ON HOMOMORPHISMS BETWEEN $C^*$–TERNARY ALGEBRAS

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\[ f \left( \frac{x+y}{2} + z \right) + f \left( \frac{x-y}{2} + z \right) = f(x) + 2f(z), \]  
\[ f \left( \frac{x+y}{2} + z \right) - f \left( \frac{x-y}{2} + z \right) = f(y), \]  
\[ 2f \left( \frac{x+y}{2} + z \right) = f(x) + f(y) + 2f(z). \]

These are applied to investigate homomorphisms between $C^*$-ternary algebras. In this paper we prove the Hyers-Ulam-Rassias stability of homomorphisms in $C^*$-ternary algebras and of derivations on $C^*$-ternary algebras for the linear combinations of the Cauchy-Jensen additive mappings (0.1), (0.2) and (0.3).

1. Introduction and preliminaries

In 1940, S. M. Ulam [27] gave a talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. Among these was the following question concerning the stability of homomorphisms.

We are given a group $G$ and a metric group $G'$ with metric $\rho(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G \to G'$ satisfies $\rho(f(xy),f(x)f(y)) < \delta$ for all $x, y \in G$, then a homomorphism $h : G \to G'$ exists with $\rho(f(x),h(x)) < \epsilon$ for all $x \in G$?

In 1941, D. H. Hyers [7] considered the case of approximately additive mappings $f : E \to E'$, where $E$ and $E'$ are Banach spaces and $f$ satisfies Hyers inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon \]


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for all \(x, y \in E\). It was shown that the limit

\[ L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]

exists for all \(x \in E\) and that \(L : E \to E'\) is the unique additive mapping satisfying

\[ \|f(x) - L(x)\| \leq \epsilon. \]

In 1978, Th. M. Rassias [19] provided a generalization of Hyers’ Theorem which allows the Cauchy difference to be unbounded.

**Theorem.** (Th. M. Rassias) Let \(f : E \to E'\) be a mapping from a normed vector space \(E\) into a Banach space \(E'\) subject to the inequality

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p) \quad (\heartsuit) \]

for all \(x, y \in E\), where \(\epsilon\) and \(p\) are constants with \(\epsilon > 0\) and \(p < 1\). Then the limit

\[ L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \]

exists for all \(x \in E\) and \(L : E \to E'\) is the unique additive mapping which satisfies

\[ \|f(x) - L(x)\| \leq \frac{2\epsilon}{2 - 2^p} \|x\|^p \quad (\diamondsuit) \]

for all \(x \in E\). If \(p < 0\) then inequality \((\heartsuit)\) holds for \(x, y \neq 0\) and \((\diamondsuit)\) for \(x \neq 0\). Also, if the mapping \(t \to f(tx)\) is continuous in \(t \in \mathbb{R}\) for each fixed \(x \in X\), then \(L\) is \(\mathbb{R}\)-linear.

In 1990, Th. M. Rassias [20] during the 27th International Symposium on Functional Equations asked the question whether such a theorem can also be proved for \(p \geq 1\). In 1991, Z. Gajda [5] following the same approach as in Th. M. Rassias [19], gave an affirmative solution to this question for \(p > 1\). It was shown by Z. Gajda [5], as well as by Th. M. Rassias and P. Šemrl [25] that one cannot prove a Th. M. Rassias’ type theorem when \(p = 1\). The counterexamples of Z. Gajda [5], as well as of Th. M. Rassias and P. Šemrl [25] have stimulated several mathematicians to invent new definitions of approximately additive or approximately linear mappings, cf. P. Găvruţa [6], S. Jung [11], who among others studied the Hyers-Ulam-Rassias stability of functional equations. The inequality \((\heartsuit)\) that was introduced for the first time by Th. M. Rassias [19] for the stability of the linear mapping between Banach spaces provided a lot of influence in the development of a generalization of the Hyers-Ulam stability concept. This new concept is known as Hyers-Ulam-Rassias stability of functional equations (cf. the books of P. Czerwik [4], D.H. Hyers, G. Isac and Th. M. Rassias [8]).

have published on various generalizations and applications of Hyers-Ulam stability and Hyers-Ulam-Rassias stability to a number of functional equations and mappings, for example: quadratic functional equation, invariant means, multiplicative mappings - superstability, bounded \( n \) th differences, convex functions, generalized orthogonality functional equation, Euler-Lagrange functional equation, Navier-Stokes equations. Several mathematicians have contributed works on these subjects; we mention a few: C. Park [12]–[17], Th. M. Rassias [21]–[24], F. Skof [26].

A. Prastaro and Th. M. Rassias [18] introduced for the first time the Hyers-Ulam-Rassias stability approach for the study of Navier-Stokes equation.

Following the terminology of [1], a non-empty set \( G \) with a ternary operation \( [\cdot, \cdot, \cdot] : G \times G \times G \to G \) is called a ternary groupoid and is denoted by \((G, [\cdot, \cdot, \cdot])\). The ternary groupoid \((G, [\cdot, \cdot, \cdot])\) is called commutative if \([x_1, x_2, x_3] = [x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}]\) for all \( x_1, x_2, x_3 \in G \) and all permutations \( \sigma \) of \( \{1, 2, 3\} \).

If a binary operation \( \circ \) is defined on \( G \) such that \([x, y, z] = (x \circ y) \circ z\) for all \( x, y, z \in G \), then we say that \([\cdot, \cdot, \cdot]\) is derived from \( \circ \). We say that \((G, [\cdot, \cdot, \cdot])\) is a ternary semigroup if the operation \([\cdot, \cdot, \cdot]\) is associative, i.e., if \([[[x, y, z], u, v]] = [x, [y, z, u], v] = [x, y, [z, u, v]]\) holds for all \( x, y, z, u, v \in G \) (see [3]).

A \( C^* \)-ternary algebra is a complex Banach space \( A \), equipped with a ternary product \([x, y, z] \mapsto [x, y, z]\) of \( A^3 \) into \( A \), which is \( \mathbb{C} \)-linear in the outer variables, conjugate \( \mathbb{C} \)-linear in the middle variable, and associative in the sense that \([x, [y, z, w, v]] = [x, [w, z, y], v] = [[[x, y, z], w], v], \) and satisfies \( \|[[x, y, z]]\| \leq \|x\| \cdot \|y\| \cdot \|z\| \) and \( \|[x, x, x]\| = \|x\|^3 \) (see [1, 28]). Every left Hilbert \( C^* \)-module is a \( C^* \)-ternary algebra via the ternary product \([x, y, z] := (x, y) z\).

If a \( C^* \)-ternary algebra \((A, [\cdot, \cdot, \cdot])\) has an identity, i.e., an element \( e \in A \) such that \( x = [x, e, e] = [e, e, x] \) for all \( x \in A \), then it is routine to verify that \( A \), endowed with \( x \circ y := [x, e, y] \) and \( x^* := [e, x, e] \), is a unital \( C^* \)-algebra. Conversely, if \((A, \circ)\) is a unital \( C^* \)-algebra, then \([x, y, z] := x \circ y^* \circ z\) makes \( A \) into a \( C^* \)-ternary algebra.

A \( \mathbb{C} \)-linear mapping \( H : A \to B \) is called a \( C^* \)-ternary algebra homomorphism if \( H([x, y, z]) = [H(x), H(y), H(z)] \) for all \( x, y, z \in A \). If, in addition, the mapping \( H \) is bijective, then the mapping \( H : A \to B \) is called a \( C^* \)-ternary algebra isomorphism. A \( \mathbb{C} \)-linear mapping \( \delta : A \to A \) is called a \( C^* \)-ternary derivation if \( \delta([x, y, z]) = [\delta(x), y, z] + [x, \delta(y), z] + [x, y, \delta(z)] \) for all \( x, y, z \in A \) (see [1]).

