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ON HOMOMORPHISMS BETWEEN C^*–TERNARY ALGEBRAS

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(*communicated by Th. Rassias*)

Abstract. C. Park in his paper [C. Park, *Isomorphisms between C^*-ternary algebras*, J. Math. Anal. Appl. 327 (2007) 101–115.], has proved the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the following Cauchy-Jensen additive mappings:

$$f\left(\frac{x+y}{2}+z\right)+f\left(\frac{x-y}{2}+z\right)=f(x)+2f(z), \tag{0.1}$$

$$f\left(\frac{x+y}{2}+z\right)-f\left(\frac{x-y}{2}+z\right)=f(y), \tag{0.2}$$

$$2f\left(\frac{x+y}{2}+z\right)=f(x)+f(y)+2f(z). \tag{0.3}$$

These are applied to investigate homomorphisms between C^*-ternary algebras. In this paper we prove the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras and of derivations on C^*-ternary algebras for the linear combinations of the Cauchy-Jensen additive mappings (0.1), (0.2) and (0.3).

1. Introduction and preliminaries

In 1940, S. M. Ulam [27] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group G and a metric group G' with metric $\rho(\cdot,\cdot)$. Given $\epsilon>0$, does there exist a $\delta>0$ such that if $f:G\to G'$ satisfies $\rho(f(xy),f(x)f(y))<\delta$ for all $x,y\in G$, then a homomorphism $h:G\to G'$ exists with $\rho(f(x),h(x))<\epsilon$ for all $x\in G$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f:E\to E'$, where E and E' are Banach spaces and f satisfies *Hyers inequality*

$$\|f(x+y)-f(x)-f(y)\|\leqslant\epsilon$$

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for all $x, y \in E$. It was shown that the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and that $L : E \to E'$ is the unique additive mapping satisfying

$$\|f(x) - L(x)\| \leqslant \epsilon.$$

In 1978, Th. M. Rassias [19] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM. (Th. M. Rassias) *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \qquad (\heartsuit)$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p \qquad (\diamondsuit)$$

for all $x \in E$. If $p < 0$ then inequality $(\heartsuit)$ holds for $x, y \neq 0$ and $(\diamondsuit)$ for $x \neq 0$. Also, if the mapping $t \to f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$, then L is $\mathbb{R}$-linear.

In 1990, Th. M. Rassias [20] during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$. In 1991, Z. Gajda [5] following the same approach as in Th. M. Rassias [19], gave an affirmative solution to this question for $p > 1$. It was shown by Z. Gajda [5], as well as by Th. M. Rassias and P. Šemrl [25] that one cannot prove a Th. M. Rassias' type theorem when $p = 1$. The counterexamples of Z. Gajda [5], as well as of Th. M. Rassias and P. Šemrl [25] have stimulated several mathematicians to invent new definitions of *approximately additive* or *approximately linear* mappings, cf. P. Găvruta [6], S. Jung [11], who among others studied the Hyers-Ulam-Rassias stability of functional equations. The inequality ($\heartsuit$) that was introduced for the first time by Th. M. Rassias [19] for the stability of the linear mapping between Banach spaces provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as *Hyers-Ulam-Rassias stability* of functional equations (cf. the books of P. Czerwik [4], D.H. Hyers, G. Isac and Th. M. Rassias [8]).

P. Găvruta [6] provided a further generalization of Th. M. Rassias' Theorem. In 1996, G. Isac and Th. M. Rassias [10] applied the Hyers-Ulam-Rassias stability theory to prove fixed point theorems and study some new applications in Nonlinear Analysis. In [9], D.H. Hyers, G. Isac and Th.M. Rassias studied the asymptoticity aspect of Hyers-Ulam stability of mappings. During past few years several mathematicians have published on various generalizations and applications of Hyers-Ulam stability and Hyers-Ulam-Rassias stability to a number of functional equations and mappings, for example : quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded nth differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation, Navier-Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: C. Park [12]–[17], Th. M. Rassias [21]–[24], F. Skof [26].

A. Prastaro and Th. M. Rassias [18] introduced for the first time the Hyers-Ulam-Rassias stability approach for the study of Navier-Stokes equation.

Following the terminology of [1], a non-empty set G with a ternary operation $[\cdot,\cdot,\cdot]: G\times G\times G \to G$ is called a *ternary groupoid* and is denoted by $(G,[\cdot,\cdot,\cdot])$. The ternary groupoid $(G,[\cdot,\cdot,\cdot])$ is called *commutative* if $[x_1,x_2,x_3] = [x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}]$ for all $x_1,x_2,x_3 \in G$ and all permutations σ of $\{1,2,3\}$.

If a binary operation $\circ$ is defined on G such that $[x,y,z] = (x\circ y)\circ z$ for all $x,y,z\in G$, then we say that $[\cdot,\cdot,\cdot]$ is derived from $\circ$. We say that $(G,[\cdot,\cdot,\cdot])$ is a *ternary semigroup* if the operation $[\cdot,\cdot,\cdot]$ is *associative*, i.e., if $[[x,y,z],u,v] = [x,[y,z,u],v] = [x,y,[z,u,v]]$ holds for all $x,y,z,u,v\in G$ (see [3]).

A C^*-ternary algebra is a complex Banach space A, equipped with a ternary product $(x,y,z)\mapsto [x,y,z]$ of A^3 into A, which is $\mathbb{C}$-linear in the outer variables, conjugate $\mathbb{C}$-linear in the middle variable, and associative in the sense that $[x,y,[z,w,v]] = [x,[w,z,y],v] = [[x,y,z],w,v]$, and satisfies $\|[x,y,z]\| \leqslant \|x\|\cdot\|y\|\cdot\|z\|$ and $\|[x,x,x]\| = \|x\|^3$ (see [1, 28]). Every left Hilbert C^*-module is a C^*-ternary algebra via the ternary product $[x,y,z] := \langle x,y\rangle z$.

If a C^*-ternary algebra $(A,[\cdot,\cdot,\cdot])$ has an identity, i.e., an element $e\in A$ such that $x = [x,e,e] = [e,e,x]$ for all $x\in A$, then it is routine to verify that A, endowed with $x\circ y := [x,e,y]$ and $x^* := [e,x,e]$, is a unital C^*-algebra. Conversely, if $(A,\circ)$ is a unital C^*-algebra, then $[x,y,z] := x\circ y^*\circ z$ makes A into a C^*-ternary algebra.

A $\mathbb{C}$-linear mapping $H: A\to B$ is called a *C^*-ternary algebra homomorphism* if

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z\in A$. If, in addition, the mapping H is bijective, then the mapping $H: A\to B$ is called a *C^*-ternary algebra isomorphism.* A $\mathbb{C}$-linear mapping $\delta: A\to A$ is called a *C^*-ternary derivation* if

$$\delta([x,y,z]) = [\delta(x),y,z] + [x,\delta(y),z] + [x,y,\delta(z)]$$

for all $x,y,z\in A$ (see [1]).

2. Stability of homomorphisms in C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|.\|_A$ and that B a C^*-ternary algebra with norm $\|.\|_B$.

