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ANTI–PERIODIC BOUNDARY VALUE PROBLEMS FOR NONLINEAR HIGHER ORDER FUNCTIONAL DIFFERENCE EQUATIONS

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Abstract. Sufficient conditions for the existence of at least one solution of anti-periodic boundary value problems for nonlinear functional difference equations are established. We allow f to be at most linear, superlinear or sublinear in obtained results.

1. Introduction

Let Z be the integers set, $[a,b]=\{a,a+1,\cdots,b\}$ for each $a,b\in Z$ with $a<b$. In this paper, we study anti-periodic boundary value problems for nonlinear functional difference equations

$$\begin{cases} \Delta x(n)=f(n,x(n),x(n+1),x(n-\tau_1(n)),\cdots,x(n-\tau_m(n)), & n\in[0,T],\\ x(0)=-x(T+1), \\ x(i)=\phi(i), \quad i\in[-\tau,-1], \\ x(i)=\psi(i), \quad i\in[T+2,T+\delta], \end{cases} \tag{1}$$

and

$$\begin{cases} \Delta^2 x(n)=f(n,x(n),x(n+1),x(n-\tau_1(n)),\cdots,x(n-\tau_m(n)), & n\in[0,T],\\ x(0)=-x(T+1), \\ x(1)=-x(T+2), \\ x(i)=\phi(i), \quad i\in[-\tau,-1], \\ x(i)=\psi(i), \quad i\in[T+3,T+\delta] \end{cases} \tag{2}$$

where $T\geqslant 1$, $\tau_i:[0,T]\to Z$, $i=1,\cdots,m$, $f(n,x,x_0,x_1,\cdots,x_{m+1})$ is continuous for each $n\in[0,T]$ with

$$\tau=\max\{\max_{n\in[0,T]}\{\tau_i(n)\}:\quad i=1,\cdots,m\},$$
$$\delta=-\min\{\min_{n\in[0,T]}\{\tau_i(n)\}:\quad i=1,\cdots,m\}.$$

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The motivation of this paper is mainly as follows:

In a recent paper [5], by using a fixed point theorem for operators on cones, Sun established existence criteria for positive solutions for the following first-order discrete periodic boundary value problem:

$$\begin{cases} \Delta x(n) = f(n, x(n+1)), & n \in [0,T], \\ x(0) = x(T+1). \end{cases}$$

In paper [6], periodic and antiperiodic boundary value problems for self-adjoint second-order difference equations were studied. Existence of eigenvalues of these two different boundary value problems was proved, numbers of their eigenvalues were calculated, and their relationships were obtained. In addition, a representation of solutions of a nonhomogeneous linear equation with initial conditions was given.

Fixed point theorems for operators on cones are also used to get positive periodic solutions of the equations

$$x(n+1) = b(n)x(n) + \lambda h(n) f(x(n-\tau(n))), \tag{3}$$

and

$$x(n+1) = b(n)x(n) - \lambda h(n) f(x(n-\tau(n))), \tag{4}$$

where $b(n)$, $h(n)$ and $\tau(n)$ are nonnegative and with period of T and $1 > b(n) > 0$ in (3) and $b(n) > 1$ in (4), T is an integer with $T \geqslant 1$. One may see [1 - 4] and the references therein.

There is no paper concerned with the solvability of problem (1) and problem (2). The purpose of this paper is to establish sufficient conditions for the existence of at least one solution of problem (1) and problem (2), respectively.

This paper is organized as follows. In Section 2, we give the main results and in Section 3, examples to illustrate the main results will be presented.

2. Main Results

Let X and Y be Banach spaces, $L:\ \mathrm{Dom}\, L \subset X \to Y$ be a Fredholm operator of index zero. If Ω is an open bounded subset of X, $\mathrm{Dom}\, L \cap \overline{\Omega} \neq \emptyset$, the map $N:\ X \to Y$ will be called $L-$ compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N:\ \overline{\Omega} \to X$ is compact.

THEOREM GM[1]. *Let X and Y be Banach spaces. Suppose $L: D(L) \subset X \to Y$ is a Fredholm operator of index zero with* $\mathrm{Ker}\, L = \{0\}$, *$N: X \to Y$ is $L-$ compact on any open bounded subset of X. If $0 \in \Omega \subset X$ is a open bounded subset and*

$$Lx \neq \lambda Nx \ \textit{for all}\ x \in D(L) \cap \partial\Omega \ \textit{and}\ \lambda \in [0,1],$$

then there is at least one $x \in \Omega$ so that $Lx = Nx$.

