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WRIGHT–CONVEXITY WITH RESPECT TO ARBITRARY MEANS

MICHAŁ LEWICKI

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Abstract. Let $I \subset \mathbb{R}$ be an open interval and $M,\ N : I^2 \longrightarrow I$ be means on I. We give sufficient conditions on means M and N under which every first Baire class solution $f : I \longrightarrow \mathbb{R}$ of the functional inequality

$$f(M(x,y)) + f(N(x,y)) \leqslant f(x) + f(y), \quad x, y \in I,$$

is convex.

1. Introduction

Let $I \subseteq \mathbb{R}$ be an open interval. By a mean we mean a function $M : I^2 \longrightarrow I$ such that

$$\min(x,y) \leqslant M(x,y) \leqslant \max(x,y), \quad x, y \in I.$$

If for all $x, y \in I$, $x \neq y$, these inequalities are strict, we call M to be a strict mean.

A function $f : I \longrightarrow \mathbb{R}$ is said to be (M,N)-Wright convex if

$$f(M(x,y)) + f(N(x,y)) \leqslant f(x) + f(y), \quad x, y \in I, \tag{1}$$

where $M,\ N : I^2 \longrightarrow I$ are means such that

$$M(x,y) + N(x,y) = x + y, \quad x, y \in I. \tag{2}$$

It can be easily derived from J. Matkowski, M. Wróbel [3] that every lower semicontinuous function satisfying (1), with strict continuous means $M,\ N : I^2 \longrightarrow I$ satisfying (2), is convex. In this note we show that this result remains true if f is of the first Baire class.

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2. Lower hull of (M,N)-Wright convex functions

Let $I \subseteq \mathbb{R}$ be an open interval. By $m_f : I \longrightarrow \bar{\mathbb{R}}$ we mean a lower hull of $f : I \longrightarrow \mathbb{R}$. In this section we will show that if f is (M,N)-Wright convex then m_f is also (M,N)-Wright convex.

We will start with the lemma.

LEMMA 2.1. *Let $I \subseteq \mathbb{R}$ be an open interval and M, $N : I^2 \longrightarrow I$ are strict continuous means on I. Further, assume that, for every fixed $u,v \in I$, $M(u,\cdot)$ and $M(\cdot,v)$ are strictly increasing functions. If a function $f : I \longrightarrow \mathbb{R}$ is locally bounded at a point $x \in I$ and is a solution of inequality (1), then f is locally bounded.*

Proof. Let us consider the set

$$B := \{y \in I : \text{f is locally bounded at a point y}\}.$$

Of course B is not empty and open.

Assume that $B \neq I$ and fix $z \in I \setminus B$. Without loss of generality we may assume that

$$\bigvee_{\epsilon>0} (z-\epsilon, z) \subseteq B.$$

Fix $u \in (z-\epsilon,z)$. Because M,N are strict means so $M(u,z), N(u,z) \in (z-\epsilon,z)$. Now, the continuity of the means gives

$$\bigvee_{\epsilon>\eta>0} \bigvee_{L_1>0} \bigwedge_{\rho\in(z-\eta,z+\eta)} [|f(M(u,\rho))| \leqslant L_1 \wedge |f(N(u,\rho))| \leqslant L_1]. \tag{3}$$

Fix a $v \in (z-\eta, z+\eta)$. Taking into account inequalities (1) and (3) we get

$$f(v) \geqslant f(M(u,v)) + f(N(u,v)) - f(u) \geqslant -2L_1 - f(u) =: L_2.$$

Hence f is bounded from below in the set $(z-\eta, z+\eta)$.

It is easy to show (by continuity of M), that

$$\bigvee_{s\in(z-\eta,z+\eta)} \bigvee_{t\in(z,z+\epsilon)} z = M(s,t).$$

The point $s \in B$, so there exists $\epsilon_s > 0$ such that $cl(s-\epsilon_s, s+\epsilon_s) \subseteq (z-\eta, z+\eta) \cap B$. Hence, by continuity of N and compactness of $cl(s-\epsilon_s, s+\epsilon_s)$, we get

$$\bigwedge_{\rho\in(s-\epsilon_s,s+\epsilon_s)} f(N(\rho,t)) \geqslant L_2, \tag{4}$$

and

$$\bigvee_{L_3>0} \bigwedge_{\rho\in(s-\epsilon_s,s+\epsilon_s)} |f(\rho)| \leqslant L_3. \tag{5}$$

Notice that the set $M((s-\epsilon_s, s+\epsilon_s), t)$ is an open neighbourhood of z. Now, fix $r \in M((s-\epsilon_s, s+\epsilon_s), t)$. There exists $w \in (s-\epsilon_s, s+\epsilon_s)$ such that $r = M(w,t)$. By (1), (4) and (5) we get

$$f(r) = f(M(w,t)) \leqslant f(w) + f(t) - f(N(w,t)) \leqslant L_3 + f(t) - L_2.$$