Let $(\alpha,\beta,\gamma)\in\mathbb{R}^3$. For a given mapping $f:A\to B$, we define

$$\begin{aligned}
C_\mu f(x,y,z) &:= f\Big(\frac{\mu x+\mu y}{2}+\mu z\Big)+f\Big(\frac{\mu x-\mu y}{2}+\mu z\Big)-\mu f(x)-2\mu f(z),\\
D_\mu f(x,y,z) &:= f\Big(\frac{\mu x+\mu y}{2}+\mu z\Big)-f\Big(\frac{\mu x-\mu y}{2}+\mu z\Big)-\mu f(y),\\
E_\mu f(x,y,z) &:= 2f\Big(\frac{\mu x+\mu y}{2}+\mu z\Big)-\mu f(x)-\mu f(y)-2\mu f(z),\\
\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z) &:= \alpha C_\mu f(x,y,z)+\beta D_\mu f(x,y,z)+\gamma E_\mu f(x,y,z)
\end{aligned}$$

for all $\mu\in\mathbb{T}^1:=\{\,\lambda\in\mathbb{C}:|\lambda|=1\,\}$ and all $x,y,z\in A$.

We will use the following lemma in this paper:

LEMMA 2.1. [14] *Let X and Y be linear spaces and let $f:X\to Y$ be an additive mapping such that $f(\mu x)=\mu f(x)$ for all $x\in X$ and all $\mu\in\mathbb{T}^1$. Then the mapping f is $\mathbb{C}$-linear.*

LEMMA 2.2. *Let X and Y be linear spaces and let α,β,γ be real numbers such that $\delta:=\alpha+\beta+2\gamma\neq 0$. Suppose that $f:X\to Y$ is a mapping satisfying*

$$\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z)=0 \tag{2.1}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y,z\in X$. Then the mapping $f:X\to Y$ is $\mathbb{C}$-linear.

Proof. Letting $x=y=z=0$ in (2.1), we get $f(0)=0$. Letting $z=0$ and replacing x and y by $2x$ and $2y$ in (2.1), respectively, we get

$$\begin{aligned}
(\alpha+\gamma)\Big[f(\mu x+\mu y)-\mu f(2x)\Big]+(\beta+\gamma)\Big[f(\mu x+\mu y)-\mu f(2y)\Big]&\\
+(\alpha-\beta)f(\mu x-\mu y)=0&
\end{aligned} \tag{2.2}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y\in X$. Letting $y=x$ in (2.2), we get $f(2\mu x)=\mu f(2x)$ for all $\mu\in\mathbb{T}^1$ and all $x\in X$. So

$$f(\mu x)=\mu f(x) \tag{2.3}$$

for all $\mu\in\mathbb{T}^1$ and all $x\in X$. Letting $x=y=z$ in (2.1), and using (2.3), we get $f(2x)=2f(x)$ for all $x\in X$. Now, we show that $f(x+y)=f(x)+f(y)$ for all $x,y\in X$.

We have two cases:

Case I. $\alpha=\beta$.

Replacing x and y by y and x in (2.2), we get

$$(\alpha+\gamma)\Big[f(\mu x+\mu y)-\mu f(2y)\Big]+(\beta+\gamma)\Big[f(\mu x+\mu y)-\mu f(2x)\Big]=0 \tag{2.4}$$

for all $\mu\in\mathbb{T}^1$ and all $x,y\in X$. Adding (2.2) to (2.4) and using (2.3), we get

$$2f(x+y)=f(2x)+f(2y)$$

for all $x,y\in X$. Since $f(2x)=2f(x)$ for all $x\in X$, then we get from the last equation that $f(x+y)=f(x)+f(y)$ for all $x,y\in X$.

Case II. $\alpha \neq \beta$.

Letting $x = 0$ in (2.2), we infer that the mapping f is odd. Replacing x and y by y and x in (2.2), we get

$$(\alpha+\gamma)\Big[f(\mu x+\mu y)-\mu f(2y)\Big]+(\beta+\gamma)\Big[f(\mu x+\mu y)-\mu f(2x)\Big] + (\alpha-\beta)f(\mu y-\mu x)=0 \tag{2.5}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X$. Adding (2.2) to (2.5) and using (2.3), we get

$$2f(x+y) = f(2x) + f(2y)$$

for all $x, y \in X$. So $f(x+y) = f(x) + f(y)$ for all $x, y \in X$.

Hence by Lemma 2.1 the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

We investigate the Hyers-Ulam-Rassias stability of homomorphisms in C^*-ternary algebras for the functional equation $\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z) = 0$.

THEOREM 2.3. *Let* α, β, γ *be real numbers with* $\delta := \alpha + \beta + 2\gamma \neq 0$, *and let* $\varphi : A^3 \to [0,\infty)$ *and* $\psi : A^3 \to [0,\infty)$ *be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0, \tag{2.6}$$

$$\lim_{n\to\infty} \frac{1}{8^n}\psi(2^n x, 2^n y, 2^n z) = 0 \tag{2.7}$$

for all $x, y, z \in A$. *Suppose that* $f : A \to B$ *is a mapping satisfying*

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z)\|_B \leqslant \varphi(x,y,z), \tag{2.8}$$

$$\Big\|f([x,y,z]) - [f(x), f(y), f(z)]\Big\|_B \leqslant \psi(x,y,z) \tag{2.9}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$. *Then there exists a unique* C^**-ternary algebra homomorphism* $H : A \to B$ *such that*

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{1}{2|\delta|}\widetilde{\varphi}(x) \tag{2.10}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.8), we get

$$\|\delta f(2x) - 2\delta f(x)\|_B \leqslant \varphi(x,x,x) \tag{2.11}$$

for all $x \in A$. If we replace x by $2^n x$ in (2.11) and divide both sides of (2.11) by $|\delta|2^{n+1}$, we get

$$\Big\|\frac{1}{2^{n+1}}f(2^{n+1}x) - \frac{1}{2^n}f(2^n x)\Big\|_B \leqslant \frac{1}{|\delta|2^{n+1}}\varphi(2^n x, 2^n x, 2^n x)$$

for all $x \in A$ and all non-negative integers n. Hence

$$\begin{aligned}\Big\|\frac{1}{2^{n+1}}f(2^{n+1}x)-\frac{1}{2^m}f(2^m x)\Big\|_B &= \Big\|\sum_{k=m}^{n}\Big[\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^k}f(2^k x)\Big]\Big\|_B\\ &\leqslant \sum_{k=m}^{n}\Big\|\frac{1}{2^{k+1}}f(2^{k+1}x)-\frac{1}{2^k}f(2^k x)\Big\|_B \\ &\leqslant \frac{1}{2|\delta|}\sum_{k=m}^{n}\frac{1}{2^k}\varphi(2^k x,2^k x,2^k x)\end{aligned} \tag{2.12}$$

for all $x \in A$ and all non-negative integers $n \geqslant m \geqslant 0$. It follows from (2.6) and (2.12) that the sequence $\{\frac{1}{2^n}f(2^n x)\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{\frac{1}{2^n}f(2^n x)\}$ converges. Thus one can define the mapping $H: A \to B$ by

$$H(x) := \lim_{n\to\infty}\frac{1}{2^n}f(2^n x)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.12) we get (2.10).

