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GRAND FURUTA INEQUALITY AND ITS VARIANT

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Abstract. The grand Furuta inequality (GFI) is understood as follows: If positive operators A and B on a Hilbert space satisfy $A \geqslant B \geqslant 0$, A is invertible and $t \in [0,1]$, then

$$A^{1-t+r} \geqslant (A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}})^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $p,\ s \geqslant 1$ and $r \geqslant t$. In this note, we present a short proof of (GFI) which is done by the usual induction on s and the use of the Furuta inequality. Furthermore we propose another simultaneous extension of the Ando-Hiai and Furuta inequalities: If $A \geqslant B \geqslant 0$, A is invertible and $t \in [0,1]$, then

$$A^t \sharp_{\frac{1-t}{p-t}} B^p \geqslant A^{-r+t} \sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p)$$

holds for $r \geqslant t$ and $p,\ s \geqslant 1$. Here $\sharp_\alpha$ is the α-geometric mean and $\natural_s$ for $s \notin [0,1]$ is of the same form as $\sharp_\alpha$.

1. Introduction. Throughout this note, A and B are positive operators on a Hilbert space. For convenience, we denote $A \geqslant 0$ (resp. $A > 0$) if A is a positive (resp. positive invertible) operator.

A real-valued function f on $[0,\infty)$ is called operator monotone if it is order-preserving, i.e., $A \geqslant B \geqslant 0$ implies $f(A) \geqslant f(B)$. The most typical example of it is the fuction: $x \to x^\alpha$ for $\alpha \in [0,1]$, which is known as the Löwner-Heinz inequality (LH). By the Kubo-Ando theory [14] on operator means, it induces the α-geometric mean $A \sharp_\alpha B$, precisely,

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}} \quad \text{for } 0 \leqslant \alpha \leqslant 1 \text{ and } A > 0.$$

Related to this, Ando-Hiai [1] showed a beautiful log-majorization theorem, whose core is the following order-preserving inequality:

ANDO-HIAI INEQUALITY. *Let α be in $[0,1]$.*

$$\text{(AH)} \qquad A \sharp_\alpha B \leqslant I \implies A^r \sharp_\alpha B^r \leqslant 1 \ \textit{for}\ r \geqslant 1.$$

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The following expression of Ando-Hiai inequality is convenient for our discussion below: If $A \geqslant B \geqslant 0$ and A is invertible, then for each $p \geqslant 1$

$$A^{-r} \natural_{\frac{1}{p}} (A^{-\frac{1}{2}} B^p A^{-\frac{1}{2}})^r \leqslant I \quad \text{for } r \geqslant 1.$$

On the other hand, Furuta paid his attention to the fact that (LH) is false for $p > 1$, i.e.,

$$A \geqslant B \geqslant 0 \not\Longrightarrow A^p \geqslant B^p.$$

He consequently found the following remedy, which includes (LH) as the case $r = 0$.

FURUTA INEQUALITY. *If $A \geqslant B \geqslant 0$, then for each $r \geqslant 0$,*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for p and q such that $p \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

By virtue of (LH), the essense of Furuta inequality is just in the optimal case $(1+r)q = p+r$, which is expressed as follows:

(FI) $\qquad A \geqslant B \geqslant 0$ implies $A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leqslant A$ for $p \geqslant 1$ and $r \geqslant 0$.

2. Grand Furuta inequality. To interpolate between Ando-Hiai inequality and Furuta inequality, Furuta [8] established the following operator inequality, which is called grand Furuta inequality (GFI). In this section, we give a short proof of it by the usual induction.

GRAND FURUTA INEQUALITY. *If $A \geqslant B \geqslant 0$ and A is invertible, then for each $t \in [0,1]$*

(GFI) $$A^{1-t+r} \geqslant \{A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $p, s \geqslant 1$ and $r \geqslant t$.