Hence function f is bounded on the set $V := (z-\eta, z+\eta) \cap M((s-\epsilon_s, s+\epsilon_s), t)$, which is the open neighbourhood of z. The contradiction so obtained completes the proof. $\square$

For $f : I \longrightarrow \mathbb{R}$ we define a lower hull $m_f : I \longrightarrow \bar{\mathbb{R}}$ (see [1]) by

$$m_f(x) := \lim_{\eta \to 0+} \varphi_x(\eta) = \sup_{\eta > 0} \varphi_x(\eta), \tag{6}$$

where

$$\varphi_x(\eta) := \inf_{w \in K(x,\eta)} f(w), \quad \eta > 0. \tag{7}$$

It is easy to see that the inequality

$$m_f(x) \leqslant f(x), \quad x \in I, \tag{8}$$

holds true.

Now we give a lemma

LEMMA 2.2. *Let $I \subseteq \mathbb{R}$ be an open interval and $M, N : I^2 \longrightarrow I$ be continuous strict means on I. If locally bounded function $f : I \longrightarrow \mathbb{R}$ is (M,N)-Wright convex then the function m_f is also (M,N)-Wright convex.*

Proof. Fix $x, y \in I$ and $\epsilon > 0$.

By definition and Lemma 2.1 m_f is a real valued function.

It follows directly from (6) and (7) that there exists $\gamma > 0$ such that

$$(*) \begin{cases} m_f(M(x,y)) - \frac{\epsilon}{4} \leqslant f(t), \quad t \in (M(x,y)-\gamma, M(x,y)+\gamma), \\ \text{and} \\ m_f(N(x,y)) - \frac{\epsilon}{4} \leqslant f(t), \quad t \in (N(x,y)-\gamma, N(x,y)+\gamma). \end{cases}$$

By continuity of M and N there exists $\eta > 0$ such that

$$K(x,\eta) \times K(y,\eta) \subset M^{-1}(K(M(x,y),\gamma)) \cap N^{-1}(K(N(x,y),\gamma)).$$

Condition (7) implies

$$(**) \begin{cases} \bigvee_{u \in K(x,\eta)} f(u) - \frac{\epsilon}{4} \leqslant \varphi_x(\eta), \\ \bigvee_{v \in K(y,\eta)} f(v) - \frac{\epsilon}{4} \leqslant \varphi_y(\eta). \end{cases}$$

We have $M(u,v) \in K(M(x,y),\gamma)$ and $N(u,v) \in K(N(x,y),\gamma)$. Finally, from $(*)$, $(**)$ and inequality (1) we get

$$m_f(x) + m_f(y) \geqslant \varphi_x(\eta) + \varphi_y(\eta) \geqslant f(u) - \frac{\epsilon}{4} + f(v) - \frac{\epsilon}{4}$$
$$\geqslant f(M(u,v)) + f(N(u,v)) - \frac{\epsilon}{2} \geqslant m_f(M(u,v)) + m_f(N(u,v)) - \epsilon.$$

Then Letting $\epsilon \longrightarrow 0+$ we obtain our assertation. □

Let us recall the main result of J. Matkowski and M. Wróbel [3].

THEOREM 2.1. *Let $M, N : I \times I \longrightarrow I$ be continuous functions satisfying*

$$x, y \in I, x \neq y \Rightarrow M(x,y), N(x,y) \in (\min(x,y), \max(x,y)),$$

and suppose that $\varphi : I \longrightarrow \mathbb{R}$ is a non-constant and continuous solution of equation

$$\varphi(M(x,y)) + \varphi(N(x,y)) = \varphi(x) + \varphi(y), \quad x, y \in I.$$

Then φ is one-to-one, and for every lower semicontinuous function $f : I \longrightarrow \mathbb{R}$ satisfying (1), the function $f \circ \varphi^{-1}$ is convex on $\varphi(I)$.

As an immediate consequence we get,

COROLLARY 2.1. *Let $I \subseteq \mathbb{R}$ be an open interval and $M, N : I^2 \longrightarrow I$ be strict continuous means on I. If a locally bounded function $f : I \longrightarrow \mathbb{R}$ is (M,N)-Wright convex then m_f is a convex function.*

Proof. According to Lemma 2.2 function m_f is (M,N)-Wright convex. It is well known (see [1]) that a lower hull of arbitrary function $f : I \longrightarrow \mathbb{R}$ is lower semicontinuous. Putting $\varphi := id_I$ in Theorem 2.1 we get the thesis. □

3. Main result

LEMMA 3.1. *Let $I \subseteq \mathbb{R}$ be an open interval. Assume that $M : I^2 \longrightarrow I$ is a strict continuous means on I, such that, for every fixed $u, v \in I$, functions $M(u,\cdot)$ and $M(\cdot,v)$ are strictly increasing. Let a set $B \subseteq I$ be such that the set $B \setminus I$ is of the first category. Then, for all $\epsilon > 0$ and $z \in I$ there exist $x, y \in K(z,\epsilon) \cap B$ such that $z = M(x,y)$.*

Proof. Fix $z \in I$ and $\epsilon > 0$.