It follows from (2.6) that

$$\begin{aligned}\Big\|\Delta^{\mu}_{\alpha,\beta,\gamma}H(x,y,z)\Big\|_B &= \lim_{n\to\infty}\frac{1}{2^n}\Big\|\Delta^{\mu}_{\alpha,\beta,\gamma}f(2^n x,2^n y,2^n z)\Big\|_B\\ &\leqslant \lim_{n\to\infty}\frac{1}{2^n}\varphi(2^n x,2^n y,2^n z)=0\end{aligned}$$

for all $x,y,z \in A$. So $\Delta^{\mu}_{\alpha,\beta,\gamma}H(x,y,z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. By Lemma 2.1 the mapping $H: A \to B$ is $\mathbb{C}$-linear.

It follows from (2.7) and (2.9) that

$$\begin{aligned}&\Big\|H([x,y,z])-[H(x),H(y),H(z)]\Big\|_B\\ &=\lim_{n\to\infty}\frac{1}{8^n}\Big\|f([2^n x,2^n y,2^n z])-[f(2^n x),f(2^n y),f(2^n z)]\Big\|_B\\ &\leqslant \lim_{n\to\infty}\frac{1}{8^n}\psi(2^n x,2^n y,2^n z)=0\end{aligned}$$

for all $x,y,z \in A$. Therefore

$$H([x,y,z]) = [H(x),H(y),H(z)]$$

for all $x,y,z \in A$. Therefore the mapping $H: A \to B$ is a C^*-ternary algebra homomorphism.

Now, let $Q: A \to B$ be another C^*-ternary algebra homomorphism satisfying

(2.10). Then we have from (2.6) that

$$\begin{aligned}\|H(x)-Q(x)\|_B &= \lim_{n\to\infty}\frac{1}{2^n}\|f(2^nx)-Q(2^nx)\|_B \\ &\leqslant \frac{1}{2|\delta|}\lim_{n\to\infty}\frac{1}{2^n}\widetilde{\varphi}(2^nx) \\ &= \frac{1}{2|\delta|}\lim_{n\to\infty}\sum_{k=n}^{\infty}\frac{1}{2^k}\varphi(2^kx,2^kx,2^kx)=0\end{aligned}$$

for all $x \in A$. So $H(x) = Q(x)$ for all $x \in A$. This proves the uniqueness of H. Thus the mapping $H : A \to B$ is a unique C^*-ternary algebra homomorphism satisfying (2.10). □

COROLLARY 2.4. *Let α,β,γ be real numbers and let $\epsilon,\theta,p_1,p_2,p_3,q_1,q_2,q_3$ be non-negative real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $0 < p_1,p_2,p_3 < 1$ and $0 < q_1,q_2,q_3 < 3$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z)\|_B \leqslant \theta(\|x\|_A^{p_1}+\|y\|_A^{p_2}+\|z\|_A^{p_3}), \tag{2.13}$$

$$\Big\|f([x,y,z])-[f(x),f(y),f(z)]\Big\|_B \leqslant \epsilon(\|x\|_A^{q_1}+\|y\|_A^{q_2}+\|z\|_A^{q_3}) \tag{2.14}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\big\|f(x)-H(x)\big\|_B \leqslant \frac{\theta}{|\delta|}\Big\{\frac{1}{2-2^{p_1}}\|x\|_A^{p_1}+\frac{1}{2-2^{p_2}}\|x\|_A^{p_2}+\frac{1}{2-2^{p_3}}\|x\|_A^{p_3}\Big\}$$

for all $x \in A$.

THEOREM 2.5. *Let α,β,γ be real numbers with $\delta := \alpha+\beta+2\gamma \neq 0$, and let $\Phi : A^3 \to [0,\infty)$ and $\Psi : A^3 \to [0,\infty)$ be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty}2^n\Phi\Big(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\Big) < \infty, \quad \lim_{n\to\infty}2^n\Phi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big)=0, \tag{2.15}$$

$$\lim_{n\to\infty}8^n\Psi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big)=0 \tag{2.16}$$

for all $x,y,z \in A$. Suppose that $f : A \to B$ is a mapping satisfying

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z)\|_B \leqslant \Phi(x,y,z), \tag{2.17}$$

$$\Big\|f([x,y,z])-[f(x),f(y),f(z)]\Big\|_B \leqslant \Psi(x,y,z) \tag{2.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\big\|f(x)-H(x)\big\|_B \leqslant \frac{1}{2|\delta|}\widetilde{\Phi}(x) \tag{2.19}$$

for all $x \in A$.

Proof. Letting $\mu = 1$ and $x = y = z$ in (2.17), we get

$$\|\delta f(2x) - 2\delta f(x)\|_B \leqslant \Phi(x,x,x) \tag{2.20}$$

for all $x \in A$. If we replace x by $\frac{x}{2^{n+1}}$ in (2.20) and multiply both sides of (2.20) to $\frac{2^n}{|\delta|}$, we get

$$\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^n f\left(\frac{x}{2^n}\right)\right\|_B \leqslant \frac{2^n}{|\delta|}\Phi\Big(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\Big)$$

for all $x \in A$ and all non-negative integers n. Hence

$$\begin{aligned}\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) - 2^m f\left(\frac{x}{2^m}\right)\right\|_B &= \left\|\sum_{k=m}^{n}\left[2^{k+1}f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right]\right\|_B \\ &\leqslant \sum_{k=m}^{n}\left\|2^{k+1}f\left(\frac{x}{2^{k+1}}\right) - 2^k f\left(\frac{x}{2^k}\right)\right\|_B \\ &\leqslant \frac{1}{2|\delta|}\sum_{k=m}^{n} 2^{k+1}\Phi\Big(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\Big)\end{aligned} \tag{2.21}$$

for all $x \in A$ and all non-negative integers $n \geqslant m \geqslant 0$. It follows from (2.15) and (2.21) that the sequence $\{2^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in A$. Since B is complete, the sequence $\{2^n f(\frac{x}{2^n})\}$ converges. Thus one can define the mapping $H : A \to B$ by

$$H(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \to \infty$ in (2.21) we get (2.19).

The rest of the proof is similar to the proof of Theorem 2.3. □

COROLLARY 2.6. *Let α, β, γ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 3$. Suppose that $f : A \to B$ is a mapping satisfying (2.13) and (2.14). Then there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that*

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{|\delta|}\Big\{\frac{1}{2^{p_1}-2}\|x\|_A^{p_1} + \frac{1}{2^{p_2}-2}\|x\|_A^{p_2} + \frac{1}{2^{p_3}-2}\|x\|_A^{p_3}\Big\}$$

for all $x \in A$.

3. Homomorphisms between C^*-ternary algebras

In this section we improve the results in Theorems 2.3, 2.4, 3.3 and 3.4 of [15].