Let $X = R^{T+\tau+\delta}$ be endowed with the norm $||x||_X = \max_{n \in [1,T+\tau+\delta]} |x(n)|$ for $x \in X$, $Y = R^{T+1}$ be endowed with the norm $||y||_Y = \max_{n\in[0,T]} |y(n)|$ for $y \in Y$. It is easy to see that X and Y are Banach spaces.

For problem (1), choose

$$\text{Dom}\,L = \left\{ x \in X: \begin{array}{ll} x(i) = 0,\ i \in [-\tau, \cdots, -1], & \\ x(i) \in R,\quad i \in [0, T+1], & x(0) = -x(T+1) \\ x(i) = 0,\quad i \in [T+2, \cdots, T+\delta], & \end{array} \right\}.$$

Set

$$L : \text{Dom}\,L \cap X \to X, \quad L \bullet x(n) = \Delta x(n), \quad n \in [0, T],$$

and $N : X \to Y$ by

$$\begin{aligned} N \bullet x(n) \;&=\; f(n, x(n) + x_0(n), x(n+1) + x_0(n+1), \\ &\qquad x(n-\tau_1(n)) + x_0(n-\tau_1(n)), \cdots, x(n-\tau_m(n)) + x_0(n-\tau_m(n))) \end{aligned}$$

$n \in [0, T]$, for all $x \in X$, where

$$x_0(n) = \begin{cases} \phi(n), \quad n \in [-\tau, -1], \\ 0, \quad n \in [0, T+1], \\ \psi(n), \quad n \in [T+2, T+\delta]. \end{cases}$$

It is easy to show that $\Delta x_0(n) = 0$ for $n \in [0, T]$ and that $x \in \text{Dom}\,L$ is a solution of $L \bullet x = N \bullet x$ implies that $x + x_0$ is a solution of problem (1) and

(i) $\text{Ker}L = \{(0, \cdots, 0) \in X\}$.

(ii) L is a Fredholm operator of index zero and N is $L-$ compact on $\overline{\Omega}$ with Ω being an open bounded nonempty subset of X.

THEOREM L1. *Suppose that there is numbers* $\beta > 0$, $\theta > 1$, *nonnegative sequences* $p(n)$, $p_i(n)$, $r(n)$ $(i = 0, \cdots, m)$, *functions* $g(n, x, x_0, \cdots, x_m)$, $h(n, x, x_0, \cdots, x_m)$ *such that*

$$f(n, x, x_0, \cdots, x_m) = g(n, x, x_0, \cdots, x_m) + h(n, x, x_0, \cdots, x_m)$$

and

$$g(n, x, x_0, x_1, \cdots, x_m)x_0 \leqslant -\beta |x_0|^{\theta+1},$$

and

$$|h(n, x, x_0, \cdots, x_m)| \leqslant p(n)|x|^\theta + \sum_{s=0}^{m} p_i(n)|x_i|^\theta + r(n),$$

for all $n \in \{0, \cdots, T\}$, $(x, x_0, x_1, \cdots, x_m) \in R^{m+2}$. *Then problem (1) has at least one solution if*

$$||p||_Y + ||p_0||_Y + (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||_Y < \beta. \tag{5}$$

Proof. Let $\Omega_1 = \{x : Lx = \lambda Nx,\ (x, \lambda) \in (\text{Dom}L) \times (0, 1)\}$, we prove that Ω_1 is bounded. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0, 1)$, so

$$\Delta y(n) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))), \tag{6}$$

where $y(n) = x(n) + x_0(n)$. So

$$[\Delta y(n)]y(n+1) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))y(n+1).$$

It is easy to see that $y(T+1)=-y(0)$ and that

$$\begin{aligned}2\sum_{n=0}^{T}[\Delta y(n)]y(n+1) &= 2\sum_{n=0}^{T}[y(n+1)^2-y(n)y(n+1)]\\ &= \sum_{n=0}^{T}[y(n+1)-y(n)]^2-y(0)^2+y(T+1)^2\geqslant 0.\end{aligned}$$

So, we get

$$\sum_{n=0}^{T} f\left(n,y(n),y(n+1),y(n-\tau_1(n)),\cdots,y(n-\tau_m(n))\right)y(n+1)\geqslant 0.$$

