1. *Let X be a real Hilbert space, and let $N : X \times X \times L \to X$ be (γ,α) − relaxed cocoercive and (β)-Lipschitz continuous in the first variable. Let $A : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \times L \to 2^X$ be A-monotone. Then*

$$\|G(u,\lambda) - G(v,\lambda)\| \leqslant \theta\|u - v\| \; \forall (u,v,\lambda) \in X \times X \times L, \tag{6}$$

where

$$\theta = \frac{1}{r-\rho m}\sqrt{s^2 - 2\rho\alpha + \rho^2\beta^2 + 2\rho\gamma\beta^2} < 1,$$

$$\left|\rho - \frac{r(1-m) - \gamma\beta^2}{\beta^2 - m^2}\right| < \frac{\sqrt{[r(1-m) - \gamma\beta^2]^2 - (s^2 - r^2)(\beta^2 - m^2)}}{\beta^2 - m^2},$$

$$r > \frac{1}{(1-m)}[\gamma\beta^2) + \sqrt{(s^2 - r^2)(\beta^2 - m^2)}], \; \beta > m,$$

$m < 1, | r |< s.$

Consequently, for each $\lambda \in L$, the mapping $G(u,\lambda)$ in light of (6) has a unique fixed point $z(\lambda)$, and hence, $z(\lambda)$ is a unique solution to (1). Thus, we have

$$G(z(\lambda),\lambda) = z(\lambda).$$

Proof. For any element $(u,\lambda) \in X \times L$, we have

$$G(u,\lambda) = J_{\rho,A}^{M(.,\lambda)}(A(u) - \rho N(u,\lambda)),$$

$$G(v,\lambda) = J_{\rho,A}^{M(.,\lambda)}(A(v) - \rho N(v,\lambda)).$$

It follows from Lemma 2 that

$$\begin{aligned}\|G(u,\lambda)-G(v,\lambda)\| &= \|J_{\rho,A}^{M(.,\lambda)}(A(u)-\rho N(u,\lambda))-J_{\rho,A}^{M(.,\lambda)}(A(v)-\rho N(v,\lambda))\| \\ &\leqslant \frac{1}{r-\rho m}\|A(u)-A(v)-\rho(N(u,\lambda))-N(v,\lambda))\|.\end{aligned}$$

The (γ,α)-relaxed cocoercivity and (β)-Lipschitz continuity of N in the first argument imply that

$$\begin{aligned}\|A(u)-A(v)-&\rho(N(u,\lambda))-N(v,\lambda))\|^2 \\ &= \|A(u)-A(v)\|^2-2\rho\langle N(u,\lambda)-N(v,\lambda),A(u)-A(v)\rangle \\ &\quad +\rho^2\|N(u,\lambda)-N(v,\lambda)\|^2 \\ &\leqslant (s^2-2\rho\alpha+\rho^2\beta^2+2\rho\gamma\beta^2)\|u-v\|^2.\end{aligned}$$

In light of above arguments, we infer that

$$\|G(u,\lambda)-G(v,\lambda)\| \leqslant \theta\|u-v\|, \tag{7}$$

where

$$\theta = \frac{1}{r-\rho m}\sqrt{s^2-2\rho r+\rho^2\beta^2+2\rho\gamma\beta^2}.$$

Since $\theta < 1,$ it concludes the proof. □

COROLLARY 1. *Let X be a real Hilbert space, and let $N : X\times L \to X$ be (γ,α) − relaxed cocoercive and (β)-Lipschitz continuous in the first variable. Let $H : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X\times L \to 2^X$ be A-monotone. Then*

$$\|G(u,\lambda)-G(v,\lambda)\| \leqslant \theta\|u-v\|\,\forall(u,v,\lambda)\in X\times X\times L, \tag{8}$$

where

$$\theta = \frac{1}{r}\sqrt{s^2-2\rho\alpha+\rho^2\beta^2+2\rho\gamma\beta^2} < 1,$$

$$\left|\rho-\frac{r-\gamma\beta^2}{\beta^2}\right| < \frac{\sqrt{[r-\gamma\beta^2]^2-(s^2-r^2)(\beta^2)}}{\beta^2},$$

$$r > [\gamma\beta^2) + \sqrt{(s^2-r^2)\beta^2]},\ s > r.$$

Consequently, for each $\lambda\in L,$ the mapping $G(u,\lambda)$ in light of (8) has a unique fixed point $z(\lambda)$, and hence, $z(\lambda)$ is a unique solution to (1). Thus, we have

$$G(z(\lambda),\lambda) = z(\lambda).$$

If the mappings $\lambda \to N(u,v,\lambda)$ and $\lambda \to J_\rho^{M(.,\lambda)}(w)$ both are continuous (or Lipschitz continuous) from L to $X,$ then the solution $z(\lambda)$ of (1) is continuous (or Lipschitz continuous) from L to $X.$