THEOREM 3.1. *Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_1 > 0$ and $q_i \neq 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying*

$$\big\|\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z)\big\|_B \leqslant \theta\|x\|_A^{p_1}\|y\|_A^{p_2}\|z\|_A^{p_3}, \tag{3.1}$$

$$\left\|f([x,y,z]) - [f(x),f(y),f(z)]\right\|_B \leqslant \epsilon \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} \tag{3.2}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x,y,z \in A$ $(x,y,z \in A\setminus\{0\}$ *when* $q_i < 0$ *for some* $1 \leqslant i \leqslant 3)$. *Then the mapping* $f : A \to B$ *is a* C^**-ternary algebra homomorphism.*

Note that we put $\|.\|_A^0 = 1$.

Proof. Since $p_1 > 0$, then by letting $x = y = z = 0$ in (3.1), we get $f(0) = 0$. Letting $x = 0$ and replacing y by $2y$ in (3.1), we get

$$\begin{aligned} \delta f(\mu y + \mu z) + (\alpha - \beta) f(\mu z - \mu y) \\ - 2\mu(\alpha + \gamma) f(z) - \mu(\beta + \gamma) f(2y) = 0 \end{aligned} \tag{3.3}$$

for all $\mu \in \mathbb{T}^1$ and all $y,z \in A$. Letting $y = z$ in (3.3), we get

$$\delta f(2\mu y) - 2\mu(\alpha + \gamma) f(y) - \mu(\beta + \gamma) f(2y) = 0 \tag{3.4}$$

for all $\mu \in \mathbb{T}^1$ and all $y \in A$. Replacing μ by $-\mu$ in (3.4), we get

$$\delta f(-2\mu y) + 2\mu(\alpha + \gamma) f(y) + \mu(\beta + \gamma) f(2y) = 0 \tag{3.5}$$

for all $\mu \in \mathbb{T}^1$ and all $y \in A$. Adding (3.4) to (3.5), we get that $f(2\mu y) + f(-2\mu y) = 0$ for all $\mu \in \mathbb{T}^1$ and all $y \in A$. So f is odd.

Now, we show that $f(\mu x) = \mu f(x)$ and $f(x+y) = f(x) + f(y)$ for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$.

We have two cases:

(**i**) Let $\alpha + \gamma \neq 0$. Letting $y = 0$ in (3.3), we get $f(\mu z) = \mu f(z)$ for all $\mu \in \mathbb{T}^1$ and all $z \in A$. Therefore it follows from (3.4) that $f(2y) = 2f(y)$ for all $y \in A$. So it follows from (3.3) that

$$\delta f(y+z) + (\alpha - \beta) f(z-y) = 2(\alpha + \gamma) f(z) + 2(\beta + \gamma) f(y) \tag{3.6}$$

for all $y,z \in A$. Replacing y by $-y$ in (3.6) and using the oddness of f, we get

$$\delta f(z-y) + (\alpha - \beta) f(z+y) = 2(\alpha + \gamma) f(z) - 2(\beta + \gamma) f(y) \tag{3.7}$$

for all $y,z \in A$. Adding (3.6) to (3.7), we get

$$f(y+z) + f(z-y) = 2f(z) \tag{3.8}$$

for all $y,z \in A$. Replacing y and z by $\frac{y-z}{2}$ and $\frac{y+z}{2}$ in (3.8), respectively, we get $f(y+z) = f(y) + f(z)$ for all $y,z \in A$.

(**ii**) Let $\alpha + \gamma = 0$. Since $\delta \neq 0$, then $\beta + \gamma \neq 0$. Letting $z = 0$ in (3.3) and using the oddness of f, we get $2f(\mu y) = \mu f(2y)$ for all $\mu \in \mathbb{T}^1$ and all $y \in A$. Hence by letting $\mu = 1$, we get $f(2y) = 2f(y)$ for all $y \in A$. So $f(\mu y) = \mu f(y)$ for all $\mu \in \mathbb{T}^1$ and all $y \in A$. It follows from (3.3) that

$$f(y+z) - f(z-y) = 2f(y) \tag{3.9}$$

for all $y,z \in A$. Replacing y and z by $\frac{y+z}{2}$ and $\frac{y-z}{2}$ in (3.9), respectively, we get $f(y+z) = f(y) + f(z)$ for all $y,z \in A$.

Hence, by Lemma 2.1 the mapping $f : A \to B$ is $\mathbb{C}$-linear.

Without any loss of generality, we may suppose that $q_1 \neq 1$. Let $q_1 > 1$. It follows from (3.2) that

$$\begin{aligned}
&\left\| f([x,y,z]) - [f(x), f(y), f(z)] \right\|_B \\
&= \lim_{n\to\infty} 2^n \left\| f\left(\left[\frac{x}{2^n}, y, z\right]\right) - \left[f\left(\frac{x}{2^n}\right), f(y), f(z)\right] \right\|_B \\
&\leqslant \epsilon \lim_{n\to\infty} \frac{2^n}{2^{nq_1}} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} = 0
\end{aligned}$$

for all $x, y, z \in A$. Therefore

$$f([x,y,z]) = [f(x), f(y), f(z)] \tag{3.10}$$

for all $x,y,z \in A$ $(x,y,z \in A \setminus \{0\}$ when $q_i < 0$ for some $2 \leqslant i \leqslant 3)$. Since $f(0) = 0$, then (3.10) holds for all $x,y,z \in A$ when $q_i < 0$ for some $2 \leqslant i \leqslant 3$. Similarly, for $q_1 < 1$, we get (3.10). So the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. □

THEOREM 3.2. *Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\alpha + \gamma \neq 0$, $p_1 > 0$ and $q_i \neq 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$ $(x,y,z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Note that we put $\|.\|_A^0 = 1$.

Proof. Letting $x = 0$ and replacing y by $2y$ in (3.1), we get

$$\begin{aligned}
&\delta f(\mu y + \mu z) + (\alpha - \beta) f(\mu z - \mu y) \\
&\quad - 2\mu(\alpha+\gamma) f(z) - \mu(\beta+\gamma) f(2y) - \mu(\alpha+\gamma) f(0) = 0
\end{aligned} \tag{3.11}$$

for all $\mu \in \mathbb{T}^1$ and all $y, z \in A$. Replacing μ by $-\mu$ in (3.11), we get

$$\begin{aligned}
&\delta f(-\mu y - \mu z) + (\alpha - \beta) f(-\mu z + \mu y) \\
&\quad + 2\mu(\alpha+\gamma) f(z) + \mu(\beta+\gamma) f(2y) + \mu(\alpha+\gamma) f(0) = 0
\end{aligned} \tag{3.12}$$

for all $\mu \in \mathbb{T}^1$ and all $y, z \in A$. Adding (3.11) to (3.12) and letting $z = 0$ in the obtained equation, we get $f(\mu y) + f(-\mu y) = 0$ for all $\mu \in \mathbb{T}^1$ and all $y \in A$. So the mapping f is odd and $f(0) = 0$. Therefore we obtain (3.3) from (3.11).

