HARDY AND RELLICH INEQUALITIES
WITH REMAINDERS

W. Desmond Evans and Roger T. Lewis

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Abstract. In this paper our primary concern is with the establishment of weighted Hardy inequalities in $L^p(\Omega)$ and Rellich inequalities in $L^2(\Omega)$ depending upon the distance to the boundary of domains $\Omega \subset \mathbb{R}^n$ with a finite diameter $D(\Omega)$. Improved constants are presented in most cases.

1. Introduction

Recently, considerable attention has been given to extensions of the multi-dimensional Hardy inequality of the form

$$\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geqslant \mu(\Omega) \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^2} d\mathbf{x} + \lambda(\Omega) \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x}, \quad u \in H_0^1(\Omega), \quad (1.1)$$

where Ω is an open connected subset of $\mathbb{R}^n$ and

$$\delta(\mathbf{x}) := dist(\mathbf{x}, \partial\Omega).$$

It is known that for $\mu(\Omega) = \frac{1}{4}$ there are smooth domains for which $\lambda(\Omega) \leqslant 0$, and for $\lambda(\Omega) = 0$, there are smooth domains for which $\mu(\Omega) < \frac{1}{4}$ - see M. Marcus, V.J. Mizel, and Y. Pinchover [8] and T. Matskewich and P.E. Sobolevskii [9]. In [2], H. Brezis and M. Marcus showed that for domains of class C^2 inequality (1.1) holds for

$$\mu(\Omega) = \frac{1}{4} \quad \text{and some} \quad \lambda(\Omega) \in (-\infty, \infty)$$

and when Ω is convex

$$\lambda(\Omega) \geqslant \frac{1}{4D(\Omega)^2} \quad (1.2)$$

in which $D(\Omega)$ is the diameter of Ω.

M. Hoffmann-Ostenhof, T. Hoffmann-Ostenhof, and A. Laptev [6] answered a question posed by H. Brezis and M. Markus in [2] by establishing the improvement to (1.2) that (1.1) holds for a convex domain Ω, with

$$\mu(\Omega) = \frac{1}{4}, \quad \lambda(\Omega) \geqslant \frac{K(n)}{4|\Omega|^{\frac{2}{n}}}, \quad \text{and} \quad K(n) := n \left[\frac{s_{n-1}}{n} \right]^{2/n} \quad (1.3)$$

in which $s_{n-1} := |\mathbb{S}^{n-1}|$ and $|\Omega|$ is the volume of Ω.

For a convex domain Ω and $\mu(\Omega) = 1/4$, a lower bound for $\lambda(\Omega)$ in (1.1) in terms of $|\Omega|$ was also obtained by S. Filippas, V. Maz'ya, and A. Tertikas in [5] as a special case of results on L^p Hardy inequalities. They prove that $\lambda(\Omega) \geqslant 3D_{int}(\Omega)^{-2}$, where $D_{int}(\Omega) = 2\sup_{x\in\Omega}\delta(x)$, the internal diameter of Ω. Since $3D_{int}(\Omega)^{-2} \geqslant \frac{3}{4n}K(n)/|\Omega|^{2/n}$, their result is an improvement of (1.3) for $n = 2, 3$, but the estimates don't compare for $n > 3$.

In this paper we show that (1.1) holds for (1.3) replaced by

$$\mu(\Omega) = \frac{1}{4} \quad \text{and} \quad \lambda(\Omega) \geqslant \frac{3K(n)}{2|\Omega|^{\frac{2}{n}}}$$

as well as proving weighted versions of the Hardy inequality in $L^p(\Omega)$ for $p > 1$.

In the case $p = 2$, the following are special cases of our results. If Ω is convex and $\sigma \in (0,1]$, then

$$\int_\Omega |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geqslant \frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \left\{ \frac{B(n,2-\sigma)}{\delta(\mathbf{x})^{2-\sigma}} + 3\left(\frac{s_{n-1}}{n|\Omega|}\right)^{\frac{2-\sigma}{n}} \right\} |u(\mathbf{x})|^2 d\mathbf{x} \quad (1.4)$$

for

$$B(n,p) := \frac{\Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{n}{2})}{\sqrt{\pi} \cdot \Gamma(\frac{n+p}{2})}. \quad (1.5)$$

If $\sigma \in [\frac{2-n}{2}, 0]$ and Ω is convex, then

$$\begin{aligned}
\int_\Omega \delta(\mathbf{x})^\sigma |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geqslant{} & \frac{n(1-\sigma)^2}{4}B(n,2-\sigma)\int_\Omega \delta(\mathbf{x})^{\sigma-2}|u(\mathbf{x})|^2 d\mathbf{x} \\
& + \frac{C_H(n,\sigma)}{|\Omega|^{\frac{2(1-\sigma)}{n}}}\int_\Omega \delta(\mathbf{x})^{|\sigma|}|u(\mathbf{x})|^2 d\mathbf{x}.
\end{aligned}$$

for $C_H(n,\sigma)$ given in (3.4).