THEOREM 2. *Let X be a real Hilbert space, and let $N : X \times L \to X$ be (γ, α) − relaxed cocoercive and (β)-Lipschitz continuous in the first variable. Let $A : X \to X$ be (r)-strongly monotone and (s)-Lipschitz continuous, and let $M : X \times L \to 2^X$ be A-monotone. Let the mappings $\lambda \to N(u, \lambda)$ and $\lambda \to J_{\rho}^{M(.,\lambda)}(v)$ are continuous (or Lipschitz continuous) from L to X for all $u, v \in X$. Then the solution $z(\lambda)$ of (1) is continuous (or Lipschitz continuous) from $L \to X$.*

Proof. From the hypotheses of the theorem, for any $\lambda, \lambda* \in L$, we have

$$\begin{aligned}
\|z(\lambda) - z(\lambda^*)\| &= \|G(z(\lambda), \lambda) - G(z(\lambda^*), \lambda^*)\| \\
&\leqslant \|G(z(\lambda), \lambda) - G(z(\lambda^*), \lambda)\| + \|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\| \\
&\leqslant \theta\|z(\lambda) - z(\lambda^*)\| + \|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\|.
\end{aligned}$$

It follows that

$$\begin{aligned}
\|G(z(\lambda^*), \lambda) - G(z(\lambda^*), \lambda^*)\| &= \|J_{\rho,A}^{M(.,\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda)) \\
&\quad -J_{\rho,A}^{M(.,\lambda^*)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*))\| \\
&\leqslant \|J_{\rho,A}^{M(.,\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda)) \\
&\quad -J_{\rho,A}^{M(.,\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*))\| \\
&\quad +\|J_{\rho,A}^{M(.,\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*)) \\
&\quad -J_{\rho,A}^{M(.,\lambda^*)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*))\| \\
&\leqslant \frac{\rho}{(r - \rho m)}\|N(z(\lambda^*), \lambda) - N(z(\lambda^*), \lambda^*)\| \\
&\quad +\|J_{\rho,A}^{M(.,\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*)) \\
&\quad -J_{\rho,A}^{M(.,\lambda^*)}(Az(\lambda^*) - \rho N(z(\lambda^*), \lambda^*))\|.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\|z(\lambda) - z(\lambda^*)\| &\leqslant \frac{\rho}{(1-\theta)(r - \rho m)}\|N(z(\lambda^*), \lambda) - N(z(\lambda^*), \lambda^*)\| \\
&\quad +\frac{1}{1-\theta}\|J_{\rho,A}^{M(.\lambda)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*)) \\
&\quad -J_{\rho,A}^{M(.,\lambda^*)}(A(z(\lambda^*)) - \rho N(z(\lambda^*), \lambda^*))\|.
\end{aligned}$$

This completes the proof. □

REFERENCES

[1] R. P. AGARWAL, Y. J. CHO AND N. J. HUANG, *Sensitivity analysis for strongly nonlinear quasivariational inclusions*, Applied Mathematics Letters **13**(2000), 19–24.

[2] Y. P. FANG AND N. J. HUANG, *H -monotone operators and system of variational inclusions*, Communications on Applied Nonlinear Analysis **11**(1)(2004), 93–101.

[3] H. IIDUKA AND W. TAKAHASHI, *Strong convergence theorem by a hybrid method for nonlinear mappings of nonexpansive and monotone type and applications*, Advances in Nonlinear Variational Inequalities **9**(1)(2006), 1–9.

[4] J. KYPARISIS, *Sensitivity analysis framework for variational Inequalities*, Mathematical Programming **38**(1987), 203–213.

[5] A. MOUDAFI, *Mixed equilibrium problems: Sensitivity analysis and algorithmic aspect*, Computers and Mathematics with Applications **44**(2002), 1099–1108.

[6] R. L. TOBIN, *Sensitivity analysis for variational inequalities*, Journal of Optimization Theory and Applications **48**(1)(1986), 191–204.

[7] R. U. VERMA, *Nonlinear variational and constrained hemivariational inequalities involving relaxed operators*, ZAMM: Z. Angew. Math. Mech. **77**(5)(1997), 387–391.

[8] R. U. VERMA, *Generalized system for relaxed cocoercive variational inequalities and projection methods*, Journal of Optimization Theory and Applications **121**(1)(2004), 203–210.

[9] R. U. VERMA, *A-monotonicity and applications to nonlinear variational inclusion problems*, Journal of Applied Mathematics and Stochastic Analysis **17**(2)(2004), 193–195.

[10] R. U. VERMA, *A-monotonicity and its role in nonlinear variational inclusions*, Journal of Optimization Theory and Applications **129**(3)(2006), 457–467.

[11] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications* II/B, Springer-Verlag, New York, New York, 1990.

[12] E. ZEIDLER, *Nonlinear Functional Analysis and its Applications* I, Springer-Verlag, New York, New York, 1986.

(Received December 17, 2006)

Ram U. Verma
The University of Toledo
Department of Mathematics
Toledo
Ohio 43606
USA
e-mail: verma99@msn.com