2. Stability of homomorphisms in \( C^* \)-ternary algebras

Throughout this section, assume that \( A \) is a \( C^* \)-ternary algebra with norm \( \|\cdot\|_A \) and that \( B \) a \( C^* \)-ternary algebra with norm \( \|\cdot\|_B \).
Let \((\alpha, \beta, \gamma) \in \mathbb{R}^3\). For a given mapping \(f : A \to B\), we define
\[
C_\mu f(x, y, z) := f\left(\frac{\mu x + \mu y}{2} + \mu z\right) + f\left(\frac{\mu x - \mu y}{2} + \mu z\right) - \mu f(x) - 2\mu f(z),
\]
\[
D_\mu f(x, y, z) := f\left(\frac{\mu x + \mu y}{2} + \mu z\right) - f\left(\frac{\mu x - \mu y}{2} + \mu z\right) - \mu f(y),
\]
\[
E_\mu f(x, y, z) := 2f\left(\frac{\mu x + \mu y}{2} + \mu z\right) - \mu f(x) - \mu f(y) - 2\mu f(z),
\]
\[
\Delta_{\alpha, \beta, \gamma}^\mu f(x, y, z) := \alpha C_\mu f(x, y, z) + \beta D_\mu f(x, y, z) + \gamma E_\mu f(x, y, z)
\]
for all \(\mu \in \mathbb{T}^1 := \{\lambda \in \mathbb{C} : |\lambda| = 1\}\) and all \(x, y, z \in A\).

We will use the following lemma in this paper:

**Lemma 2.1.** [14] Let \(X\) and \(Y\) be linear spaces and let \(f : X \to Y\) be an additive mapping such that \(f(\mu x) = \mu f(x)\) for all \(x \in X\) and all \(\mu \in \mathbb{T}^1\). Then the mapping \(f\) is \(\mathbb{C}\)-linear.

**Lemma 2.2.** Let \(X\) and \(Y\) be linear spaces and let \(\alpha, \beta, \gamma\) be real numbers such that \(\delta := \alpha + \beta + 2\gamma \neq 0\). Suppose that \(f : X \to Y\) is a mapping satisfying
\[
\Delta_{\alpha, \beta, \gamma}^\mu f(x, y, z) = 0
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x, y, z \in X\). Then the mapping \(f : X \to Y\) is \(\mathbb{C}\)-linear.

**Proof.** Letting \(x = y = z = 0\) in (2.1), we get \(f(0) = 0\). Letting \(z = 0\) and replacing \(x\) and \(y\) by \(2x\) and \(2y\) in (2.1), respectively, we get
\[
(\alpha + \gamma)\left[f(\mu x + \mu y) - \mu f(2x)\right] + (\beta + \gamma)\left[f(\mu x + \mu y) - \mu f(2y)\right] + (\alpha - \beta)f(\mu x - \mu y) = 0
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x, y \in X\). Letting \(y = x\) in (2.2), we get \(f(2\mu x) = \mu f(2x)\) for all \(\mu \in \mathbb{T}^1\) and all \(x \in X\). So
\[
f(\mu x) = \mu f(x)
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x \in X\). Letting \(x = y = z\) in (2.1), and using (2.3), we get \(f(2x) = 2f(x)\) for all \(x \in X\). Now, we show that \(f(x + y) = f(x) + f(y)\) for all \(x, y \in X\).

We have two cases:

Case I. \(\alpha = \beta\).

Replacing \(x\) and \(y\) by \(y\) and \(x\) in (2.2), we get
\[
(\alpha + \gamma)\left[f(\mu x + \mu y) - \mu f(2y)\right] + (\beta + \gamma)\left[f(\mu x + \mu y) - \mu f(2x)\right] = 0
\]
for all \(\mu \in \mathbb{T}^1\) and all \(x, y \in X\). Adding (2.2) to (2.4) and using (2.3), we get
\[
2f(x + y) = f(2x) + f(2y)
\]
for all \(x, y \in X\). Since \(f(2x) = 2f(x)\) for all \(x \in X\), then we get from the last equation that \(f(x + y) = f(x) + f(y)\) for all \(x, y \in X\).
Case II. $\alpha \neq \beta$.

Letting $x = 0$ in (2.2), we infer that the mapping $f$ is odd. Replacing $x$ and $y$ by $y$ and $x$ in (2.2), we get

$$
(\alpha + \gamma)[f(\mu x + \mu y) - \mu f(2y)] + (\beta + \gamma)[f(\mu y + \mu y) - \mu f(2x)]
$$

$$
+ (\alpha - \beta)f(\mu y - \mu x) = 0
$$

(2.5)

for all $\mu \in \mathbb{T}^1$ and all $x, y \in X$. Adding (2.2) to (2.5) and using (2.3), we get

$$
2f(x + y) = f(2x) + f(2y)
$$

for all $x, y \in X$. So $f(x + y) = f(x) + f(y)$ for all $x, y \in X$.

Hence by Lemma 2.1 the mapping $f : X \to Y$ is $\mathbb{C}$-linear. □

We investigate the Hyers-Ulam-Rassias stability of homomorphisms in $C^*$-ternary algebras for the functional equation $\Delta^\mu_{\alpha, \beta, \gamma} f(x, y, z) = 0$.

**Theorem 2.3.** Let $\alpha, \beta, \gamma$ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\varphi : A^3 \to [0, \infty)$ and $\psi : A^3 \to [0, \infty)$ be functions such that

$$
\tilde{\varphi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \varphi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, 2^n y, 2^n z) = 0,
$$

$$
\lim_{n \to \infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0
$$

(2.6)

for all $x, y, z \in A$. Suppose that $f : A \to B$ is a mapping satisfying

$$
\|\Delta^\mu_{\alpha, \beta, \gamma} f(x, y, z)\|_B \leq \varphi(x, y, z),
$$

$$
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \varphi(x, y, z)
$$

(2.7)

for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. Then there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

$$
\|f(x) - H(x)\|_B \leq \frac{1}{2|\delta|} \tilde{\varphi}(x)
$$

(2.8)

for all $x \in A$.

**Proof.** Letting $\mu = 1$ and $x = y = z$ in (2.8), we get

$$
\|\delta f(2x) - 2\tilde{\delta} f(x)\|_B \leq \varphi(x, x, x)
$$

(2.9)

for all $x \in A$. If we replace $x$ by $2^n x$ in (2.11) and divide both sides of (2.11) by $|\delta| 2^{n+1}$, we get

$$
\left\| \frac{1}{2^{n+1}} f(2^{n+1} x) - \frac{1}{2^n} f(2^n x) \right\|_B \leq \frac{1}{|\delta| 2^{n+1}} \varphi(2^n x, 2^n x, 2^n x)
$$

(2.10)
for all $x \in A$ and all non-negative integers $n$. Hence

$$
\left\| \frac{1}{2^{n+1}} f \left( 2^{n+1}x \right) - \frac{1}{2^n} f \left( 2^m x \right) \right\|_B = \left\| \sum_{k=0}^{n} \left[ \frac{1}{2^{k+1}} f \left( 2^{k+1}x \right) - \frac{1}{2^k} f \left( 2^k x \right) \right] \right\|_B 
\leq \sum_{k=0}^{n} \left\| \frac{1}{2^{k+1}} f \left( 2^{k+1}x \right) - \frac{1}{2^k} f \left( 2^k x \right) \right\|_B
\leq \frac{1}{2^|\delta|} \sum_{k=0}^{n} \frac{1}{2^k} \phi(2^k x, 2^k x, 2^k x)
$$

(2.12)

for all $x \in A$ and all non-negative integers $n \geq m \geq 0$. It follows from (2.6) and (2.12) that the sequence $\left\{ \frac{1}{2^n} f \left( 2^n x \right) \right\}$ is a Cauchy sequence for all $x \in A$. Since $B$ is complete, the sequence $\left\{ \frac{1}{2^n} f \left( 2^n x \right) \right\}$ converges. Thus one can define the mapping $H : A \rightarrow B$ by

$$
H(x) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f \left( 2^n x \right)
$$

for all $x \in A$. Moreover, letting $m = 0$ and passing the limit $n \rightarrow \infty$ in (2.12) we get (2.10).