Similar results for weighted forms of the Hardy inequality in $L^p(\Omega)$ are given in section 4. Finally, we comment that our one-dimensional inequalities in §2 lead to improved constants for the Rellich inequality obtained by G. Barbatis in [1] for $n \geqslant 4$.

2. One-dimensional inequalities

As is the case in [6], our proofs are based on one-dimensional Hardy-type inequalities coupled with the use of the mean-distance function introduced by Davies to extend to higher dimensions; see [4]. The basic one-dimensional inequality is as follows:

LEMMA 1. *Let* $u \in C_0^1(0, 2b)$, $\rho(t) := \min\{t, 2b - t\}$ *and let* $f \in C^1[0,b]$ *be monotonic on* $[0,b]$. *Then for* $p > 1$

$$\int_0^{2b} |f'(\rho(t))||u(t)|^p dt \leqslant p^p \int_0^{2b} \frac{|f(\rho(t)) - f(b)|^p}{|f'(\rho(t))|^{p-1}}|u'(t)|^p dt. \quad (2.1)$$

Proof. First let $u := v\chi_{(0,b]}$, the restriction to $(0,b]$ of some $v \in C_0^1(0,2b)$. For any constant c

$$-\int_0^b [f(t) - c]'|u(t)|^p dt = -[f(t) - c]|u(t)|^p \Big|_0^b + \int_0^b [f(t) - c]\frac{p}{2}[|u(t)|^2]^{\frac{p}{2}-1}[|u(t)|^2]' dt.$$

By choosing $c = f(b)$, we have that

$$-\int_0^b f'(t)|u(t)|^p dt = p \int_0^b [f(t) - f(b)]|u(t)|^{p-2}\mathfrak{Re}[\overline{u(t)}u'(t)] dt. \qquad (2.2)$$

Similarly, for $u = v\chi_{[b,2b)}$, $v \in C_0^1(0,2b)$, we have

$$-\int_b^{2b} f'(2b - s)|u(s)|^p ds = p \int_b^{2b} [f(2b - s) - f(b)]|u(s)|^{p-2}\mathfrak{Re}[\overline{u(s)}u'(s)] ds.$$

Therefore, since f is monotonic, for any $u \in C_0^1(0,2b)$

$$\int_0^{2b} |f'(\rho(t))||u(t)|^p dt = p \int_0^{2b} |f(\rho(t)) - f(b)||u(t)|^{p-2}\mathfrak{Re}[\overline{u(t)}u'(t)] dt$$

$$\leqslant p \int_0^b |f'(\rho(t))|^{\frac{p-1}{p}}|u(t)|^{p-1}\frac{|f(\rho(t)) - f(b)|}{|f'(\rho(t))|^{\frac{p-1}{p}}}|u'(t)| dt$$

$$\leqslant p \left[\int_0^b |f'(\rho(t))||u(t)|^p dt\right]^{\frac{p-1}{p}} \left[\int_0^b \frac{|f(\rho(t)) - f(b)|^p}{|f'(\rho(t))|^{p-1}}|u'(t)|^p dt\right]^{\frac{1}{p}}$$

on applying Hölder's inequality. Inequality (2.1) now follows. $\qquad \square$

The next lemma provides the one-dimensional result needed to improve (1.3), which was proved in [6].

LEMMA 2. *Let $\sigma \leqslant 1$ and define $\mu(t) := 2b - \rho(t)$. For all $u \in C_0^1(0,2b)$*

$$\int_0^{2b} \rho(t)^\sigma |u'(t)|^2 dt \geqslant \left(\frac{1-\sigma}{2}\right)^2 \int_0^{2b} \rho(t)^{\sigma-2}\left[1 + k(\sigma)\left(\frac{2\rho(t)}{\mu(t)}\right)^{1-\sigma}\right]^2 |u(t)|^2 dt, \qquad (2.3)$$

for

$$k(\sigma) := \begin{cases} [1 - 2^{\frac{1}{\sigma}-1}]^{-\sigma}, & \sigma < 0, \\ 1, & \sigma \in [0,1]. \end{cases}$$

Proof. On setting $f(t) = t^{\sigma-1}$ in (2.1) we get

$$|1 - \sigma|^p \int_0^{2b} \rho(t)^{\sigma-2}|u(t)|^p dt \leqslant p^p \int_0^{2b} \rho(t)^{p+\sigma-2}\left|1 - \left[\frac{\rho(t)}{b}\right]^{1-\sigma}\right|^p |u'(t)|^p dt. \quad (2.4)$$

With $u \in C_0^1(0, 2b)$, let $p = 2$ and substitute $v(t) = [1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}]u(t)$ in (2.4). We claim that this gives

$$\int_0^{2b} \rho^\sigma(t)|v'(t)|^2 dt \geq \left(\frac{1-\sigma}{2}\right)^2 \int_0^{2b} \rho(t)^{\sigma-2}\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]^{-2}|v(t)|^2 dt \quad (2.5)$$

for any real number σ. The substitution gives

$$\rho(t)^{\sigma/2}v'(t) = -(1-\sigma)b^{\sigma-1}\rho(t)^{-\sigma/2}\rho'(t)u(t) + \rho(t)^{\sigma/2}\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]u'(t).$$