It follows that

$$\begin{aligned}&\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\ &\quad\leqslant -\sum_{n=0}^{T} g(n,y(n),y(n+1),y(n-\tau_1(n)),\cdots,y(n-\tau_m(n)))y(n+1)\\ &\quad\leqslant \sum_{n=0}^{T} h(n,y(n),y(n+1),y(n-\tau_1(n)),\cdots,y(n-\tau_m(n)))y(n+1)\\ &\quad\leqslant \sum_{n=0}^{T} |h(n,y(n),y(n+1),y(n-\tau_1(n)),\cdots,y(n-\tau_m(n)))|\,|y(n+1)|\\ &\quad\leqslant \sum_{n=0}^{T} p(n)|y(n+1)||y(n)|^{\theta}+\sum_{n=0}^{T} p_0(n)|y(n+1)|^{\theta+1}\\ &\qquad+\sum_{i=1}^{m}\sum_{n=0}^{T} p_i(n)|y(n-\tau_i(n))|^{\theta}|y(n+1)|+\sum_{n=0}^{T} r(n)|y(n+1)|\\ &\quad\leqslant ||p||_Y\sum_{n=0}^{T}|y(n+1)||y(n)|^{\theta}+||p_0||_Y\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\ &\qquad+\sum_{i=1}^{m}||p_i||_Y\sum_{n=0}^{T}|y(n-\tau_i(n))|^{\theta}|y(n+1)|+||r||_Y\sum_{n=0}^{T}|y(n+1)|.\end{aligned}$$

Hence

$$\begin{aligned}\beta\sum_{n=0}^{T}|y(n+1)|^{\theta+1} &\leqslant ||p||_Y\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\left(\sum_{n=0}^{T}|y(n)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\\ &\quad+||p_0||_Y\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\ &\quad+\sum_{i=1}^{m}||p_i||_Y\left(\sum_{n=0}^{T}|y(n-\tau_i(n))|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\ &\quad+||r||_Y(T+1)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\ &= (||p||_Y+||p_0||_Y)\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\ &\quad\sum_{i=1}^{m}||p_i||_Y\left(\sum_{u\in\{n-\tau_i(n)-1:\ n=0,\cdots,T\}}|y(u+1)|^{\theta+1}\right)^{\frac{\theta}{\theta+1}}\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\
&+||r||_Y(T+1)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\
\leqslant\ &(||p||_Y+||p_0||_Y)\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\
&(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||_Y\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\
&+||r||_Y(T+1)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}\\
=\ &(||p||_Y+||p_0||_Y-(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||_Y)\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\\
&+||r||_Y(T+1)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}.
\end{aligned}
$$

We get

$$
\begin{aligned}
\left(\beta-||p||_Y-||p_0||_Y-(T+1)^{\frac{\theta}{\theta+1}}\sum_{i=1}^{m}||p_i||_Y\right)\sum_{u=0}^{T}|y(u+1)|^{\theta+1}\\
\leqslant ||r||_Y(T+1)^{\frac{\theta}{\theta+1}}\left(\sum_{n=0}^{T}|y(n+1)|^{\theta+1}\right)^{\frac{1}{\theta+1}}.
\end{aligned}
$$

It follows from (5) that there is $M_1>0$ such that $\sum_{u=0}^{T}|y(u+1)|^{\theta+1}\leqslant M_1$.

Hence $|y(n+1)|\leqslant M_1^{1/(\theta+1)}$ for all $n\in\{0,\cdots,T\}$. Thus we get

$$|x(n+1)|\leqslant|y(n+1)|+|x_0(n+1)|\leqslant M_1^{1/(\theta+1)}+||x_0||_X,\quad n\in[0,\cdots,T].$$

Hence $||x||_X\leqslant M_1^{1/(\theta+1)}+||x_0||_X$. So Ω_1 is bounded.

Let $\Omega\supset\overline{\Omega_1}$ be an open bounded subset of X, it is easy to see that $Lx\neq\lambda Nx$ for all $x\in\operatorname{Dom}L\cap\partial\Omega$ and $\lambda\in[0,1]$.