It follows from (2.6) that

$$
\left\| \Delta_{\alpha,\beta,\gamma}^\mu H(x, y, z) \right\|_B = \lim_{n \rightarrow \infty} \frac{1}{2^n} \left\| \Delta_{\alpha,\beta,\gamma}^\mu f \left( 2^n x, 2^n y, 2^n z \right) \right\|_B
\leq \lim_{n \rightarrow \infty} \frac{1}{2^n} \phi(2^n x, 2^n y, 2^n z) = 0
$$

for all $x, y, z \in A$. So $\Delta_{\alpha,\beta,\gamma}^\mu H(x, y, z) = 0$ for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A$. By Lemma 2.1 the mapping $H : A \rightarrow B$ is $\mathbb{C}$-linear.

It follows from (2.7) and (2.9) that

$$
\left\| H([x, y, z]) - [H(x), H(y), H(z)] \right\|_B
= \lim_{n \rightarrow \infty} \frac{1}{8^n} \left\| f \left( [2^n x, 2^n y, 2^n z] \right) - [f(2^n x), f(2^n y), f(2^n z)] \right\|_B
\leq \lim_{n \rightarrow \infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0
$$

for all $x, y, z \in A$. Therefore

$$
H([x, y, z]) = [H(x), H(y), H(z)]
$$

for all $x, y, z \in A$. Therefore the mapping $H : A \rightarrow B$ is a $C^*$-ternary algebra homomorphism.

Now, let $Q : A \rightarrow B$ be another $C^*$-ternary algebra homomorphism satisfying
(2.10). Then we have from (2.6) that
\[
\|H(x) - Q(x)\|_B = \lim_{n \to \infty} \frac{1}{2^n} \|f(2^n x) - Q(2^n x)\|_B
\]
\[
\leq \frac{1}{2|\delta|} \lim_{n \to \infty} \frac{1}{2^n} \tilde{\alpha}(2^n x)
\]
\[
= \frac{1}{2|\delta|} \lim_{n \to \infty} \sum_{k=n}^{\infty} \frac{1}{2^k} \tilde{\alpha}(2^k x, 2^k x, 2^k x) = 0
\]
for all \(x \in A\). So \(H(x) = Q(x)\) for all \(x \in A\). This proves the uniqueness of \(H\). Thus the mapping \(H : A \to B\) is a unique \(C^*\)-ternary algebra homomorphism satisfying (2.10). \(\square\)

**COROLLARY 2.4.** Let \(\alpha, \beta, \gamma\) be real numbers and let \(\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3\) be non-negative real numbers such that \(\delta := \alpha + \beta + 2\gamma \neq 0\), \(0 < p_1, p_2, p_3 < 1\) and \(0 < q_1, q_2, q_3 < 3\). Suppose that \(f : A \to B\) is a mapping satisfying
\[
\|\Delta_{\alpha, \beta, \gamma} f(x, y, z)\|_B \leq \theta (\|x\|_{T^1}^2 + \|y\|_{T^1}^2 + \|z\|_{T^1}^2),
\]  
(2.13)
\[
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \epsilon (\|x\|_{T^1}^2 + \|y\|_{T^1}^2 + \|z\|_{T^1}^2)
\]  
(2.14)
for all \(\mu \in T^1\) and all \(x, y, z \in A\). Then there exists a unique \(C^*\)-ternary algebra homomorphism \(H : A \to B\) such that
\[
\|f(x) - H(x)\|_B \leq \frac{\theta}{|\delta|} \left\{ \frac{1}{2 - 2^{p_1}} \|x\|_{T^1}^2 + \frac{1}{2 - 2^{p_2}} \|x\|_{T^1}^2 + \frac{1}{2 - 2^{p_3}} \|x\|_{T^1}^2 \right\}
\]
for all \(x \in A\).

**THEOREM 2.5.** Let \(\alpha, \beta, \gamma\) be real numbers with \(\delta := \alpha + \beta + 2\gamma \neq 0\), and let \(\Phi : A^3 \to [0, \infty)\) and \(\Psi : A^3 \to [0, \infty)\) be functions such that
\[
\tilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\left( \frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n} \right) < \infty, \quad \lim_{n \to \infty} 2^n \Phi\left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0,
\]  
(2.15)
\[
\lim_{n \to \infty} 8^n \Psi\left( \frac{x}{2^n}, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0
\]  
(2.16)
for all \(x, y, z \in A\). Suppose that \(f : A \to B\) is a mapping satisfying
\[
\|\Delta_{\alpha, \beta, \gamma} f(x, y, z)\|_B \leq \Phi(x, y, z),
\]  
(2.17)
\[
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \Psi(x, y, z)
\]  
(2.18)
for all \(\mu \in T^1\) and all \(x, y, z \in A\). Then there exists a unique \(C^*\)-ternary algebra homomorphism \(H : A \to B\) such that
\[
\|f(x) - H(x)\|_B \leq \frac{1}{2|\delta|} \tilde{\Phi}(x)
\]  
(2.19)
for all \(x \in A\).
Proof. Letting \( \mu = 1 \) and \( x = y = z \) in (2.17), we get
\[
\|\delta f(2x) − 2\delta f(x)\|_B \leq \Phi(x, x, x) \tag{2.20}
\]
for all \( x \in A \). If we replace \( x \) by \( \frac{x}{2^n} \) in (2.20) and multiply both sides of (2.20) to \( \frac{2^n}{|\delta|} \), we get
\[
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) − 2^nf\left(\frac{x}{2^n}\right)\right\|_B \leq \frac{2^n}{|\delta|}\Phi\left(\frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}, \frac{x}{2^{n+1}}\right)
\]
for all \( x \in A \) and all non-negative integers \( n \). Hence
\[
\left\|2^{n+1}f\left(\frac{x}{2^{n+1}}\right) − 2^mf\left(\frac{x}{2^m}\right)\right\|_B = \left\|\sum_{k=m}^{n} \left[2^{k+1}f\left(\frac{x}{2^{k+1}}\right) − 2^kf\left(\frac{x}{2^k}\right)\right]\right\|_B
\]
\[
\leq \sum_{k=m}^{n} \left\|2^{k+1}f\left(\frac{x}{2^{k+1}}\right) − 2^kf\left(\frac{x}{2^k}\right)\right\|_B \tag{2.21}
\]
\[
\leq \frac{1}{2|\delta|} \sum_{k=m}^{n} \left\|2^{k+1}\Phi\left(\frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}, \frac{x}{2^{k+1}}\right)\right\|_B
\]
for all \( x \in A \) and all non-negative integers \( n \geq m \geq 0 \). It follows from (2.15) and (2.21) that the sequence \( \{2^nf\left(\frac{x}{2^n}\right)\} \) is a Cauchy sequence for all \( x \in A \). Since \( B \) is complete, the sequence \( \{2^nf\left(\frac{x}{2^n}\right)\} \) converges. Thus one can define the mapping \( H : A \to B \) by
\[
H(x) := \lim_{n \to \infty} 2^nf\left(\frac{x}{2^n}\right)
\]
for all \( x \in A \). Moreover, letting \( m = 0 \) and passing the limit \( n \to \infty \) in (2.21) we get (2.19).

The rest of the proof is similar to the proof of Theorem 2.3. \( \square \)

Corollary 2.6. Let \( \alpha, \beta, \gamma \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3 \) be non-negative real numbers such that \( \delta := \alpha + \beta + 2\gamma \neq 0 \), \( p_1, p_2, p_3 > 1 \) and \( q_1, q_2, q_3 > 3 \). Suppose that \( f : A \to B \) is a mapping satisfying (2.13) and (2.14). Then there exists a unique \( C^\ast \) -ternary algebra homomorphism \( H : A \to B \) such that
\[
\|f(x) − H(x)\|_B \leq \frac{\theta}{|\delta|} \left\{ \frac{1}{2^{p_1} − 2} ||x||_A^{p_1} + \frac{1}{2^{p_2} − 2} ||x||_A^{p_2} + \frac{1}{2^{p_3} − 2} ||x||_A^{p_3} \right\}
\]
for all \( x \in A \).

3. Homomorphisms between \( C^\ast \) -ternary algebras

In this section we improve the results in Theorems 2.3, 2.4, 3.3 and 3.4 of [15].