Consequently,

$$\rho(t)^\sigma|v'(t)|^2 = (1-\sigma)^2 b^{2\sigma-2}\rho(t)^{-\sigma}|u(t)|^2 + \rho(t)^\sigma\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]^2|u'(t)|^2$$

$$- (1-\sigma)b^{\sigma-1}\rho'(t)\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right][|u|^2]'$$

which implies that

$$\int_0^{2b} \rho(t)^\sigma|v'(t)|^2 dt = \int_0^{2b} \rho(t)^\sigma\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]^2|u'(t)|^2 dt$$

$$+ \int_0^{2b} (1-\sigma)^2 b^{2\sigma-2}\rho(t)^{-\sigma}|u(t)|^2 dt$$

$$+ (1-\sigma)b^{\sigma-1}\int_0^{2b}\frac{d}{dt}\left[\rho'(t)\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]\right]|u|^2 dt$$

$$= \int_0^{2b} \rho(t)^\sigma\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]^2|u'(t)|^2 dt \quad (2.6)$$

since $\rho'(t) = 1$ in $(0, b)$ and -1 in $(b, 2b)$. Therefore, (2.5) follows from (2.4).

Since $2b = \mu(t) + \rho(t)$

$$\left[1 - \left(\frac{\rho(t)}{b}\right)^{1-\sigma}\right]^{-2} = \left[1 + \frac{\rho(t)^{1-\sigma}}{b^{1-\sigma} - \rho(t)^{1-\sigma}}\right]^2$$

$$= \left[1 + 2^{1-\sigma}\left(\frac{\rho(t)}{\mu(t)}\right)^{1-\sigma}k_\sigma\left(\frac{\rho(t)}{\mu(t)}\right)\right]^2 \quad (2.7)$$

for

$$k_\sigma(x) := \frac{1}{(1+x)^{1-\sigma} - (2x)^{1-\sigma}}, \qquad x \in [0, 1), \qquad \sigma \neq 1.$$

For $\sigma < 1$, $k_\sigma(x) > 0$ in $(0, 1)$, $k_\sigma(0) = 1$ and $k_\sigma(x) \to \infty$ as $x \to 1^-$. By examining the derivative of $k_\sigma(x)$

$$k'_\sigma(x) = \frac{-(1 - \sigma)((1 + x)^{-\sigma} - 2^{1-\sigma}x^{-\sigma})}{[(1 + x)^{1-\sigma} - (2x)^{1-\sigma}]^2}$$

we see that

$$\lim_{x \to 0^+} k'_\sigma(x) = \begin{cases} -(1 - \sigma), & \sigma < 0, \\ 1, & \sigma = 0, \\ \infty, & 0 < \sigma < 1. \end{cases}$$

For $\sigma < 0$, $k_\sigma(x)$ is minimized at

$$x_\sigma := 1/(2^{1-\frac{1}{\sigma}} - 1) < 1.$$

Calculations show that

$$k_\sigma(x_\sigma) = \left[1 - 2^{\frac{1}{\sigma}-1}\right]^{-\sigma} =: k(\sigma).$$

For $\sigma \in [0, 1)$, $k'_\sigma(x)$ is never zero in $(0, 1)$ indicating that $k_\sigma(x)$ is minimized at $x = 0$ for $\sigma \in [0, 1)$ and $x \in [0, 1)$. The inequality (2.3) now follows. $\qquad\square$

In order to treat the case in which $p \neq 2$, we make use of the methods of Tidblom [11] and prove a weighted version of Theorem 1.1 in [11].

LEMMA 3. *Let* $u \in C_0^1(0, 2b)$, $p \in (1, \infty)$, *and* $\sigma \leqslant p - 1$. *Then*

$$\int_0^{2b} \rho(t)^\sigma |u'(t)|^p dt \geqslant \left[\frac{p - \sigma - 1}{p}\right]^p \int_0^{2b} \left\{\rho(t)^{\sigma-p} + (p - 1)b^{\sigma-p}\right\} |u(t)|^p dt.$$

Proof. We may assume that $\sigma \neq p - 1$ since otherwise the conclusion is trivial. According to (2.2) for a monotonic function f and a positive function g,

$$\int_0^b |f'(t)||u(t)|^p dt \leqslant \int_0^b p|f(t) - f(b)||u(t)|^{p-1}|u'(t)|dt$$

$$\leqslant p \left[\int_0^b g(t)|u'(t)|^p dt\right]^{1/p} \left[\int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{\frac{1}{p-1}} |u(t)|^p dt\right]^{1-\frac{1}{p}}.$$

Consequently,

$$p^p \int_0^b g(t)|u'(t)|^p dt \geqslant \frac{\left(\int_0^b |f'(t)||u(t)|^p dt\right)^p}{\left(\int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)}\right)^{1/(p-1)} |u(t)|^p dt\right)^{p-1}}.$$