It follows from the proof of case (i) of Theorem 3.1 that the mapping $f : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.3. *Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\alpha + \gamma \neq 0$, $p_2 > 0$ and $q_i \neq 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$ $(x,y,z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Note that we put $\|.\|_A^0 = 1$.

Proof. Letting $y=0$ and replacing x by $2x$ in (3.1), we get

$$\begin{aligned}2(\alpha+\gamma)f(\mu x+\mu z)-\mu(\alpha+\gamma)f(2x)\\ -2\mu(\alpha+\gamma)f(z)-\mu(\beta+\gamma)f(0)=0\end{aligned} \tag{3.13}$$

for all $\mu \in \mathbb{T}^1$ and all $x,z \in A$. Replacing μ by $-\mu$ in (3.13), we get

$$\begin{aligned}2(\alpha+\gamma)f(-\mu x-\mu z)+\mu(\alpha+\gamma)f(2x)\\ +2\mu(\alpha+\gamma)f(z)+\mu(\beta+\gamma)f(0)=0\end{aligned} \tag{3.14}$$

for all $\mu \in \mathbb{T}^1$ and all $x,z \in A$. Adding (3.13) to (3.14), we infer that the mapping f is odd and $f(0)=0$. So it follows from (3.13)that

$$2f(\mu x+\mu z)=\mu f(2x)+2\mu f(z) \tag{3.15}$$

for all $\mu \in \mathbb{T}^1$ and all $x,z \in A$. Letting $z=0$ in (3.15), we get $2f(\mu x)=\mu f(2x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So

$$f(2x)=2f(x), \quad f(\mu x)=\mu f(x) \tag{3.16}$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. It follows from (3.15) and (3.16) that $f(x+z)=f(x)+f(z)$ for all $x,z \in A$. Hence, by Lemma 2.1 the mapping $f:A\to B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.4. *Let $\alpha,\beta,\gamma,q_1,q_2,q_3$ be real numbers and let $\epsilon,\theta,p_1,p_2,p_3$ be non-negative real numbers such that $\alpha+\gamma\neq 0$, $p_3>0$ and $q_i\neq 1$ for some $1\leqslant i\leqslant 3$. Suppose that $f:A\to B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x,y,z\in A$ $(x,y,z\in A\setminus\{0\}$ when $q_i<0$ for some $1\leqslant i\leqslant 3)$. Then the mapping $f:A\to B$ is a C^*-ternary algebra homomorphism.*

Note that we put $\|.\|_A^0=1$.

Proof. Letting $z=0$ and replacing x and y by $2x$ and $2y$ in (3.1), respectively, we get

$$\begin{aligned}\delta f(\mu x+\mu y)+(\alpha-\beta)f(\mu x-\mu y)-\mu(\alpha+\gamma)f(2x)\\ -\mu(\beta+\gamma)f(2y)-2\mu(\alpha+\gamma)f(0)=0\end{aligned} \tag{3.17}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Replacing μ by $-\mu$ in (3.17), we get

$$\begin{aligned}\delta f(-\mu x-\mu y)+(\alpha-\beta)f(-\mu x+\mu y)+\mu(\alpha+\gamma)f(2x)\\ +\mu(\beta+\gamma)f(2y)+2\mu(\alpha+\gamma)f(0)=0\end{aligned} \tag{3.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Adding (3.17) to (3.18), we get

$$\delta\Big[f(\mu x+\mu y)+f(-\mu x-\mu y)\Big]+(\alpha-\beta)\Big[f(\mu x-\mu y)+f(-\mu x+\mu y)\Big]=0$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Letting $\mu=1$ and $y=0$ in the last equation, we infer that the mapping f is odd and so $f(0)=0$. Therefore by letting $y=0$ in (3.17), we get $2f(\mu x)=\mu f(2x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Hence

$$f(2x)=2f(x), \quad f(\mu x)=\mu f(x)$$

for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So we have the following equation from (3.17),

$$\delta f(x+y)+(\alpha-\beta)f(x-y)=2(\alpha+\gamma)f(x)+2(\beta+\gamma)f(y) \tag{3.19}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Now, we show that the mapping f is additive.

Replacing y by $-y$ in (3.19) and using the oddness of f, we get

$$\delta f(x-y)+(\alpha-\beta)f(x+y)=2(\alpha+\gamma)f(x)-2(\beta+\gamma)f(y) \tag{3.20}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Adding (3.19) to (3.20), we get

$$f(x+y)+f(x-y)=2f(x)$$

for all $x \in A$. Replacing x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, we get $f(x+y)=f(x)+f(y)$ for all $x,y \in A$. Therefore by Lemma 2.1 the mapping $f : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.5. *Let $\alpha,\beta,\gamma,q_1,q_2,q_3$ be real numbers and let $\epsilon,\theta,p_1,p_2,p_3$ be non-negative real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $p_3 > 0$ and $q_i \neq 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$ $(x,y,z \in A\setminus\{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Then the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism.*

Note that we put $\|.\|_A^0 = 1$.

Proof. Since $p_3 > 0$, then by letting $x=y=z=0$ in (3.1), we get $f(0)=0$.

If $\alpha+\gamma \neq 0$, then the result follows from Theorem 3.4.

Now, let $\alpha+\gamma=0$. So $\delta=\beta+\gamma$. Letting $z=0$ and replacing x and y by $2x$ and $2y$ in (3.1), respectively, we get

$$f(\mu x+\mu y)-f(\mu x-\mu y)=\mu f(2y) \tag{3.21}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y \in A$. Letting $y=x$ in (3.21), we get $f(2\mu x)=\mu f(2x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Therefore $f(0)=0$ and $f(\mu x)=\mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So the mapping f is odd. It follows from (3.21) that

$$f(x+y)-f(x-y)=f(2y)$$

for all $x,y \in A$. Replacing x and y by $\frac{x-y}{2}$ and $\frac{x+y}{2}$, respectively, we get $f(x+y)=f(x)+f(y)$ for all $x,y \in A$. Therefore by Lemma 2.1 the mapping $f : A \to B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

4. Homomorphisms between unital C^* -ternary algebras

Throughout this section, assume that A is a C^* -ternary algebra with norm $\|.\|_A$ and that B a unital C^* -ternary algebra with norm $\|.\|_B$ and unit e'.

We investigate homomorphisms between unital C^* -ternary algebras, associated to the functional equation $\Delta^{\mu}_{\alpha,\beta,\gamma}f(x,y,z) = 0$.