Theorem 3.1. Let \( \alpha, \beta, \gamma, q_1, q_2, q_3 \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3 \) be non-negative real numbers such that \( \delta := \alpha + \beta + 2\gamma \neq 0 \), \( p_1 > 0 \) and \( q_i \neq 1 \) for some \( 1 \leq i \leq 3 \). Suppose that \( f : A \to B \) is a mapping satisfying
\[
\left\|\Delta_{\alpha, \beta, \gamma}^xf(x, y, z)\right\|_B \leq \theta||x||_A^{p_1}||y||_A^{p_2}||z||_A^{p_3},\tag{3.1}
\]
\[ \|f([x,y,z]) - [f(x), f(y), f(z)]\|_B \leq \epsilon \|x\|_A^q \|y\|_A^q \|z\|_A^q \]  

(3.2)

for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

Note that we put \( \|\cdot\|_A = 1 \).

**Proof.** Since \( p_1 > 0 \), then by letting \( x = y = z = 0 \) in (3.1), we get \( f(0) = 0 \).

Letting \( x = 0 \) and replacing \( y \) by \( 2y \) in (3.1), we get

\[ \delta f(\mu y + \mu z) + (\alpha - \beta)f(\mu z - \mu y) - 2\mu(\alpha + \gamma)f(z) - \mu(\beta + \gamma)f(2y) = 0 \]  

(3.3)

for all \( \mu \in \mathbb{T}^1 \) and all \( y, z \in A \). Letting \( y = z \) in (3.3), we get

\[ \delta f(2\mu y) - 2\mu(\alpha + \gamma)f(y) - \mu(\beta + \gamma)f(2y) = 0 \]  

(3.4)

for all \( \mu \in \mathbb{T}^1 \) and all \( y \in A \). Replacing \( \mu \) by \(-\mu \) in (3.4), we get

\[ \delta f(-2\mu y) + 2\mu(\alpha + \gamma)f(y) + \mu(\beta + \gamma)f(2y) = 0 \]  

(3.5)

for all \( \mu \in \mathbb{T}^1 \) and all \( y \in A \). Adding (3.4) to (3.5), we get that \( f(2\mu y) + f(-2\mu y) = 0 \) for all \( \mu \in \mathbb{T}^1 \) and all \( y \in A \). So \( f \) is odd.

Now, we show that \( f(\mu x) = \mu f(x) \) and \( f(x + y) = f(x) + f(y) \) for all \( \mu \in \mathbb{T}^1 \) and all \( x, y \in A \).

We have two cases:

(i) Let \( \alpha + \gamma \neq 0 \). Letting \( y = 0 \) in (3.3), we get \( f(\mu z) = \mu f(z) \) for all \( \mu \in \mathbb{T}^1 \) and all \( z \in A \). Therefore it follows from (3.4) that \( f(2y) = 2f(y) \) for all \( y \in A \). So it follows from (3.3) that

\[ \delta f(y + z) + (\alpha - \beta)f(z - y) = 2(\alpha + \gamma)f(z) + 2(\beta + \gamma)f(y) \]  

(3.6)

for all \( y, z \in A \). Replacing \( y \) by \(-y \) in (3.6) and using the oddness of \( f \), we get

\[ \delta f(z - y) + (\alpha - \beta)f(z + y) = 2(\alpha + \gamma)f(z) - 2(\beta + \gamma)f(y) \]  

(3.7)

for all \( y, z \in A \). Adding (3.6) to (3.7), we get

\[ f(y + z) + f(z - y) = 2f(z) \]  

(3.8)

for all \( y, z \in A \). Replacing \( y \) and \( z \) by \( \frac{y - z}{2} \) and \( \frac{y + z}{2} \) in (3.8), respectively, we get \( f(y + z) = f(y) + f(z) \) for all \( y, z \in A \).

(ii) Let \( \alpha + \gamma = 0 \). Since \( \delta \neq 0 \), then \( \beta + \gamma \neq 0 \). Letting \( z = 0 \) in (3.3) and using the oddness of \( f \), we get \( 2f(\mu y) = \mu f(2y) \) for all \( \mu \in \mathbb{T}^1 \) and all \( y \in A \). Hence by letting \( \mu = 1 \), we get \( f(2y) = 2f(y) \) for all \( y \in A \). So \( f(\mu y) = \mu f(y) \) for all \( \mu \in \mathbb{T}^1 \) and all \( y \in A \). It follows from (3.3) that

\[ f(y + z) - f(z - y) = 2f(y) \]  

(3.9)

for all \( y, z \in A \). Replacing \( y \) and \( z \) by \( \frac{y - z}{2} \) and \( \frac{y + z}{2} \) in (3.9), respectively, we get \( f(y + z) = f(y) + f(z) \) for all \( y, z \in A \).

Hence, by Lemma 2.1 the mapping \( f : A \to B \) is \( C \)-linear.
Without any loss of generality, we may suppose that \( q_1 \neq 1 \). Let \( q_1 > 1 \). It follows from (3.2) that

\[
\left\| f ([x, y, z]) - [f (x), f (y), f (z)] \right\|_B
= \lim_{n \to \infty} 2^n \left\| f \left( \left[ \frac{x}{2^n}, y, z \right] \right) - \left[ f \left( \frac{x}{2^n} \right), f (y), f (z) \right] \right\|_B
\leq \epsilon \lim_{n \to \infty} 2^n \| x \|_A^q \| y \|_A^q \| z \|_A^q = 0
\]

for all \( x, y, z \in A \). Therefore

\[
f ([x, y, z]) = [f (x), f (y), f (z)] \tag{3.10}
\]

for all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 2 \leq i \leq 3 \). Since \( f (0) = 0 \), then (3.10) holds for all \( x, y, z \in A \) when \( q_i < 0 \) for some \( 2 \leq i \leq 3 \). Similarly, for \( q_1 < 1 \), we get (3.10). So the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism. \( \square \)

**THEOREM 3.2.** Let \( \alpha, \beta, \gamma, q_1, q_2, q_3 \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3 \) be non-negative real numbers such that \( \alpha + \gamma \neq 0 \), \( p_1 > 0 \) and \( q_i \neq 1 \) for some \( 1 \leq i \leq 3 \). Suppose that \( f : A \to B \) is a mapping satisfying (3.1) and (3.2) for all \( \mu \in T^1 \) and all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

Note that we put \( \| . \|_A^0 = 1 \).

**Proof.** Letting \( x = 0 \) and replacing \( y \) by \( 2y \) in (3.1), we get

\[
\delta f (\mu y + \mu z) + (\alpha - \beta) f (\mu z - \mu y)
- 2\mu (\alpha + \gamma) f (z) - \mu (\beta + \gamma) f (2y) - \mu (\alpha + \gamma) f (0) = 0 \tag{3.11}
\]

for all \( \mu \in T^1 \) and all \( y, z \in A \). Replacing \( \mu \) by \( -\mu \) in (3.11), we get

\[
\delta f (-\mu y - \mu z) + (\alpha - \beta) f (-\mu z + \mu y)
+ 2\mu (\alpha + \gamma) f (z) + \mu (\beta + \gamma) f (2y) + \mu (\alpha + \gamma) f (0) = 0 \tag{3.12}
\]

for all \( \mu \in T^1 \) and all \( y, z \in A \). Adding (3.11) to (3.12) and letting \( z = 0 \) in the obtained equation, we get \( f (\mu y) + f (-\mu y) = 0 \) for all \( \mu \in T^1 \) and all \( y \in A \). So the mapping \( f \) is odd and \( f (0) = 0 \). Therefore we obtain (3.3) from (3.11).

It follows from the proof of case (i) of Theorem 3.1 that the mapping \( f : A \to B \) is \( \mathbb{C} \)-linear.