Now, as in [11], using a corollary to Young's inequality, namely

$$A^p/B^{p-1} \geqslant pA - (p - 1)B,$$

with $A = \int_0^b |f'(t)||u(t)|^p dt, B = \int_0^b \left(\frac{|f(t)-f(b)|^p}{g(t)} \right)^{1/p-1} |u(t)|^p dt$, it follows that

$$p^p \int_0^b g(t)|u'(t)|^p dt \geqslant \int_0^b \left\{ p|f'(t)| - (p-1)\left(\frac{|f(t)-f(b)|^p}{g(t)} \right)^{1/(p-1)} \right\} |u(t)|^p dt.$$

Choose $f(t) = t^{\sigma-p+1}$ and $g(t) = (p-\sigma-1)^{-(p-1)}t^\sigma$. Then

$$\left(\frac{|f(t)-f(b)|^p}{g(t)} \right)^{1/(p-1)} = (p-\sigma-1)\left[\frac{|t^{\sigma-p+1} - b^{\sigma-p+1}|^p}{t^\sigma} \right]^{\frac{1}{p-1}}$$

$$= (p-\sigma-1)t^{\sigma-p}\left[\left(1 - \left(\frac{t}{b} \right)^{p-\sigma-1} \right)^p \right]^{\frac{1}{p-1}}.$$

Consequently, for $t \in (0,b)$

$$p|f'(t)| - (p-1)\left(\frac{|f(t)-f(b)|^p}{g(t)} \right)^{1/(p-1)}$$

$$= (p-\sigma-1)\left\{ pt^{\sigma-p} - (p-1)t^{\sigma-p}\left[\left(1 - \left(\frac{t}{b} \right)^{p-\sigma-1} \right)^p \right]^{\frac{1}{p-1}} \right\}$$

$$= (p-\sigma-1)t^{\sigma-p}\left\{ 1 + (p-1)\left(1 - \left[1 - \left(\frac{t}{b} \right)^{p-\sigma-1} \right]^{\frac{p}{p-1}} \right) \right\}$$

$$\geqslant (p-\sigma-1)t^{\sigma-p}\left\{ 1 + (p-1)\left(\frac{t}{b} \right)^{p-\sigma-1} \right\}$$

$$\geqslant (p-\sigma-1)\left\{ t^{\sigma-p} + (p-1)\left(\frac{1}{b^{p-\sigma}} \right) \right\}.$$

and the inequality follows. In the inequality above we have used the fact that

$$\left[1 - \left(\frac{t}{b} \right)^{p-\sigma-1} \right]^{\frac{p}{p-1}} \leqslant 1 - \left(\frac{t}{b} \right)^{p-\sigma-1}.$$

The proof is completed by following the last part of the proof of Lemma 1. □

For a certain range of values taken by σ, $\sigma \in [-c_\sigma, 1)$ with $c_\sigma > 0$, the inequality in $L^2(\Omega)$ given by Lemma 2 gives a better bound than Lemma 3 with $p = 2$. In fact for $\sigma < 1$

$$\rho(t)^{\sigma-2}\left[1 + k(\sigma)\left(\frac{2\rho(t)}{\mu(t)} \right)^{1-\sigma} \right]^2 = \rho(t)^{\sigma-2} + \frac{2^{2-\sigma}k(\sigma)}{\rho(t)\mu(t)^{1-\sigma}} + \frac{2^{2-2\sigma}k(\sigma)^2}{\rho(t)^\sigma\mu(t)^{2-2\sigma}}$$

with

$$\frac{2^{2-\sigma}k(\sigma)}{\rho(t)\mu(t)^{1-\sigma}} + \frac{2^{2-2\sigma}k(\sigma)^2}{\rho(t)^\sigma\mu(t)^{2-2\sigma}}$$

$$\geqslant \begin{cases} \frac{5}{2\sigma}b^{\sigma-2}, & \sigma \in [0,1), \\ \left[2 - \sigma + b^\sigma k(\sigma)\rho(t)^{|\sigma|} \right]k(\sigma)b^{\sigma-2}, & \sigma < 0. \end{cases} \tag{2.8}$$

Since $k(\sigma)$ decreases to 0 as $|\sigma| \to \infty$ and $k(-3) \approx 0.22$, then the left-hand side of (2.8) is greater than $b^{\sigma-2}$ for $\sigma \in [-3, 1)$.

3. A Hardy inequality in $L^2(\Omega)$

We need the following notation (c.f.[6]). For each $\mathbf{x} \in \Omega$ and $v \in \mathbb{S}^{n-1}$,

$$\tau_v(\mathbf{x}) := \min\{s > 0 : \mathbf{x} + sv \notin \Omega\};$$

$$D_v(\mathbf{x}) := \tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x});$$

$$\rho_v(\mathbf{x}) := \min\{\tau_v(\mathbf{x}), \tau_{-v}(\mathbf{x})\};$$

$$\mu_v(\mathbf{x}) := \max\{\tau_v(\mathbf{x}), \tau_{-v}(\mathbf{x})\} = D_v(\mathbf{x}) - \rho_v(\mathbf{x});$$

$$D(\Omega) := \sup_{\mathbf{x} \in \Omega, \, v \in \mathbb{S}^{n-1}} D_v(\mathbf{x});$$

$$\Omega_{\mathbf{x}} := \{\mathbf{y} \in \Omega : \mathbf{x} + t(\mathbf{y} - \mathbf{x}) \in \Omega, \, \forall t \in [0, 1]\}.$$

Note that $D(\Omega)$ is the diameter of Ω and $\Omega_{\mathbf{x}}$ is the part of Ω which can be "seen" from the point $\mathbf{x} \in \Omega$. The volume of $\Omega_{\mathbf{x}}$ is denoted by $|\Omega_{\mathbf{x}}|$.