Thus by Theorem GM, $Lx=Nx$ has at least one solution in $\operatorname{Dom}L\cap\overline{\Omega}$, so $x+x_0$ is a solution of problem (1).The proof is completed. □

THEOREM L2. *Suppose that there is numbers $\beta>0$, $\theta>1$, nonnegative sequences $p(n)$, $p_i(n)$, $r(n)$ $(i=0,\cdots,m)$, functions $g(n,x,x_0,\cdots,x_m)$, $h(n,x,x_0,\cdots,x_m)$ such that*

$$f(n,x,x_0,\cdots,x_m)=g(n,x,x_0,\cdots,x_m)+h(n,x,x_0,\cdots,x_m)$$

and

$$g(n,x,x_0,x_1,\cdots,x_m)x \geqslant \beta|x|^{\theta+1},$$

and

$$|h(n,x,x_0,\cdots,x_m)| \leqslant p(n)|x|^{\theta} + \sum_{s=0}^{m} p_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{0,\cdots,T\}$, $(x,x_0,x_1,\cdots,x_m) \in R^{m+2}$. *Then problem (1) has at least one solution if*

$$||p||_Y + ||p_0||_Y + (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||_Y < \beta. \tag{7}$$

Proof. Let $\Omega_1 = \{x : Lx = \lambda Nx,\ (x,\lambda) \in (\mathrm{Dom}L) \times (0,1)\}$. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0,1)$, so

$$\Delta^2 y(n) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))), \tag{8}$$

where $y(n) = x(n) + x_0(n)$. So

$$[\Delta^2 y(n)]y(n) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))y(n).$$

It is easy to see from $y(T+1) = -y(0)$ that

$$\begin{aligned} 2\sum_{n=0}^{T}[\Delta y(n)]y(n) &= 2\sum_{n=0}^{T}[y(n+1)y(n) - y(n)^2] \\ &= -\sum_{n=0}^{T}[y(n+1)-y(n)]^2 - y(0)^2 + y(T+1)^2 \leqslant 0. \end{aligned}$$

So, we get

$$\sum_{n=0}^{T-1} f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))y(n) \leqslant 0.$$

The remainder of the proof of is just similar to that of the proof of Theorem L1 and is omitted. □

Let $X = R^{T+\tau+\delta+1}$ be endowed with the norm $||x||_X = \max_{n\in[1,T+\tau+\delta+1]} |x(n)|$ for $x \in X$, $Y = R^{T+1}$ be endowed with the norm $||y||_Y = \max_{n\in[0,T]} |y(n)|$ for $y \in Y$. It is easy to see that X and Y are Banach spaces.

For problem (2), choose

$$\mathrm{Dom}\,L = \left\{ x \in: \begin{array}{ll} x(i) = 0,\ i \in [-\tau,\cdots,-1], & \\ x(i) \in R, \quad i \in [0, T+2], & x(0) = -x(T+1), x(1) = -x(T+2) \\ x(i) = 0, \quad i \in [T+3,\cdots,T+\delta], & \end{array} \right\}.$$

Set

$$L : \mathrm{Dom}\,L \cap X \to X, \quad L \bullet x(n) = \Delta^2 x(n), \quad n \in [0,T],$$

and $N : X \to Y$ by

$$\begin{aligned} N \bullet x(n) &= f(n, x(n) + x_0(n), x(n+1) + x_0(n+1), \\ &\qquad x(n-\tau_1(n)) + x_0(n-\tau_1(n)), \cdots, x(n-\tau_m(n)) + x_0(n-\tau_m(n))) \end{aligned}$$

$n \in [0,T]$, for all $x \in X$, where

$$x_0(n) = \begin{cases} \phi(n), & n \in [-\tau,-1], \\ 0, & n \in [0,T+2], \\ \psi(n), & n \in [T+3,T+\delta]. \end{cases}$$

It is easy to show that $\Delta x_0(n) = 0$ for $n \in [0,T]$ and that $x \in \mathrm{Dom}\, L$ is a solution of $L \bullet x = N \bullet x$ implies that $x + x_0$ is a solution of problem (2) and

(*i*) $\mathrm{Ker} L = \{(0,\cdots,0) \in X\}$.

(*ii*) L is a Fredholm operator of index zero and N is $L-$ compact on $\overline{\Omega}$ with Ω being an open bounded nonempty subset of X.

We have the following result.