The rest of the proof is similar to the proof of Theorem 3.1. \( \square \)

**THEOREM 3.3.** Let \( \alpha, \beta, \gamma, q_1, q_2, q_3 \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3 \) be non-negative real numbers such that \( \alpha + \gamma \neq 0 \), \( p_2 > 0 \) and \( q_i \neq 1 \) for some \( 1 \leq i \leq 3 \). Suppose that \( f : A \to B \) is a mapping satisfying (3.1) and (3.2) for all \( \mu \in T^1 \) and all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

Note that we put \( \| . \|_A^0 = 1 \).
Proof. Letting $y = 0$ and replacing $x$ by $2x$ in (3.1), we get

$$2(\alpha + \gamma)f(\mu x + \mu z) - \mu(\alpha + \gamma)f(2x) - 2\mu(\alpha + \gamma)f(z) - \mu(\beta + \gamma)f(0) = 0$$

(3.13)

for all $\mu \in T^1$ and all $x, z \in A$. Replacing $\mu$ by $-\mu$ in (3.13), we get

$$2(\alpha + \gamma)f(-\mu x - \mu z) + \mu(\alpha + \gamma)f(2x) + 2\mu(\alpha + \gamma)f(z) + \mu(\beta + \gamma)f(0) = 0$$

(3.14)

for all $\mu \in T^1$ and all $x, z \in A$. Adding (3.13) to (3.14), we infer that the mapping $f$ is odd and $f(0) = 0$. So it follows from (3.13) that

$$2f(\mu x + \mu z) = \mu f(2x) + 2\mu f(z)$$

(3.15)

for all $\mu \in T^1$ and all $x, z \in A$. Letting $z = 0$ in (3.15), we get $2f(\mu x) = \mu f(2x)$ for all $\mu \in T^1$ and all $x \in A$. So

$$f(2x) = 2f(x), \quad f(\mu x) = \mu f(x)$$

(3.16)

for all $\mu \in T^1$ and all $x \in A$. It follows from (3.15) and (3.16) that $f(x + z) = f(x) + f(z)$ for all $x, z \in A$. Hence, by Lemma 2.1 the mapping $f : A \rightarrow B$ is $C^*$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

**THEOREM 3.4.** Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\alpha + \gamma \neq 0$, $p_3 > 0$ and $q_i \neq 1$ for some $1 \leq i \leq 3$. Suppose that $f : A \rightarrow B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in T^1$ and all $x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leq i \leq 3$. Then the mapping $f : A \rightarrow B$ is a $C^*$-ternary algebra homomorphism.

Note that we put $\|I\|^0 = 1$.

Proof. Letting $z = 0$ and replacing $x$ and $y$ by $2x$ and $2y$ in (3.1), respectively, we get

$$\delta f(\mu x + \mu y) + (\alpha - \beta)f(\mu x - \mu y) - \mu(\alpha + \gamma)f(2x) - \mu(\beta + \gamma)f(2y) - 2\mu(\alpha + \gamma)f(0) = 0$$

(3.17)

for all $\mu \in T^1$ and all $x, y \in A$. Replacing $\mu$ by $-\mu$ in (3.17), we get

$$\delta f(-\mu x - \mu y) + (\alpha - \beta)f(-\mu x + \mu y) + \mu(\alpha + \gamma)f(2x) + \mu(\beta + \gamma)f(2y) + 2\mu(\alpha + \gamma)f(0) = 0$$

(3.18)

for all $\mu \in T^1$ and all $x, y \in A$. Adding (3.17) to (3.18), we get

$$\delta \left[ f(\mu x + \mu y) + f(-\mu x - \mu y) \right] + (\alpha - \beta) \left[ f(\mu x - \mu y) + f(-\mu x + \mu y) \right] = 0$$

for all $\mu \in T^1$ and all $x, y \in A$. Letting $\mu = 1$ and $y = 0$ in the last equation, we infer that the mapping $f$ is odd and so $f(0) = 0$. Therefore by letting $y = 0$ in (3.17), we get $2f(\mu x) = \mu f(2x)$ for all $\mu \in T^1$ and all $x \in A$. Hence

$$f(2x) = 2f(x), \quad f(\mu x) = \mu f(x)$$
for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So we have the following equation from (3.17),
\[
\delta f(x + y) + (\alpha - \beta)f(x - y) = 2(\alpha + \gamma)f(x) + 2(\beta + \gamma)f(y)
\] (3.19)
for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Now, we show that the mapping $f$ is additive.
Replacing $y$ by $-y$ in (3.19) and using the oddness of $f$, we get
\[
\delta f(x - y) + (\alpha - \beta)f(x + y) = 2(\alpha + \gamma)f(x) - 2(\beta + \gamma)f(y)
\] (3.20)
for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Adding (3.19) to (3.20), we get
\[
f(x + y) + f(x - y) = 2f(x)
\]
for all $x \in A$. Replacing $x$ and $y$ by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, we get $f(x + y) = f(x) + f(y)$ for all $x, y \in A$. Therefore by Lemma 2.1 the mapping $f : A \rightarrow B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □

**Theorem 3.5.** Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_3 > 0$ and $q_i \neq 1$ for some $1 \leq i \leq 3$. Suppose that $f : A \rightarrow B$ is a mapping satisfying (3.1) and (3.2) for all $\mu \in \mathbb{T}^1$ and all $x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $1 \leq i \leq 3$.

Then the mapping $f : A \rightarrow B$ is a $\mathbb{C}^*$-ternary algebra homomorphism.

Note that we put $\|\cdot\|_A^0 = 1$.

**Proof.** Since $p_3 > 0$, then by letting $x = y = z = 0$ in (3.1), we get $f(0) = 0$. If $\alpha + \gamma \neq 0$, then the result follows from Theorem 3.4.

Now, let $\alpha + \gamma = 0$. So $\delta = \beta + \gamma$. Letting $z = 0$ and replacing $x$ and $y$ by $2x$ and $2y$ in (3.1), respectively, we get
\[
f(\mu x + \mu y) - f(\mu x - \mu y) = \mu f(2y)
\] (3.21)
for all $\mu \in \mathbb{T}^1$ and all $x, y \in A$. Letting $y = x$ in (3.21), we get $f(2\mu x) = 2\mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. Therefore $f(0) = 0$ and $f(\mu x) = \mu f(x)$ for all $\mu \in \mathbb{T}^1$ and all $x \in A$. So the mapping $f$ is odd. It follows from (3.21) that
\[
f(x + y) - f(x - y) = f(2y)
\]
for all $x, y \in A$. Replacing $x$ and $y$ by $\frac{x+y}{2}$ and $\frac{x-y}{2}$, respectively, we get $f(x + y) = f(x) + f(y)$ for all $x, y \in A$. Therefore by Lemma 2.1 the mapping $f : A \rightarrow B$ is $\mathbb{C}$-linear.

The rest of the proof is similar to the proof of Theorem 3.1. □
4. Homomorphisms between unital $C^*$-ternary algebras

Throughout this section, assume that $A$ is a $C^*$-ternary algebra with norm $\|\cdot\|_A$ and that $B$ a unital $C^*$-ternary algebra with norm $\|\cdot\|_B$ and unit $e'$.

We investigate homomorphisms between unital $C^*$-ternary algebras, associated to the functional equation $\Delta_\alpha^H(f(x,y,z)) = 0$.

**THEOREM 4.1.** Let $\alpha, \beta, \gamma$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $0 < p_1, p_2, p_3 < 1$, $0 < q_1, q_2 < 2$ and $0 < q_3 < 3$. Suppose that $f : A \to B$ is a mapping satisfying (2.13) and (2.14). If there exists a real number $\lambda > 1$ ($0 < \lambda < 1$) and an element $x_0 \in A$ such that $\lim_{n \to \infty} 1 \lambda^n f(\lambda^n x_0) = e'$ ($\lim_{n \to \infty} \lambda^n f(\frac{\lambda^n x_0}{\lambda^n}) = e'$), then the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism.