Let $d\omega(v)$ denote the normalized measure on $\mathbb{S}^{n-1}$ (so that $1 = \int_{\mathbb{S}^{n-1}} d\omega(v)$) and define

$$\rho(\mathbf{x}; s) := \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^s d\omega(v). \tag{3.1}$$

Hence $\rho^{-1/2}(\mathbf{x}; -2) = \rho(\mathbf{x})$ the "mean-distance" function introduced by Davies in [4]. For

$$B(n, p) := \int_{\mathbb{S}^{n-1}} |\cos(\mathbf{e}, v)|^p d\omega(v) = \frac{\Gamma(\frac{p+1}{2}) \cdot \Gamma(\frac{n}{2})}{\sqrt{\pi} \cdot \Gamma(\frac{n+p}{2})}, \quad \mathbf{e} \in \mathbb{R}^n, \tag{3.2}$$

it is known that

$$\rho(\mathbf{x}; -p) := \int_{\mathbb{S}^{n-1}} \frac{1}{\rho_v(\mathbf{x})^p} d\omega(v) \geqslant \frac{B(n, p)}{\delta(\mathbf{x})^p} \tag{3.3}$$

for convex domains Ω – see Exercise 5.7 in [4], [3], and [11]. Note that $B(n, 2) = n^{-1}$. This fact can be applied to most of the results below when Ω is convex.

For a Hardy inequality in $L^2(\Omega)$ with weights we will need to define

$$C_H(n, \sigma) := n\left(\frac{s_{n-1}}{n}\right)^{\frac{2(1-\sigma)}{n}} k(\sigma)[2^{|\sigma|} + 2^{2|\sigma|-1}k(\sigma)](1 - \sigma)^2 \tag{3.4}$$

for $\sigma \in [\frac{2-n}{2}, 0]$ and $n \geqslant 2$ where as given in Lemma 2

$$k(\sigma) := \begin{cases} [1 - 2^{\frac{1}{\sigma}-1}]^{-\sigma}, & \sigma < 0, \\ 1, & \sigma \in [0, 1]. \end{cases}$$

Note that $C_H(n, 0) = \frac{3}{2}K(n)$ for $K(n)$ defined in (1.3).

THEOREM 1. *If* $\frac{2-n}{2} \leqslant \sigma \leqslant 0$, *then for any* $u \in C_0^1(\Omega)$

$$\int_\Omega \delta(\mathbf{x})^\sigma |\nabla u(\mathbf{x})|^2 d\mathbf{x} \geqslant \frac{n(1 - \sigma)^2}{4} \int_\Omega \rho(\mathbf{x}; \sigma - 2)|u(\mathbf{x})|^2 d\mathbf{x}$$

$$+ C_H(n, \sigma) \int_\Omega \frac{\delta(\mathbf{x})^{|\sigma|}}{|\Omega_{\mathbf{x}}|^{\frac{2(1-\sigma)}{n}}} |u(\mathbf{x})|^2 d\mathbf{x}. \tag{3.5}$$

If $0 < \sigma \leqslant 1$, then

$$\int_\Omega |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geqslant \frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \left\{ \rho(\mathbf{x}; \sigma-2) + 3\left(\frac{s_{n-1}}{n|\Omega_\mathbf{x}|}\right)^{\frac{2-\sigma}{n}} \right\} |u(\mathbf{x})|^2 d\mathbf{x}. \quad (3.6)$$

If Ω is convex, then for any $u \in C_0^1(\Omega)$

$$\begin{aligned}
\int_\Omega \delta(\mathbf{x})^\sigma |\nabla u(\mathbf{x})|^2 d\mathbf{x} \; &\geqslant \; \frac{n(1-\sigma)^2}{4} B(n, 2-\sigma) \int_\Omega \delta(\mathbf{x})^{\sigma-2} |u(\mathbf{x})|^2 d\mathbf{x} \\
&\quad + \frac{C_H(n,\sigma)}{|\Omega|^{\frac{2(1-\sigma)}{n}}} \int_\Omega \delta(\mathbf{x})^{|\sigma|} |u(\mathbf{x})|^2 d\mathbf{x}.
\end{aligned}$$

when $\sigma \in [\frac{2-n}{2}, 0]$ and

$$\int_\Omega |\nabla u(\mathbf{x})|^2 \, d\mathbf{x} \geqslant \frac{2^\sigma n(1-\sigma)^2}{4D(\Omega)^\sigma} \int_\Omega \left\{ B(n, 2-\sigma)\delta(\mathbf{x})^{\sigma-2} + 3\left(\frac{s_{n-1}}{n|\Omega|}\right)^{\frac{2-\sigma}{n}} \right\} |u(\mathbf{x})|^2 d\mathbf{x}.$$

when $\sigma \in (0, 1]$.