THEOREM 4.1. *Let α,β,γ be real numbers and let $\epsilon,\theta,p_1,p_2,p_3,q_1,q_2,q_3$ be non-negative real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $0 < p_1,p_2,p_3 < 1$, $0 < q_1,q_2 < 2$ and $0 < q_3 < 3$. Suppose that $f : A \to B$ is a mapping satisfying (2.13) and (2.14). If there exists a real number $\lambda > 1$ $(0 < \lambda < 1)$ and an element $x_0 \in A$ such that $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$ $(\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e')$, then the mapping $f : A \to B$ is a C^* -ternary algebra homomorphism.*

Proof. By Corollary 2.4 there exists a unique C^* -ternary algebra homomorphism $H : A \to B$ such that

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{|\delta|}\Big\{\frac{1}{2-2^{p_1}}\|x\|_A^{p_1} + \frac{1}{2-2^{p_2}}\|x\|_A^{p_2} + \frac{1}{2-2^{p_3}}\|x\|_A^{p_3}\Big\} \tag{4.1}$$

for all $x \in A$. It follows from (4.1) that

$$H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x), \qquad \Big(H(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n})\Big) \tag{4.2}$$

for all $x \in A$ and all real number $\lambda > 1$ $(0 < \lambda < 1)$. Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. It follows from (2.14) that

$$\begin{aligned}
&\Big\|[H(x),H(y),H(z)] - [H(x),H(y),f(z)]\Big\|_B \\
&= \Big\|H[x,y,z] - [H(x),H(y),f(z)]\Big\|_B \\
&= \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \Big\|f([\lambda^n x, \lambda^n y, z]) - \Big[f(\lambda^n x), f(\lambda^n y), f(z)\Big]\Big\|_B \\
&\leqslant \epsilon \lim_{n\to\infty} \frac{1}{\lambda^{2n}} \Big[\lambda^{nq_1}\|x\|_A^{q_1} + \lambda^{nq_2}\|y\|_A^{q_2} + \|z\|_A^{q_3})\Big] = 0
\end{aligned}$$

for all $x,y,z \in A$. So $[H(x),H(y),H(z)] = [H(x),H(y),f(z)]$ for all $x,y,z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(z) = f(z)$ for all $z \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. Therefore the mapping $f : A \to B$ is a C^* -ternary algebra homomorphism. □

REMARK 4.2. Theorem 4.1 will be valid if we replace the conditions $0 < q_1, q_2 < 2$ and $0 < q_3 < 3$ by $0 < q_2, q_3 < 2$ and $0 < q_1 < 3$, respectively.

THEOREM 4.3. *Let α,β,γ be real numbers and let $\epsilon,\theta,p_1,p_2,p_3,q_1,q_2,q_3$ be non-negative real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $p_1,p_2,p_3 > 1$ and*

$q_1, q_2, q_3 > 2$. *Suppose that* $f : A \to B$ *is a mapping satisfying (2.13) and*

$$\begin{aligned}\Big\| f([x,y,z]) - [f(x), f(y), f(z)]\Big\|_B \\ \leqslant \epsilon\Big(\|x\|_A^{q_1}\|y\|_A^{q_2} + \|y\|_A^{q_2}\|z\|_A^{q_3} + \|x\|_A^{q_1}\|z\|_A^{q_3}\Big)\end{aligned} \tag{4.3}$$

for all $\mu \in \mathbb{T}^1$ *and all* $x, y, z \in A$. *If there exists a real number* $\lambda > 1$ $(0 < \lambda < 1)$ *and an element* $x_0 \in A$ *such that* $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$ $(\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e')$, *then the mapping* $f : A \to B$ *is a* C^**-ternary algebra homomorphism.*

Proof. By Theorem 2.5 there exists a unique C^*-ternary algebra homomorphism $H : A \to B$ such that

$$\big\|f(x) - H(x)\big\|_B \leqslant \frac{\theta}{|\delta|}\Big\{\frac{1}{2^{p_1}-2}\|x\|_A^{p_1} + \frac{1}{2^{p_2}-2}\|x\|_A^{p_2} + \frac{1}{2^{p_3}-2}\|x\|_A^{p_3}\Big\} \tag{4.4}$$

for all $x \in A$. It follows from (4.4) that

$$H(x) = \lim_{n\to\infty} \lambda^n f(\frac{x}{\lambda^n}), \qquad \Big(H(x) = \lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x)\Big) \tag{4.5}$$

for all $x \in A$ and all real number $\lambda > 1$ $(0 < \lambda < 1)$. Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n\to\infty} \lambda^n f(\frac{x_0}{\lambda^n}) = e'$. It follows from (4.3) that

$$\begin{aligned}&\Big\|[H(x), H(y), H(z)] - [H(x), H(y), f(z)]\Big\|_B \\ &= \Big\|H[x,y,z] - [H(x), H(y), f(z)]\Big\|_B \\ &= \lim_{n\to\infty} \lambda^{2n}\Big\|f\Big(\Big[\frac{x}{\lambda^n}, \frac{y}{\lambda^n}, z\Big]\Big) - \Big[f\Big(\frac{x}{\lambda^n}\Big), f\Big(\frac{y}{\lambda^n}\Big), f(z)\Big]\Big\|_B \\ &\leqslant \epsilon \lim_{n\to\infty} \lambda^{2n}\Big[\frac{1}{\lambda^{n(q_1+q_2)}}\|x\|_A^{q_1}\|y\|_A^{q_2} + \frac{1}{\lambda^{nq_2}}\|y\|_A^{q_2}\|z\|_A^{q_3} + \frac{1}{\lambda^{nq_1}}\|x\|_A^{q_1}\|z\|_A^{q_3})\Big] = 0\end{aligned}$$

for all $x, y, z \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(z) = f(z)$ for all $z \in A$ when $0 < \lambda < 1$ and $\lim_{n\to\infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. Therefore the mapping $f : A \to B$ is a C^*-ternary algebra homomorphism. □

5. Stability of derivations on C^*-ternary algebras

Throughout this section, assume that A is a C^*-ternary algebra with norm $\|\cdot\|_A$.

In this section we prove the Hyers-Ulam-Rassias stability of derivations on C^*-ternary algebras for the functional equation $\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z) = 0$.

THEOREM 5.1. *Let* α, β, γ *be real numbers with* $\delta := \alpha + \beta + 2\gamma \neq 0$, *and let* $\varphi : A^3 \to [0,\infty)$ *and* $\psi : A^3 \to [0,\infty)$ *be functions such that*

$$\widetilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n}\varphi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n\to\infty} \frac{1}{2^n}\varphi(2^n x, 2^n y, 2^n z) = 0, \tag{5.1}$$

$$\lim_{n\to\infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0 \tag{5.2}$$

for all $x, y, z \in A$. Suppose that $f : A \to A$ is a mapping satisfying

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z)\|_A \leqslant \varphi(x,y,z), \tag{5.3}$$

$$\Big\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \Big\|_A \leqslant \psi(x,y,z) \tag{5.4}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique C^*-ternary algebra derivation $D : A \to A$ such that

$$\big\| f(x) - D(x) \big\|_A \leqslant \frac{1}{2|\delta|} \widetilde{\varphi}(x) \tag{5.5}$$

for all $x \in A$.