**Proof.** By Corollary 2.4 there exists a unique $C^*$-ternary algebra homomorphism $H : A \to B$ such that

$$\|f(x) - H(x)\|_B \leq \frac{\theta}{|\delta|} \left\{ \frac{1}{2 - 2p_1} \|x\|_{p_1}^{p_1} + \frac{1}{2 - 2p_2} \|x\|_{p_2}^{p_2} + \frac{1}{2 - 2p_3} \|x\|_{p_3}^{p_3} \right\}$$  \quad (4.1)$$

for all $x \in A$. It follows from (4.1) that

$$H(x) = \lim_{n \to \infty} \frac{1}{\lambda^n} f(\lambda^n x), \quad \left( H(x) = \lim_{n \to \infty} \lambda^n f(\frac{x}{\lambda^n}) \right)$$  \quad (4.2)$$

for all $x \in A$ and all real number $\lambda > 1$ ($0 < \lambda < 1$). Therefore by the assumption, we get that $H(x_0) = e'$. Let $\lambda > 1$ and $\lim_{n \to \infty} \frac{1}{\lambda^n} f(\lambda^n x_0) = e'$. It follows from (2.14) that

$$\left\| [H(x), H(y), H(z)] - [H(x), H(y), f(z)] \right\|_B$$

$$= \left\| H[x, y, z] - [H(x), H(y), f(z)] \right\|_B$$

$$= \lim_{n \to \infty} \frac{1}{\lambda^n} \left\| f(\lambda^n x, \lambda^n y, \lambda^n z) - f(\lambda^n x, f(\lambda^n y), f(\lambda^n z)) \right\|_B$$

$$\leq \epsilon \lim_{n \to \infty} \frac{1}{\lambda^n} \left[ \lambda^{nq_1} \|x\|_{p_1}^{q_1} + \lambda^{nq_2} \|y\|_{p_2}^{q_2} + \|z\|_{p_3}^{q_3} \right] = 0$$

for all $x, y, z \in A$. So $[H(x), H(y), H(z)] = [H(x), H(y), f(z)]$ for all $x, y, z \in A$. Letting $x = y = x_0$ in the last equality, we get $f(z) = H(z)$ for all $z \in A$. Similarly, one can show that $H(z) = f(z)$ for all $z \in A$ when $0 < \lambda < 1$ and $\lim_{n \to \infty} \lambda^n f(\frac{\lambda^n x_0}{\lambda^n}) = e'$. Therefore the mapping $f : A \to B$ is a $C^*$-ternary algebra homomorphism. \quad $\square$

**REMARK 4.2.** Theorem 4.1 will be valid if we replace the conditions $0 < q_1, q_2 < 2$ and $0 < q_3 < 3$ by $0 < q_2, q_3 < 2$ and $0 < q_1 < 3$, respectively.

**THEOREM 4.3.** Let $\alpha, \beta, \gamma$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3$ be non-negative real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_1, p_2, p_3 > 1$ and
\( q_1, q_2, q_3 > 2 \). Suppose that \( f : A \to B \) is a mapping satisfying (2.13) and
\[
\|f([x, y, z]) - [f(x), f(y), f(z)]\|_B \leq \epsilon \left( \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} + \|y\|_A^{q_4} \|z\|_A^{q_5} + \|x\|_A^{q_6} \|z\|_A^{q_7} \right)
\] (4.3)
for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \). If there exists a real number \( \lambda > 1 \) \((0 < \lambda < 1)\) and an element \( x_0 \in A \) such that \( \lim_{n \to \infty} \lambda^n f \left( \frac{x}{\lambda^n} \right) = e' \) \( (\lim_{n \to \infty} \frac{1}{\lambda^n} f (\lambda^n x_0) = e') \), then the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism.

**Proof.** By Theorem 2.5 there exists a unique \( C^* \)-ternary algebra homomorphism \( H : A \to B \) such that
\[
\|f(x) - H(x)\|_B \leq \frac{\theta}{|\delta|} \left( \frac{1}{2p_1 - 2} \|x\|_A^{p_1} + \frac{1}{2p_2 - 2} \|x\|_A^{p_2} + \frac{1}{2p_3 - 2} \|x\|_A^{p_3} \right)
\] (4.4)
for all \( x \in A \). It follows from (4.4) that
\[
H(x) = \lim_{n \to \infty} \lambda^n f \left( \frac{x}{\lambda^n} \right), \quad \left( H(x) = \lim_{n \to \infty} \frac{1}{\lambda^n} f (\lambda^n x) \right)
\] (4.5)
for all \( x \in A \) and all real number \( \lambda > 1 \) \((0 < \lambda < 1)\). Therefore by the assumption, we get that \( H(x_0) = e' \). Let \( \lambda > 1 \) and \( \lim_{n \to \infty} \lambda^n f \left( \frac{x}{\lambda^n} \right) = e' \). It follows from (4.3) that
\[
\left\| [H(x), H(y), H(z)] - [H(x), H(y), f(z)] \right\|_B = \lim_{n \to \infty} \lambda^n \left[ f \left( \frac{x}{\lambda^n}, \frac{y}{\lambda^n}, \frac{z}{\lambda^n} \right) - f \left( \frac{x}{\lambda^n}, \frac{y}{\lambda^n}, f(z) \right) \right]_B
\]
\[
\leq \epsilon \lim_{n \to \infty} \lambda^n \left[ \frac{1}{\lambda^{n(q_1 + q_2)}} \|x\|_A^{q_1} \|y\|_A^{q_2} \|z\|_A^{q_3} + \frac{1}{\lambda^{nq_1}} \|y\|_A^{q_3} \|z\|_A^{q_1} + \frac{1}{\lambda^{nq_3}} \|x\|_A^{q_3} \|y\|_A^{q_1} \right] = 0
\]
for all \( x, y, z \in A \). So \([H(x), H(y), H(z)] = [H(x), H(y), f(z)]\) for all \( x, y, z \in A \). Letting \( x = y = x_0 \) in the last equality, we get \( f(z) = H(z) \) for all \( z \in A \). Similarly, one can show that \( H(z) = f(z) \) for all \( z \in A \) when \( 0 < \lambda < 1 \) and \( \lim_{n \to \infty} \frac{1}{\lambda^n} f (\lambda^n x_0) = e' \). Therefore the mapping \( f : A \to B \) is a \( C^* \)-ternary algebra homomorphism. \( \square \)

5. **Stability of derivations on \( C^* \)-ternary algebras**

Throughout this section, assume that \( A \) is a \( C^* \)-ternary algebra with norm \( \| \cdot \|_A \).

In this section we prove the Hyers-Ulam-Rassias stability of derivations on \( C^* \)-ternary algebras for the functional equation \( \Lambda_{\alpha, \beta, \gamma} f(x, y, z) = 0 \).

**Theorem 5.1.** Let \( \alpha, \beta, \gamma \) be real numbers with \( \delta := \alpha + \beta + 2\gamma \neq 0 \), and let \( \phi : A^3 \to [0, \infty) \) and \( \psi : A^3 \to [0, \infty) \) be functions such that
\[
\tilde{\phi}(x) := \sum_{n=0}^{\infty} \frac{1}{2^n} \phi(2^n x, 2^n x, 2^n x) < \infty, \quad \lim_{n \to \infty} \frac{1}{2^n} \phi(2^n x, 2^n y, 2^n z) = 0,
\] (5.1)
\[
\lim_{n \to \infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0
\]  
(5.2)

for all \( x, y, z \in A \). Suppose that \( f : A \to A \) is a mapping satisfying

\[
\|\Delta^\mu_{\alpha, \beta, \gamma} f (x, y, z)\|_A \leq \varphi(x, y, z),
\]

(5.3)

\[
\left\| f ([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \right\|_A \leq \psi(x, y, z)
\]  
(5.4)

for all \( \mu \in T^1 \) and all \( x, y, z \in A \). Then there exists a unique \( C^* \)-ternary algebra derivation \( D : A \to A \) such that

\[
\left\| f(x) - D(x) \right\|_A \leq \frac{1}{2|\delta|} \tilde{\varphi}(x)
\]  
(5.5)

for all \( x \in A \).