Proof. Let $\partial_v u$, $v \in \mathbb{S}^{n-1}$, denote the derivative of u in the direction of v, i.e., $\partial_v u = v \cdot (\nabla u)$. It follows from Lemma 2 that for $\sigma \in (-\infty, 1]$

$$\int_\Omega \rho_v^\sigma(\mathbf{x})|\partial_v u|^2 d\mathbf{x} \geqslant \left(\frac{1-\sigma}{2}\right)^2 \int_\Omega \rho_v(\mathbf{x})^{\sigma-2} \left(1 + k(\sigma)\left[\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})}\right]^{(1-\sigma)}\right)^2 |u(\mathbf{x})|^2 d\mathbf{x}.$$
$$(3.7)$$

Expanding the integrand in (3.7), we have

$$\begin{aligned}
\rho_v(\mathbf{x})^{\sigma-2} &\left(1 + k(\sigma)\left[\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})}\right]^{(1-\sigma)}\right)^2 \\
&= \rho_v(\mathbf{x})^{\sigma-2} + 2^{2-\sigma}\frac{k(\sigma)\rho_v(\mathbf{x})^{-\sigma}}{(\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x}))^{1-\sigma}} + 2^{2(1-\sigma)}k(\sigma)^2 \frac{\rho_v(\mathbf{x})^{-\sigma}}{\mu_v(\mathbf{x})^{2(1-\sigma)}}.
\end{aligned} \quad (3.8)$$

If $\sigma \leqslant 0$

$$\begin{aligned}
\rho_v(\mathbf{x})^{\sigma-2} &\left[1 + k(\sigma)\left(\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})}\right)^{(1-\sigma)}\right]^2 \\
&\geqslant \rho_v(\mathbf{x})^{\sigma-2} + 2^{2-\sigma}\frac{k(\sigma)\delta(\mathbf{x})^{|\sigma|}}{(\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x}))^{1-\sigma}} + 2^{2(1-\sigma)}k(\sigma)^2 \frac{\delta(\mathbf{x})^{|\sigma|}}{\tau_v(\mathbf{x})^{2(1-\sigma)} + \tau_{-v}(\mathbf{x})^{2(1-\sigma)}}
\end{aligned}$$
$$(3.9)$$

since $\rho_v(\mathbf{x})^{-\sigma} \geqslant \delta(\mathbf{x})^{|\sigma|}$ in this case. As in [6], we note that since

$$\int_{\mathbb{S}^{n-1}} (\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x}))^{1-\sigma} d\omega(v) \leqslant \int_{\mathbb{S}^{n-1}} (\tau_v(\mathbf{x}))^{2(1-\sigma)} d\omega(v)$$

$$\leqslant \left[\int_{\mathbb{S}^{n-1}} (\tau_v(\mathbf{x}))^n d\omega(v) \right]^{\frac{2(1-\sigma)}{n}}$$

$$= \left[\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right]^{\frac{2(1-\sigma)}{n}}$$

for $\sigma \geqslant \frac{2-n}{2}$, then

$$\int_{\mathbb{S}^{n-1}} \frac{1}{(\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x}))^{1-\sigma}} d\omega(v) \geqslant \left[\int_{\mathbb{S}^{n-1}} (\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x}))^{1-\sigma} d\omega(v) \right]^{-1}$$

$$\geqslant \left[\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right]^{-\frac{2(1-\sigma)}{n}}.$$

For the third term in inequality (3.9)

$$\int_{\mathbb{S}^{n-1}} \left(\tau_v(\mathbf{x})^{2(1-\sigma)} + \tau_{-v}(\mathbf{x})^{2(1-\sigma)} \right) d\omega(v) = 2 \int_{\mathbb{S}^{n-1}} \tau_v(\mathbf{x})^{2(1-\sigma)} d\omega(v)$$

implying that for $\sigma \geqslant \frac{2-n}{2}$

$$\int_{\mathbb{S}^{n-1}} \left(\tau_v(\mathbf{x})^{2(1-\sigma)} + \tau_{-v}(\mathbf{x})^{2(1-\sigma)} \right)^{-1} d\omega(v) \geqslant \frac{1}{2} \left[\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right]^{-\frac{2(1-\sigma)}{n}}.$$

Consequently, for $\frac{2-n}{2} \leqslant \sigma \leqslant 0$ we have that

$$\int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^{\sigma-2} \left[1 + k(\sigma) \left(\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})} \right)^{(1-\sigma)} \right]^2 d\omega(v)$$

$$\geqslant \rho(\mathbf{x}; \sigma - 2) + C_H(n, \sigma)\delta(\mathbf{x})^{|\sigma|} / \left[n|\Omega_{\mathbf{x}}|^{\frac{2(1-\sigma)}{n}} \right]. \tag{3.10}$$