Proof. By the proof of Theorem 2.3, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (5.5) and

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. It follows from (5.2) and (5.4) that

$$\begin{aligned}
&\Big\| D[x,y,z] - [D(x),y,z] - [x,D(y),z] - [x,y,D(z)] \Big\|_A \\
&= \lim_{n\to\infty} \frac{1}{8^n} \Big\| f[2^n x, 2^n y, 2^n z] - [f(2^n x), 2^n y, 2^n z] \\
&\quad - [2^n x, f(2^n y), 2^n z] - [2^n x, 2^n y, f(2^n z)] \Big\|_A \\
&\leqslant \lim_{n\to\infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0
\end{aligned}$$

for all $x, y, z \in A$. So

$$D[x,y,z] = [D(x),y,z] + [x,D(y),z] + [x,y,D(z)]$$

for all $x, y, z \in A$. Therefore the mapping $D : A \to A$ is a C^*-ternary algebra derivation. $\square$

THEOREM 5.2. *Let α, β, γ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\varphi : A^3 \to [0, \infty)$ be a function satisfying (5.1). Suppose that the function $\psi : A^3 \to [0, \infty)$ satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, 2^n y, z) = 0$;
(ii) $\lim_{n\to\infty} \frac{1}{4^n} \psi(x, 2^n y, 2^n z) = 0$;
(iii) $\lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, y, 2^n z) = 0$

for all $x, y, z \in A$. Let $f : A \to A$ be a mapping satisfying (5.3) and (5.4). Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

Proof. By the proof of Theorem 2.3, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. We show that if the mapping ψ satisfies in one of the conditions (i), (ii) or (iii), then $f = D$.

Let ψ satisfy in (i) (we have a similar proof if ψ satisfies in (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}
&\Big\| D[x,y,z] - [D(x),y,z] - [x,D(y),z] - [x,y,f(z)] \Big\|_A \\
&= \lim_{n\to\infty} \frac{1}{4^n} \Big\| f[2^n x, 2^n y, z] - [f(2^n x), 2^n y, z] \\
&\quad - [2^n x, f(2^n y), z] - [2^n x, 2^n y, f(z)] \Big\|_A \\
&\leqslant \lim_{n\to\infty} \frac{1}{4^n} \psi(2^n x, 2^n y, z) = 0
\end{aligned}$$

for all $x,y,z \in A$. Therefore

$$D([x,y,z]) = [D(x),y,z] + [x,D(y),z] + [x,y,f(z)] \tag{5.6}$$

for all $x,y,z \in A$. Replacing z by $2z$ in (5.6), we get

$$2D([x,y,z]) = 2[D(x),y,z] + 2[x,D(y),z] + [x,y,f(2z)] \tag{5.7}$$

for all $x,y,z \in A$. It follows from (5.6) and (5.7) that

$$[x,y,f(2z) - 2f(z)] = 0$$

for all $x,y,z \in A$. Letting $x = y = f(2z) - 2f(z)$ in the last equation, we get

$$\|f(2z) - 2f(z)\|_A^3 = \Big\| \Big[f(2z) - 2f(z), f(2z) - 2f(z), f(2z) - 2f(z) \Big] \Big\|_A = 0$$

for all $z \in A$. So $f(2z) = 2f(z)$ for all $z \in A$. By using induction, we infer that $f(2^n z) = 2^n f(z)$ for all $z \in A$ and all $n \in \mathbb{Z}$. Therefore $D(x) = f(x)$ for all $x \in A$. Hence it follows from (5.6) that the mapping $f : A \to A$ is a C^*-ternary derivation. ☐

THEOREM 5.3. *Let α, β, γ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\varphi : A^3 \to [0,\infty)$ be a function satisfying (5.1). Suppose that the function $\psi : A^3 \to [0,\infty)$ satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} \frac{1}{2^n} \psi(2^n x, y, z) = 0$;

(ii) $\lim_{n\to\infty} \frac{1}{2^n} \psi(x, 2^n y, z) = 0$;

(iii) $\lim_{n\to\infty} \frac{1}{2^n} \psi(x, y, 2^n z) = 0$

for all $x,y,z \in A$. Let $f : A \to A$ be a mapping satisfying (5.3) and (5.4). Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

Proof. By the proof of Theorem 2.3, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. We show that if the mapping ψ satisfies in one of the conditions (i), (ii) or (iii), then $f = D$.

Let ψ satisfy in (i) (we have a similar proof if ψ satisfies in (ii) or (iii)). It follows from (5.4) that

$$\begin{aligned}
&\Big\| D[x,y,z] - [D(x),y,z] - [x,f(y),z] - [x,y,f(z)] \Big\|_A \\
&= \lim_{n\to\infty} \frac{1}{2^n} \Big\| f[2^n x,y,z] - [f(2^n x),y,z] \\
&\quad - [2^n x,f(y),z] - [2^n x,y,f(z)] \Big\|_A \\
&\leqslant \lim_{n\to\infty} \frac{1}{2^n} \psi(2^n x,y,z) = 0
\end{aligned}$$

for all $x,y,z \in A$. Therefore

$$D([x,y,z]) = [D(x),y,z] + [x,f(y),z] + [x,y,f(z)] \tag{5.8}$$

for all $x,y,z \in A$.

The rest of the proof is similar to the proof Theorem 5.2. □

COROLLARY 5.4. *Let α,β,γ be real numbers and let $\epsilon,\theta \geqslant 0$, $p_1,p_2,p_3,q_1,q_2,q_3 > 0$ be real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $p_1,p_2,p_3 < 1$ and $q_i < 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z)\|_A \leqslant \theta(\|x\|_A^{p_1} + \|y\|_A^{p_2} + \|z\|_A^{p_3}), \tag{5.9}$$

$$\begin{aligned}
&\Big\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \Big\|_A \\
&\qquad \leqslant \epsilon(\|x\|_A^{q_1} + \|y\|_A^{q_2} + \|z\|_A^{q_3})
\end{aligned} \tag{5.10}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

THEOREM 5.5. *Let α,β,γ be real numbers with $\delta := \alpha+\beta+2\gamma \neq 0$, and let $\Phi : A^3 \to [0,\infty)$ and $\Psi : A^3 \to [0,\infty)$ be functions such that*

$$\widetilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\Big(\frac{x}{2^n},\frac{x}{2^n},\frac{x}{2^n}\Big) < \infty, \quad \lim_{n\to\infty} 2^n \Phi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) = 0, \tag{5.11}$$

$$\lim_{n\to\infty} 8^n \Psi\Big(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n}\Big) = 0 \tag{5.12}$$

for all $x,y,z \in A$. Suppose that $f : A \to A$ is a mapping satisfying

$$\|\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z)\|_A \leqslant \Phi(x,y,z), \tag{5.13}$$

$$\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \leqslant \Psi(x,y,z) \tag{5.14}$$

for all $\mu \in \mathbb{T}^1$ and all $x,y,z \in A$. Then there exists a unique C^*-ternary algebra derivation $D : A \to A$ such that

$$\left\| f(x) - D(x) \right\|_A \leqslant \frac{1}{2|\delta|}\widetilde{\Phi}(x) \tag{5.15}$$

for all $x \in A$.