THEOREM L3. *Suppose that there is numbers $\beta > 0$, $\theta > 1$, nonnegative sequences $p(n)$, $p_i(n)$, $r(n)$ $(i = 0,\cdots,m)$, functions $g(n,x,x_0,\cdots,x_m)$, $h(n,x,x_0,\cdots,x_m)$ such that*

$$f(n,x,x_0,\cdots,x_m) = g(n,x,x_0,\cdots,x_m) + h(n,x,x_0,\cdots,x_m)$$

and

$$g(n,x,x_0,x_1,\cdots,x_m)x_0 \geqslant \beta |x_0|^{\theta+1},$$

and

$$|h(n,x,x_0,\cdots,x_m)| \leqslant p(n)|x|^{\theta} + \sum_{s=0}^{m} p_i(n)|x_i|^{\theta} + r(n),$$

for all $n \in \{0,\cdots,T\}$, $(x,x_0,x_1,\cdots,x_m) \in R^{m+2}$. Then problem (2) has at least one solution if

$$||p||_Y + ||p_0||_Y + (T+1)^{\frac{\theta}{\theta+1}} \sum_{i=1}^{m} ||p_i||_Y < \beta. \tag{9}$$

Proof. Let $\Omega_1 = \{x : Lx = \lambda N x,\ (x,\lambda) \in [(\mathrm{Dom}\, L \setminus \mathrm{Ker}\, L)] \times (0,1)\}$, we prove that Ω_1 is bounded. For $x \in \Omega_1$, we have $L \bullet x = \lambda N \bullet x$, $\lambda \in (0,1)$, so

$$\Delta^2 y(n) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))), \tag{10}$$

where $y(n) = x(n) + x_0(n)$. So

$$[\Delta^2 y(n)]y(n+1) = \lambda f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n))y(n+1).$$

It is easy to see that $y(T+1) = -y(0)$ and $y(1) = -y(T+2)$ and that

$$\begin{aligned} 2\textstyle\sum_{n=0}^{T}[\Delta^2 y(n)]y(n+1) &= 2\textstyle\sum_{n=0}^{T}[y(n+2)y(n+1) - 2y(n+1)^2 + y(n)y(n+1)] \\ &= -\textstyle\sum_{n=0}^{T}\big((y(n+2)-y(n+1))^2 + (y(n+1)-y(n))^2 \\ &\quad -y(0)^2 + y(1)^2 + y(T+1)^2 - y(T+2)^2\big) \\ &\leqslant 0. \end{aligned}$$

So, we get

$$\sum_{n=0}^{T} f(n, y(n), y(n+1), y(n-\tau_1(n)), \cdots, y(n-\tau_m(n)) y(n+1) \leqslant 0.$$

The remainder of the proof is just similar to that of Theorem L2 and is omitted. □

3. Examples

In this section, we present some examples to illustrate the main results in Section 2.

EXAMPLE 3.1. Consider the following problem

$$\begin{cases} \Delta x(n) = p_0(n)[x(n)]^{2k+1} + \beta[x(n+1)]^{2k+1} \\ \qquad\qquad + \sum_{i=1}^{m} p_i(n+1)[x(n-\tau_i(n))]^{2k+1} + r(n), \\ x(0) = -x(T+1), \\ x(i) = \phi(i), \quad i \in [-\tau, -1], \\ x(i) = \psi(i), \quad i \in [T+2, \delta] \end{cases} \tag{11}$$

where $k \geqslant 0$ an integer, $\beta < 0$, $p(n), p_i(n), r(n), \tau_i(n)$ $(i = 1, \cdots, m)$ are sequences. Corresponding to (1), we find

$$f(n, x_0, \cdots, x_m) = p_0(n)x^{2m+1} + \beta x_0^{2m+1} + \sum_{i=1}^{m} p_i(n) x_i^{2m+1} + r(n).$$

It is easy to see that (11) has at least one solution for every $r(n)$ if $||p||_Y + ||p_0||_Y + (T+1)^{\frac{2k+1}{2k+2}} \sum_{i=1}^{m} ||p_i||_Y < -\beta$.

EXAMPLE 3.2. Consider the following problem

$$\begin{cases} \Delta^2 x(n) = \beta[x(n)]^{2k+1} + \alpha[x(n+1)]^{2k+1} \\ \qquad\qquad + \gamma[x(n+2)]^{2k+1} + r(n), \quad n \in [0, T], \\ x(0) = -x(T+1), \\ x(1) = -x(T+2), \end{cases} \tag{12}$$

where $k \geqslant 0$ an integer, $\alpha > 0$. It follows from Theorem L3 that (12) has at least one solution for each r if $|\beta| + (T+1)^{\frac{2k+1}{2k+2}} |\gamma| < \alpha$.

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