Upon combining this fact with (3.7) we have

$$\left(\frac{1-\sigma}{2} \right)^2 \int_{\Omega} \left\{ \rho(\mathbf{x}; \sigma - 2) + \frac{C_H(n, \sigma)\delta(\mathbf{x})^{|\sigma|}}{n|\Omega_{\mathbf{x}}|^{2(1-\sigma)/n}} \right\} |u(\mathbf{x})|^2 d\mathbf{x}$$

$$\leqslant \int_{\Omega} \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^{\sigma} |\partial_v u(\mathbf{x})|^2 d\omega(v) d\mathbf{x} \tag{3.11}$$

$$= \int_{\Omega} \delta(\mathbf{x})^{\sigma} \int_{\mathbb{S}^{n-1}} |\cos(v, \nabla u(\mathbf{x}))|^2 d\omega(v) |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

for $\sigma \leqslant 0$. Since

$$\int_{\mathbb{S}^{n-1}} |\cos(v, \alpha)|^2 d\omega(v) = \frac{1}{n} \tag{3.12}$$

for any fixed $\alpha \in \mathbb{S}^{n-1}$ (see Tidblom [11], p.2270), inequality (3.5) follows.

For $0 < \sigma \leqslant 1$, we consider first the third term on the right-hand side of (3.8). We have

$$\int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^{\sigma} \mu_v(\mathbf{x})^{2(1-\sigma)} d\omega(v)$$

$$\leqslant \int_{\mathbb{S}^{n-1}} 2^{-\sigma} \left(\tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x}) \right)^{\sigma} \left(\tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x}) \right)^{2(1-\sigma)} d\omega(v)$$

$$= 2^{-\sigma} \left\| \tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x}) \right\|_{L^{2-\sigma}(\mathbb{S}^{n-1})}^{2-\sigma}$$

$$\leqslant 2^{-\sigma} \left[\left\| \tau_v(\mathbf{x}) \right\|_{L^{2-\sigma}(\mathbb{S}^{n-1})} + \left\| \tau_{-v}(\mathbf{x}) \right\|_{L^{2-\sigma}(\mathbb{S}^{n-1})} \right]^{2-\sigma}$$

$$= 2^{2(1-\sigma)} \int_{\mathbb{S}^{n-1}} \tau_v(\mathbf{x})^{2-\sigma} d\omega(v)$$

$$\leqslant 2^{2(1-\sigma)} \left[\int_{\mathbb{S}^{n-1}} \left(\tau_v(\mathbf{x}) \right)^n d\omega(v) \right]^{\frac{2-\sigma}{n}}$$

$$= 2^{2(1-\sigma)} \left[\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right]^{\frac{2-\sigma}{n}}$$

for $n \geqslant 2$ by the Minkowski and Hölder inequalities. Therefore, the term

$$\int_{\mathbb{S}^{n-1}} \frac{\rho_v(\mathbf{x})^{-\sigma} d\omega(v)}{\mu_v(\mathbf{x})^{2(1-\sigma)}} \geqslant 2^{2(\sigma-1)} \left(\frac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \right)^{\frac{2-\sigma}{n}}.$$

Similarly, in the second term of (3.8)

$$\int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x}) \mu_v(\mathbf{x})^{1-\sigma} d\omega(v) \leqslant \frac{1}{2} \int_{\mathbb{S}^{n-1}} (\tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x}))(\tau_v(\mathbf{x}) + \tau_{-v}(\mathbf{x}))^{1-\sigma} d\omega(v)$$

$$\leqslant 2^{1-\sigma} \left[\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right]^{\frac{2-\sigma}{n}}$$

as before implying that

$$\int_{\mathbb{S}^{n-1}} \frac{d\omega(v)}{\rho_v(\mathbf{x}) \mu_v(\mathbf{x})^{1-\sigma}} \geqslant 2^{\sigma-1} \left(\frac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \right)^{\frac{2-\sigma}{n}}.$$

For $0 < \sigma < 1$ we now have that

$$\int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^{\sigma-2} \left[1 + k(\sigma) \left(\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})} \right)^{(1-\sigma)} \right]^2 d\omega(v) \geqslant \rho(\mathbf{x}; \sigma - 2) + 3 \left(\frac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \right)^{\frac{2-\sigma}{n}}$$

since $k(\sigma) = 1$ in this case. Consequently,

$$\int_{\Omega} \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^{\sigma} |\cos(v, \nabla u(\mathbf{x}))|^2 d\omega(v) |\nabla u(\mathbf{x})|^2 d\mathbf{x}$$

$$\geqslant \left(\frac{1-\sigma}{2} \right)^2 \int_{\Omega} \left[\rho(\mathbf{x}; \sigma - 2) + 3 \left(\frac{s_{n-1}}{n|\Omega_{\mathbf{x}}|} \right)^{\frac{2-\sigma}{n}} \right] |u(\mathbf{x})|^2 d\mathbf{x}.$$

According to (3.12) it follows that

$$\int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^\sigma |\cos(v, \nabla u(\mathbf{x}))|^2 d\omega(v) \leqslant \frac{D(\Omega)^\sigma}{2^\sigma n}.$$

Therefore, (3.6) holds.