Proof. We prove it by the induction on s. For this, we first prove it for $1 \leqslant s \leqslant 2$: Since $(X^*C^2X)^s = X^*C(CXX^*C)^{s-1}CX$ for arbitrary X and $C \geqslant 0$, and $0 \leqslant s-1 \leqslant 1$, (LH) implies that

$$A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2} = A^{\frac{r-t}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} A^{-t} B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}} A^{\frac{r-t}{2}}$$

$$\leqslant A^{\frac{r-t}{2}} B^{\frac{p}{2}} (B^{\frac{p}{2}} B^{-t} B^{\frac{p}{2}})^{s-1} B^{\frac{p}{2}} A^{\frac{r-t}{2}} = A^{\frac{r-t}{2}} B^{(p-t)s+t} A^{\frac{r-t}{2}}.$$

Furthermore it follows from (LH) and (FI) that

$$\{A^{r/2}(A^{-t/2} B^p A^{-t/2})^s A^{r/2}\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant \{A^{\frac{r-t}{2}} B^{(p-t)s+t} A^{\frac{r-t}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \leqslant A^{1-t+r}$$

by noting that $(p-t)s+t+(r-t) = (p-t)s+r$. Hence (GFI) is proved for $1 \leqslant s \leqslant 2$.

Next, under the assumption (GFI) holds for some $s \geqslant 1$, we now prove that (GFI) holds for $s+1$. Since (GFI) holds for s, we take $r = t$ in it. Thus we have

$$A \geqslant \{A^{t/2}(A^{-t/2} B^p A^{-t/2})^s A^{t/2}\}^{\frac{1}{(p-t)s+t}}.$$

Put $C=\{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}^{\frac{1}{(p-t)s+t}}$, that is, $A\geqslant C$. By using that $s\geqslant 1$ if and only if $1\leqslant\frac{s+1}{s}\leqslant 2$ and that (GFI) for $1\leqslant s\leqslant 2$ has been proved, we obtain that

$$\begin{aligned}
A^{1-t+r} &\geqslant \{A^{r/2}(A^{-t/2}C^{(p-t)s+t}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{\{(p-t)s+t-t\}(\frac{s+1}{s})+r}}\\
&= \{A^{r/2}(A^{-t/2}C^{(p-t)s+t}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}\\
&= \{A^{r/2}(A^{-t/2}\{A^{t/2}(A^{-t/2}B^pA^{-t/2})^sA^{t/2}\}A^{-t/2})^{\frac{s+1}{s}}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}\\
&= \{A^{r/2}(A^{-t/2}B^pA^{-t/2})^{s+1}A^{r/2}\}^{\frac{1-t+r}{(p-t)(s+1)+r}}.
\end{aligned}$$

This means that (GFI) holds for $s+1$, and so the proof is complete. □

3. A variant of grand Furuta inequality. We use the notation $A\natural_r B = A^{\frac12}(A^{\frac12}BA^{\frac12})^rA^{\frac12}$ for $r\notin[0,\ 1]$. (GFI) is expressed as follows:

If $A\geqslant B\geqslant 0$, A is invertible and $0\leqslant t\leqslant 1$, then

$$\text{(GFI)}\qquad A^{-r+t}\ \sharp_{\frac{1-t+r}{(p-t)s+r}}\ (A^t\ \natural_s\ B^p)\leqslant A$$

holds for $r\geqslant t$ and $p,s\geqslant 1$.

One of the authors analysed Furuta inequality as a satellite [11] (cf.[12]):

(SF) $A\geqslant B\geqslant 0$ and $A>0$ imply $A^{-r}\ \sharp_{\frac{1+r}{p+r}}\ B^p\leqslant B\leqslant A$ for $p\geqslant 1$ and $r\geqslant 0.$

Based on (SF), we improved (GFI):

If $A\geqslant B\geqslant 0$, A is invertible and $0\leqslant t\leqslant 1$, then

$$\text{(SGF)}\qquad A^{-r+t}\ \sharp_{\frac{1-t+r}{(p-t)s+r}}\ (A^t\ \natural_s\ B^p)\leqslant B$$

holds for $r\geqslant t$ and $p,s\geqslant 1$.