**Proof.** By the proof of Theorem 2.3, there exists a unique \( \mathbb{C} \)-linear mapping \( D : A \to A \) satisfying (5.5) and

\[
D(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in A \). It follows from (5.2) and (5.4) that

\[
\left\| D[x, y, z] - [D(x), y, z] - [x, D(y), z] - [x, y, D(z)] \right\|_A
\]

\[
= \lim_{n \to \infty} \frac{1}{8^n} \left\| f(2^n x, 2^n y, 2^n z) - [f(2^n x), 2^n y, 2^n z] - [2^n x, f(2^n y), 2^n z] - [2^n x, 2^n y, f(2^n z)] \right\|_A
\]

\[
\leq \lim_{n \to \infty} \frac{1}{8^n} \psi(2^n x, 2^n y, 2^n z) = 0
\]

for all \( x, y, z \in A \). So

\[
D[x, y, z] = [D(x), y, z] + [x, D(y), z] + [x, y, D(z)]
\]

for all \( x, y, z \in A \). Therefore the mapping \( D : A \to A \) is a \( C^* \)-ternary algebra derivation. \( \square \)

**Theorem 5.2.** Let \( \alpha, \beta, \gamma \) be real numbers with \( \delta := \alpha + \beta + 2\gamma \neq 0 \), and let \( \varphi : A^3 \to [0, \infty) \) be a function satisfying (5.1). Suppose that the function \( \psi : A^3 \to [0, \infty) \) satisfies in one of the following conditions

(i) \( \lim_{n \to \infty} \frac{1}{2^n} \psi(2^n x, 2^n y, z) = 0 \);

(ii) \( \lim_{n \to \infty} \frac{1}{2^n} \psi(x, 2^n y, 2^n z) = 0 \);

(iii) \( \lim_{n \to \infty} \frac{1}{2^n} \psi(2^n x, y, 2^n z) = 0 \)

for all \( x, y, z \in A \). Let \( f : A \to A \) be a mapping satisfying (5.3) and (5.4). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation.
Proof. By the proof of Theorem 2.3, there exists a \( C \)-linear mapping \( D : A \to A \) defined by

\[
D(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)
\]

for all \( x \in A \). We show that if the mapping \( \psi \) satisfies in one of the conditions (i), (ii) or (iii), then \( f = D \).

Let \( \psi \) satisfy in (i) (we have a similar proof if \( \psi \) satisfies in (ii) or (iii)). It follows from (5.4) that

\[
\left\| D[y, z, x] - [D(x), y, z] - [x, D(y), z] - [x, y, f(z)] \right\|_A
= \lim_{n \to \infty} \frac{1}{4^n} \left\| f[2^n x, 2^n y, z] - f(2^n x, 2^n y, z)\right. - [2^n x, f(2^n y), z] - [2^n x, 2^n y, f(z)] \left\|_A
\leq \lim_{n \to \infty} \frac{1}{4^n} \psi(2^n x, 2^n y, z) = 0
\]

for all \( x, y, z \in A \). Therefore

\[
D([x, y, z]) = [D(x), y, z] + [x, D(y), z] + [x, y, f(z)] \tag{5.6}
\]

for all \( x, y, z \in A \). Replacing \( z \) by \( 2z \) in (5.6), we get

\[
2D([x, y, z]) = 2[D(x), y, z] + 2[x, D(y), z] + [x, y, f(2z)] \tag{5.7}
\]

for all \( x, y, z \in A \). It follows from (5.6) and (5.7) that

\[
[x, y, f(2z) - 2f(z)] = 0
\]

for all \( x, y, z \in A \). Letting \( x = y = f(2z) - 2f(z) \) in the last equation, we get

\[
\left\| f(2z) - 2f(z) \right\|_A^2 = \left\| [f(2z) - 2f(z), f(2z) - 2f(z), f(2z) - 2f(z)] \right\|_A = 0
\]

for all \( z \in A \). So \( f(2z) = 2f(z) \) for all \( z \in A \). By using induction, we infer that \( f(2^n z) = 2^n f(z) \) for all \( z \in A \) and all \( n \in \mathbb{Z} \). Therefore \( D(x) = f(x) \) for all \( x \in A \). Hence it follows from (5.6) that the mapping \( f : A \to A \) is a \(*\)-ternary derivation. \( \square \)

**Theorem 5.3.** Let \( \alpha, \beta, \gamma \) be real numbers with \( \delta := \alpha + \beta + 2\gamma \neq 0 \), and let \( \varphi : A^3 \to [0, \infty) \) be a function satisfying (5.1). Suppose that the function \( \psi : A^3 \to [0, \infty) \) satisfies in one of the following conditions

(i) \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(2^n x, y, z) = 0 \);
(ii) \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(x, 2^n y, z) = 0 \);
(iii) \( \lim_{n \to \infty} \frac{1}{2^n} \varphi(x, y, 2^n z) = 0 \)

for all \( x, y, z \in A \). Let \( f : A \to A \) be a mapping satisfying (5.3) and (5.4). Then the mapping \( f : A \to A \) is a \(*\)-ternary algebra derivation.
Proof. By the proof of Theorem 2.3, there exists a $\mathbb{C}$-linear mapping $D : A \to A$ defined by

$$D(x) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n x)$$

for all $x \in A$. We show that if the mapping $\psi$ satisfies in one of the conditions (i), (ii) or (iii), then $f = D$.

Let $\psi$ satisfy in (i) (we have a similar proof if $\psi$ satisfies in (ii) or (iii) ). It follows from (5.4) that

$$\left\| D[x, y, z] - [D(x), y, z] - [x, f(y), z] - [x, y, f(z)] \right\|_A$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \left\| f(2^n x, y, z) - [f(2^n x), y, z] - [2^n x, f(y), z] - [2^n x, y, f(z)] \right\|_A$$

$$\leq \lim_{n \to \infty} \frac{1}{2^n} \psi(2^n x, y, z) = 0$$

for all $x, y, z \in A$. Therefore

$$D([x, y, z]) = [D(x), y, z] + [x, f(y), z] + [x, y, f(z)]$$

(5.8)

for all $x, y, z \in A$.

The rest of the proof is similar to the proof Theorem 5.2. □

**COROLLARY 5.4.** Let $\alpha, \beta, \gamma$ be real numbers and let $\epsilon, \theta > 0$, $p_1, p_2, p_3, q_1, q_2, q_3 > 0$ be real numbers such that $\delta := \alpha + \beta + 2\gamma \neq 0$, $p_1, p_2, p_3 < 1$ and $q_i < 1$ for some $1 \leq i \leq 3$. Suppose that $f : A \to A$ is a mapping satisfying

$$\|\Delta_{\alpha, \beta, \gamma}^\mu f(x, y, z)\|_A \leq \theta(\|x\|_{A}^{p_1} + \|y\|_{A}^{p_2} + \|z\|_{A}^{p_3}),$$

$$\|f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)]\|_A \leq \epsilon(\|x\|_{A}^{q_1} + \|y\|_{A}^{q_2} + \|z\|_{A}^{q_3})$$

(5.9)

(5.10)

for all $\mu \in \mathbb{T}^3$ and all $x, y, z \in A$. Then the mapping $f : A \to A$ is a $C^*$-ternary algebra derivation.

**THEOREM 5.5.** Let $\alpha, \beta, \gamma$ be real numbers with $\delta := \alpha + \beta + 2\gamma \neq 0$, and let $\Phi : A^3 \to [0, \infty)$ and $\Psi : A^3 \to [0, \infty)$ be functions such that

$$\tilde{\Phi}(x) := \sum_{n=1}^{\infty} 2^n \Phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) < \infty, \quad \lim_{n \to \infty} 2^n \Phi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 0,$$

(5.11)

$$\lim_{n \to \infty} 8^n \Psi\left(\frac{x}{2^n}, \frac{x}{2^n}, \frac{x}{2^n}\right) = 0$$

(5.12)

for all $x, y, z \in A$. Suppose that $f : A \to A$ is a mapping satisfying

$$\|\Delta_{\alpha, \beta, \gamma}^\mu f(x, y, z)\|_A \leq \Phi(x, y, z),$$

(5.13)
\begin{equation}
\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \leq \Psi(x,y,z) \tag{5.14}
\end{equation}
for all \( \mu \in \mathbb{T}^3 \) and all \( x, y, z \in A \). Then there exists a unique \( C^* \)-ternary algebra derivation \( D : A \to A \) such that

\begin{equation}
\left\| f(x) - D(x) \right\|_A \leq \frac{1}{2|\delta|} \tilde{\Phi}(x) \tag{5.15}
\end{equation}
for all \( x \in A \).