By continuity of M there exist $i, m \in K(z,\epsilon)$ such that $z = M(i,m)$. We show now that for every $w \in (i,m)$ there exists exactly one $w' \in (i,m)$ such that

$$z = M(w,w'), \tag{9}$$

holds.

We may assume that $w < z$. Function $M(\cdot,m)$ is strictly increasing so $M(i,m) < M(w,m)$. As the function $M(w,\cdot)$ is continuous and strictly increasing mean, there exists exactly one $w' < m$ such that $z = M(w, w')$. Due to (9) we may define a function $\varphi : (i, m) \longrightarrow \varphi((i, m))$ such that

$$z = M(u, \varphi(u)), \quad u \in (i, m). \tag{10}$$

It is obvious that the function φ is continuous and strictly decreasing. Hence φ is a homeomorphism.

Define a set $V := (i, m) \cap \varphi((i, m))$. Observe that V is an open interval, such that $z \in V$. As $\varphi(B \cap (i, m))$ is a second category set in V, we get $\varphi(B \cap (i, m)) \cap B \neq \emptyset$. Fix $y \in \varphi(B \cap (i, m)) \cap B$. There exists $x \in B \cap (i, m)$ such that $y = \varphi(x)$. Now, by (10) we have

$$z = M(x, \varphi(x)) = M(x, y)$$

That ends the proof. □

The main result of this paper is contained in the following theorem:

THEOREM 3.1. *Let $I \subseteq \mathbb{R}$ be an open interval. Assume that $M, N : I^2 \longrightarrow I$ are strict continuous means on I. Further, assume that for every fixed $u, v \in I$, functions $M(u, \cdot)$ and $M(\cdot, v)$ are strictly increasing. If a function $f : I \longrightarrow \mathbb{R}$ is of the first Baire class and (M, N)-Wright convex then f is convex.*

Proof. Fix $z \in I$. By Corollary 2.1 and (8) it is enough to show that the inequality

$$f(z) \leqslant m_f(z), \quad z \in I, \tag{11}$$

holds.

Fix $\eta > 0$. By the definition of m_f there exist $\delta > 0$ such that

$$\bigwedge_{u,v \in K(z,\delta)} m_f(z) - \frac{\eta}{3} \leqslant f(N(u, v)). \tag{12}$$

As f is of the first Baire class, it has a continuity point. Thus, by Lemma 2.1 f is locally bounded. Hence m_f is real valued function and, by Corollary 2.1, it is a convex function. This implies that m_f is continuous. Hence, there exists $\gamma > 0$ such that

$$\bigwedge_{s \in K(z,\gamma)} m_f(s) \leqslant m_f(z) + \frac{\eta}{3}. \tag{13}$$

Put $\epsilon := \min\{\delta, \gamma\}$.

Function f is of the first Baire class, so the set

$$B := \{u \in I : f \text{ is continuous at a point u}\} \cap K(z, \epsilon),$$

is such that $K(z, \epsilon) \setminus B$ is of the first category.

By Lemma 3.1, there exist $x, y \in K(z, \epsilon) \cap B$ such that $M(x, y) = z$. Because x, y are continuity points of function f we get

$$f(z) + f(N(x, y)) \leqslant f(x) + f(y) = m_f(x) + m_f(y). \tag{14}$$

Taking into account (12) we get

$$f(z)+m_f(z)-\frac{\eta}{3}\leqslant f(z)+f(N(x,y)).$$

Now, by (14)

$$f(z)+m_f(z)-\frac{\eta}{3}\leqslant m_f(x)+m_f(y).$$

Therefore, by (13) we have

$$f(z)+m_f(z)-\frac{\eta}{3}\leqslant m_f(z)+\frac{\eta}{3}+m_f(z)+\frac{\eta}{3},$$

that is,

$$f(z)\leqslant m_f(z)+\eta,$$

for any $\eta>0$. Last statement is equivalent to the inequality (11). This completes the proof. □

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Michał Lewicki
Institute of Mathematics
Silesian University
ul. Bankowa 14
PL-40 007 Katowice
Poland
e-mail: m_lewicki@wp.pl