Under such situation, we now propose a variant of grand Furuta inequality which contains (SF) and (AH).

THEOREM. *If $A\geqslant B\geqslant 0$ and A is invertible, then for each $p\geqslant 1$ and $0\leqslant t\leqslant 1$,*

$$A^{-r+t}\ \sharp_{\frac{1-t+r}{(p-t)s+r}}\ (A^t\ \natural_s\ B^p)\leqslant A^t\ \sharp_{\frac{1-t}{p-t}}\ B^p$$

holds for $r\geqslant t$ and $s\geqslant 1$.

Proof. First of all, we cite two useful formulas on $\natural_a$ (and $\sharp_a$), which are frequently used below; for $X,\ Y,>0$ and $a,\ b\in\mathbb{R}$

(i) transposition: $X\natural_a Y=Y\natural_{1-a}X$,

(ii) multiplicativity: $X\natural_{ab}Y=X\natural_a(X\natural_b Y)$.

Now, if $A\geqslant B\geqslant 0$, then $A^t\geqslant B^t$ by the Löwner-Heinz inequality. It is equivalent to $A^{-t}\ \sharp_{\frac{t}{p}}\ A^{-\frac t2}B^pA^{-\frac t2}\leqslant I.$ Recall our recent result in [5] on a generalization of (AH): Let α be in $[0,1]$. Then

$$\text{(GAH)}\qquad A\ \sharp_\alpha\ B\leqslant I\implies A^r\ \sharp_{\frac{\alpha r}{\alpha r+(1-\alpha)s}}\ B^s\leqslant 1\ \text{ for } r,s\geqslant 1.$$

By this, we have

$$A^{-tr_1} \sharp_{\frac{\frac{tr_1}{p}}{(1-\frac{t}{p})s+\frac{tr_1}{p}}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s \leqslant I \text{ for } r_1,\ s \geqslant 1$$

If we take $r_1 = \frac{r}{t} \geqslant 1$, then

$$A^{-r} \sharp_{\frac{r}{(p-t)s+r}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s = (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^s \sharp_{\frac{(p-t)s}{(p-t)s+r}} A^{-r} \leqslant I,$$

or equivalently

$$(A^t \natural_s B^p) \sharp_{\frac{(p-t)s}{(p-t)s+r}} A^{-r+t} \leqslant A^t.$$

Hence we have the conclusion by the following calculations.

$$\begin{aligned} & A^{-r+t}\sharp_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) = (A^t \natural_s B^p) \sharp_{\frac{(p-t)s-(1-t)}{(p-t)s+r}} A^{-r+t} \\ = \ & (A^t \natural_s B^p) \sharp_{\frac{(p-t)s-(1-t)}{(p-t)s}} ((A^t \natural_s B^p) \sharp_{\frac{(p-t)s}{(p-t)s+r}} A^{-r+t}) \\ \leqslant \ & (A^t \natural_s B^p) \sharp_{\frac{(p-t)s-(1-t)}{(p-t)s}} A^t \\ = \ & A^t \sharp_{\frac{1-t}{(p-t)s}} (A^t \natural_s B^p) = A^t \sharp_{\frac{1-t}{p-t}} B^p. \quad \square \end{aligned}$$

REMARK. In Theorem, if we take $t = 0$ and $s = 1$, then we have (SF); on the other hand, if we take $t = 1$ and $r = s$, then we have the following inequality equivalent to (AH) for $\alpha = \frac{1}{p}$, as stated in the first section: If $A \geqslant B \geqslant 0$ and A is invertible, then for each $p \geqslant 1$,

$$A^{-r+1} \sharp_{\frac{1}{p}} (A \natural_r B^p) \leqslant A$$

holds for $r \geqslant 1$.

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