\textbf{Proof.} By the proof of Theorem 2.5, there exists a unique \( \mathbb{C} \)-linear mapping \( D : A \to A \) satisfying (5.15) and

\[ D(x) := \lim_{n \to \infty} 2^n f\left( \frac{x}{2^n} \right) \]
for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 5.1. \( \square \)

\textbf{THEOREM 5.6.} Let \( \alpha, \beta, \gamma \) be real numbers with \( \delta := \alpha + \beta + 2\gamma \neq 0 \), and let \( \Phi : A^3 \to [0, \infty) \) be a function satisfying (5.11). Suppose that the function \( \Psi : A^3 \to [0, \infty) \) satisfies in one of the following conditions

(i) \( \lim_{n \to \infty} 4^n \Psi\left( \frac{x}{2^n}, \frac{y}{2^n}, z \right) = 0; \)

(ii) \( \lim_{n \to \infty} 4^n \Psi\left( x, \frac{y}{2^n}, \frac{z}{2^n} \right) = 0; \)

(iii) \( \lim_{n \to \infty} 4^n \Psi\left( \frac{x}{2^n}, y, \frac{z}{2^n} \right) = 0 \)

for all \( x, y, z \in A \). Let \( f : A \to A \) be a mapping satisfying (5.13) and (5.14). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation.

\textbf{Proof.} By the proof of Theorem 2.5, there exists a \( \mathbb{C} \)-linear mapping \( D : A \to A \) defined by

\[ D(x) := \lim_{n \to \infty} 2^n f\left( \frac{x}{2^n} \right) \]
for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 5.2. \( \square \)

\textbf{COROLLARY 5.7.} Let \( \alpha, \beta, \gamma \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3, q_1, q_2, q_3 \) be non-negative real numbers such that \( \delta := \alpha + \beta + 2\gamma \neq 0 \), \( p_1, p_2, p_3 > 1 \) and \( q_1, q_2, q_3 > 2 \). Suppose that \( f : A \to A \) is a mapping satisfying (5.9) and

\begin{equation}
\left\| f([x,y,z]) - [f(x),y,z] - [x,f(y),z] - [x,y,f(z)] \right\|_A \leq \epsilon \left( \|x\|_{A}^{q_1} \|y\|_{A}^{q_2} + \|y\|_{A}^{q_2} \|z\|_{A}^{q_3} + \|x\|_{A}^{q_1} \|z\|_{A}^{q_3} \right) \tag{5.16}
\end{equation}
for all \( x, y, z \in A \). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation.

\textbf{THEOREM 5.8.} Let \( \alpha, \beta, \gamma \) be real numbers with \( \delta := \alpha + \beta + 2\gamma \neq 0 \), and let \( \Phi : A^3 \to [0, \infty) \) be a function satisfying (5.11). Suppose that the function \( \Psi : A^3 \to [0, \infty) \) satisfies in one of the following conditions

(i) \( \lim_{n \to \infty} 2^n \Psi\left( \frac{x}{2^n}, y, \frac{z}{2^n} \right) = 0; \)
(ii) \( \lim_{n \to \infty} 2^n \Psi(x, \frac{y}{2^n}, z) = 0; \)
(iii) \( \lim_{n \to \infty} 2^n \Psi(x, y, \frac{z}{2^n}) = 0 \)
for all \( x, y, z \in A \). Let \( f : A \to A \) be a mapping satisfying (5.13) and (5.14). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation.

**Proof.** By the proof of Theorem 2.5, there exists a \( \mathbb{C} \)-linear mapping \( D : A \to A \) defined by
\[
D(x) := \lim_{n \to \infty} 2^n f \left( \frac{x}{2^n} \right)
\]
for all \( x \in A \).

The rest of the proof is similar to the proof of Theorem 5.3. \( \square \)

**THEOREM 5.9.** Let \( \alpha, \beta, \gamma, q_1, q_2, q_3 \) be real numbers and let \( \epsilon, \theta, p_1, p_2, p_3 \) be non-negative real numbers such that \( \delta := \alpha + \beta + 2\gamma \neq 0 \), \( p_1 + p_3 > 0 \) and \( q_i \neq 1 \) for some \( 1 \leq i \leq 3 \). Suppose that \( f : A \to A \) is a mapping satisfying
\[
\left\| \Delta_{\alpha, \beta, \gamma}^\mu f(x, y, z) \right\|_A \leq \theta \|x\|_{A}^{p_1} \|y\|_{A}^{p_2} \|z\|_{A}^{p_3},
\]
\[
\left\| f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \right\|_A \leq \epsilon \|x\|_{A}^{q_1} \|y\|_{A}^{q_2} \|z\|_{A}^{q_3}
\]
for all \( \mu \in \mathbb{T}^1 \) and all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Then the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation.

Note that we put \( \| \cdot \|_A^0 = 1 \).

**Proof.** Without any loss of generality, we can assume that \( q_1 \neq 1 \). Since \( p_1 + p_3 > 0 \), then we can let \( p_1 > 0 \) \( (p_3 > 0) \). Therefore it follows from the proof of Theorem 3.1 (Theorem 3.5) that the mapping \( f : A \to A \) is \( \mathbb{C} \)-linear. Let \( q_1 < 1 \). It follows from (5.18) that
\[
\left\| f([x, y, z]) - [f(x), y, z] - [x, f(y), z] - [x, y, f(z)] \right\|_A = \lim_{n \to \infty} \frac{1}{2^n} \left\| f[2^n x, y, z] - [f(2^n x), y, z] - [2^n x, f(y), z] - [2^n x, y, f(z)] \right\|_A \leq \epsilon \lim_{n \to \infty} \frac{2^{nq_1}}{2^n} \|x\|_{A}^{q_1} \|y\|_{A}^{q_2} \|z\|_{A}^{q_3} = 0
\]
for all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Therefore
\[
f([x, y, z]) = [f(x), y, z] + [x, f(y), z] + [x, y, f(z)]
\]
for all \( x, y, z \in A \) \( (x, y, z \in A \setminus \{0\} \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Since \( f(0) = 0 \), then (5.19) holds for all \( x, y, z \in A \) when \( q_i < 0 \) for some \( 1 \leq i \leq 3 \). Similarly, we get (5.19) when \( q_1 > 1 \). So the mapping \( f : A \to A \) is a \( C^* \)-ternary algebra derivation. \( \square \)
THEOREM 5.10. Let $\alpha, \beta, \gamma, q_1, q_2, q_3$ be real numbers and let $\epsilon, \theta, p_1, p_2, p_3$ be non-negative real numbers such that $\alpha + \gamma \neq 0$, $p_2 > 0$ and $q_i \neq 1$ for some $1 \leq i \leq 3$. Suppose that $f : A \rightarrow A$ is a mapping satisfying (5.17) and (5.18). Then the mapping $f : A \rightarrow B$ is a $C^*$-ternary algebra derivation.

Note that we put $\|\|_A^0 = 1$.

Proof. Without any loss of generality, we can assume that $q_1 \neq 1$. It follows from the proof of Theorem 3.3 that the mapping $f : A \rightarrow A$ is $\mathbb{C}$-linear. Let $q_1 > 1$. It follows from (5.18) that

$$\left\| f ([x, y, z]) - [f (x), y, z] - [x, f (y), z] - [x, y, f (z)] \right\|_A$$

$$= \lim_{n \to \infty} 2^n \left\| f \left( \left\{ \frac{x}{2^n}, y, z \right\} \right) - f \left( \left\{ \frac{x}{2^n}, y, z \right\} \right) - \left[ \frac{x}{2^n}, f (y), z \right] - \left[ \frac{x}{2^n}, y, f (z) \right] \right\|_A$$

$$\leq \epsilon \lim_{n \to \infty} \frac{2^n}{2 n q_1} \|x\|_A \|y\|_A \|z\|_A^q = 0$$

for all $x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $2 \leq i \leq 3$. Therefore

$$f ([x, y, z]) = [f (x), y, z] + [x, f (y), z] + [x, y, f (z)] \quad (5.20)$$

for all $x, y, z \in A \setminus \{0\}$ when $q_i < 0$ for some $2 \leq i \leq 3$. Since $f (0) = 0$, then (5.19) holds for all $x, y, z \in A$ when $q_i < 0$ for some $1 \leq i \leq 3$. Similarly, we get (5.20) when $q_1 < 1$. So the mapping $f : A \rightarrow A$ is a $C^*$-ternary algebra derivation. \[\square\]

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ON HOMOMORPHISMS BETWEEN $C^*$ -TERNARY ALGEBRAS


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