Proof. By the proof of Theorem 2.5, there exists a unique $\mathbb{C}$-linear mapping $D : A \to A$ satisfying (5.15) and

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.1. □

THEOREM 5.6. *Let α, β, γ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\Phi : A^3 \to [0,\infty)$ be a function satisfying (5.11). Suppose that the function $\Psi : A^3 \to [0,\infty)$ satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} 4^n \Psi(\frac{x}{2^n}, \frac{y}{2^n}, z) = 0$;

(ii) $\lim_{n\to\infty} 4^n \Psi(x, \frac{y}{2^n}, \frac{z}{2^n}) = 0$;

(iii) $\lim_{n\to\infty} 4^n \Psi(\frac{x}{2^n}, y, \frac{z}{2^n}) = 0$

for all $x,y,z \in A$. Let $f : A \to A$ be a mapping satisfying (5.13) and (5.14). Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

Proof. By the proof of Theorem 2.5, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.2. □

COROLLARY 5.7. *Let α, β, γ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_1, p_2, p_3 > 1$ and $q_1, q_2, q_3 > 2$. Suppose that $f : A \to A$ is a mapping satisfying (5.9) and*

$$\begin{aligned} &\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \\ &\quad \leqslant \epsilon \Big(\|x\|_A^{q_1} \|y\|_A^{q_2} + \|y\|_A^{q_2} \|z\|_A^{q_3} + \|x\|_A^{q_1} \|z\|_A^{q_3} \Big) \end{aligned} \tag{5.16}$$

for all $x,y,z \in A$. Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

THEOREM 5.8. *Let α, β, γ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\Phi : A^3 \to [0,\infty)$ be a function satisfying (5.11). Suppose that the function $\Psi : A^3 \to [0,\infty)$ satisfies in one of the following conditions*

(i) $\lim_{n\to\infty} 2^n \Psi(\frac{x}{2^n}, y, z) = 0$;

(ii) $\lim_{n\to\infty} 2^n\Psi(x, \frac{y}{2^n}, z) = 0$;
(iii) $\lim_{n\to\infty} 2^n\Psi(x, y, \frac{z}{2^n}) = 0$
for all $x, y, z \in A$. Let $f : A \to A$ be a mapping satisfying (5.13) and (5.14). Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

Proof. By the proof of Theorem 2.5, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n\to\infty} 2^n f\left(\frac{x}{2^n}\right)$$

for all $x \in A$.

The rest of the proof is similar to the proof of Theorem 5.3. □

THEOREM 5.9. *Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\delta := \alpha+\beta+2\gamma \neq 0$, $p_1 + p_3 > 0$ and $q_i \neq 1$ for some $1 \leqslant i \leqslant 3$. Suppose that $f : A \to A$ is a mapping satisfying*

$$\left\|\Delta^{\mu}_{\alpha,\beta,\gamma} f(x,y,z)\right\|_A \leqslant \theta \|x\|_A^{p_1}\|y\|_A^{p_2}\|z\|_A^{p_3}, \tag{5.17}$$

$$\begin{aligned}\Big\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\Big\|_A \\ \leqslant \epsilon\|x\|_A^{q_1}\|y\|_A^{q_2}\|z\|_A^{q_3}\end{aligned} \tag{5.18}$$

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$ $(x, y, z \in A\setminus\{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Then the mapping $f : A \to A$ is a C^*-ternary algebra derivation.

Note that we put $\|.\|_A^0 = 1$.

Proof. Without any loss of generality, we can assume that $q_1 \neq 1$. Since $p_1+p_3 > 0$, then we can let $p_1 > 0$ $(p_3 > 0)$. Therefore it follows from the proof of Theorem 3.1 (Theorem 3.5) that the mapping $f : A \to A$ is $\mathbb{C}$-linear. Let $q_1 < 1$. It follows from (5.18) that

$$\begin{aligned}&\Big\|f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)]\Big\|_A \\ &= \lim_{n\to\infty}\frac{1}{2^n}\Big\|f[2^n x,y,z] - [f(2^n x),y,z] \\ &\quad - [2^n x,f(y),z] - [2^n x,y,f(z)]\Big\|_A \\ &\leqslant \epsilon \lim_{n\to\infty}\frac{2^{nq_1}}{2^n}\|x\|_A^{q_1}\|y\|_A^{q_2}\|z\|_A^{q_3} = 0\end{aligned}$$

for all $x, y, z \in A$ $(x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Therefore

$$f([x,y,z]) = [f(x),y,z] + [x,f(y),z] + [x,y,f(z)] \tag{5.19}$$

for all $x, y, z \in A$ $(x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3)$. Since $f(0) = 0$, then (5.19) holds for all $x, y, z \in A$ when $q_i < 0$ for some $1 \leqslant i \leqslant 3$. Similarly, we get (5.19) when $q_1 > 1$. So the mapping $f : A \to A$ is a C^*-ternary algebra derivation. □

THEOREM 5.10. *Let* $\alpha,\beta,\gamma,q_1,q_2,q_3$ *be real numbers and let* $\epsilon,\theta,p_1,p_2,p_3$ *be non-negative real numbers such that* $\alpha+\gamma\neq 0$, $p_2>0$ *and* $q_i\neq 1$ *for some* $1\leqslant i\leqslant 3$. *Suppose that* $f:A\to A$ *is a mapping satisfying (5.17) and (5.18). Then the mapping* $f:A\to B$ *is a* C^* *-ternary algebra derivation.*

Note that we put $\|.\|_A^0=1$.

Proof. Without any loss of generality, we can assume that $q_1\neq 1$. It follows from the proof of Theorem 3.3 that the mapping $f:A\to A$ is $\mathbb{C}$-linear. Let $q_1>1$. It follows from (5.18) that

$$\begin{aligned}
&\Big\|f([x,y,z])-[f(x),y,z]-[x,f(y),z]-[x,y,f(z)]\Big\|_A\\
&=\lim_{n\to\infty}2^n\Big\|f\Big(\Big[\frac{x}{2^n},y,z\Big]\Big)-\Big[f\Big(\frac{x}{2^n}\Big),y,z\Big]\\
&\quad-\Big[\frac{x}{2^n},f(y),z\Big]-\Big[\frac{x}{2^n},y,f(z)\Big]\Big\|_A\\
&\leqslant\epsilon\lim_{n\to\infty}\frac{2^n}{2^{nq_1}}\|x\|_A^{q_1}\|y\|_A^{q_2}\|z\|_A^{q_3}=0
\end{aligned}$$

for all $x,y,z\in A$ $(x,y,z\in A\setminus\{0\}$ when $q_i<0$ for some $2\leqslant i\leqslant 3)$. Therefore

$$f([x,y,z])=[f(x),y,z]+[x,f(y),z]+[x,y,f(z)] \tag{5.20}$$

for all $x,y,z\in A$ $(x,y,z\in A\setminus\{0\}$ when $q_i<0$ for some $2\leqslant i\leqslant 3)$. Since $f(0)=0$, then (5.19) holds for all $x,y,z\in A$ when $q_i<0$ for some $1\leqslant i\leqslant 3$. Similarly, we get (5.20) when $q_1<1$. So the mapping $f:A\to A$ is a C^*-ternary algebra derivation. □

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