The inequalities in the statement of the theorem for the case of a convex domain Ω follow from (3.3) and the fact that $|\Omega_\mathbf{x}| = |\Omega|$ for all $\mathbf{x} \in \Omega$. □

4. An $L^p(\Omega)$ inequality

With the guidance of Tidblom's analysis for the Hardy inequality in [11], L^p versions of the weighted Hardy theorem in the last section can be proved by similar techniques. When $\sigma = 0$, the next theorem reduces to Theorem 2.1 of [11].

THEOREM 2. *Let $u \in C_0^1(\Omega)$ and $p \in (1, \infty)$. If $\sigma \leqslant 0$, then for $B(n,p)$ defined in (3.2)*

$$\int_\Omega \delta(\mathbf{x})^\sigma |\nabla u(\mathbf{x})|^p d\mathbf{x} \geqslant$$

$$\frac{[|p - \sigma - 1|/p]^p}{B(n,p)} \int_\Omega \left\{ \rho(\mathbf{x}; \sigma - p) + (p-1) \left[\frac{s_{n-1}}{n|\Omega_\mathbf{x}|} \right]^{\frac{p-\sigma}{n}} \right\} |u(\mathbf{x})|^p d\mathbf{x} \qquad (4.1)$$

and if $\sigma \in [0, p-1]$, then

$$\int_\Omega |\nabla u(\mathbf{x})|^p d\mathbf{x} \geqslant$$

$$\frac{2^\sigma[|p - \sigma - 1|/p]^p}{B(n,p) D(\Omega)^\sigma} \int_\Omega \left\{ \rho(\mathbf{x}; \sigma - p) + (p-1) \left[\frac{s_{n-1}}{n|\Omega_\mathbf{x}|} \right]^{\frac{p-\sigma}{n}} \right\} |u(\mathbf{x})|^p d\mathbf{x}. \qquad (4.2)$$

If Ω is convex, $\rho(\mathbf{x}, \sigma - p)$ can be replaced in (4.1) and (4.2) by the term $B(n, p - \sigma)/\delta(\mathbf{x})^{p-\sigma}$ (in view of (3.3)) and $|\Omega_x|$ by $|\Omega|$.

Proof. From Lemma 3 we have that for $\sigma \leqslant p-1$, any $v \in \mathbb{S}^{n-1}$, and $u \in C_0^1(\Omega)$

$$\int_\Omega \rho_v(\mathbf{x})^\sigma |\partial_v u(\mathbf{x})|^p d\mathbf{x}$$

$$\geqslant \left[\frac{|p - \sigma - 1|}{p} \right]^p \int_\Omega \left\{ \rho_v(\mathbf{x})^{\sigma-p} + \frac{(p-1)2^{p-\sigma}}{D_v(\mathbf{x})^{p-\sigma}} \right\} |u(\mathbf{x})|^p d\mathbf{x}. \qquad (4.3)$$

If $\sigma \leqslant 0$ we bound $\rho_v(\mathbf{x})^\sigma$ for any $v \in \mathbb{S}^{n-1}$ by $\delta(\mathbf{x})^\sigma$ in the first integral above. If $\sigma > 0$, we bound it by $D(\Omega)^\sigma/2^\sigma$. As in [11] we may use the fact that

$$\int_{\mathbb{S}^{n-1}} |\partial_v u(\mathbf{x})|^p d\omega(v) = B(n,p)|\nabla u(\mathbf{x})|^p. \qquad (4.4)$$

After bounding $\rho_v(\mathbf{x})^\sigma$ as described above, integrate in (4.3) over $\mathbb{S}^{n-1}$ with respect to $d\omega(v)$. In order to evaluate the integral of $(2/D_v(\mathbf{x}))^{p-\sigma}$, we proceed as in [11]. Since $\sigma \leqslant p - 1$, then $f(t) = t^{\sigma-p}$ is convex for $t > 0$ and we have that

$$\int_{\mathbb{S}^{n-1}} \left(\frac{2}{D_v(\mathbf{x})}\right)^{p-\sigma} d\omega(v) \geqslant \left(\int_{\mathbb{S}^{n-1}} \frac{D_v(\mathbf{x})}{2} d\omega(v)\right)^{\sigma-p} \geqslant \left(\frac{n|\Omega_{\mathbf{x}}|}{s_{n-1}}\right)^{-\frac{p-\sigma}{n}} \quad (4.5)$$

by Jensen's inequality and Lemma 2.1 of [11]. The conclusion follows. $\qquad\square$

5. Rellich's inequality

The methods described above with Proposition 1 below can be used to prove a weighted Rellich inequality which, for $n \geqslant 4$ and without weights, improves the constant given in a Rellich inequality proved recently by Barbatis ([1], Theorem 1.2). A comparison is made below. The methods used by Barbatis depends upon the identity (5.2) first proved by M.P. Owen ([10], see the proof of Theorem 2.3). In order to incorporate weights, our proof requires the point-wise identity (5.1) which does not follow from the proof of Owen.

PROPOSITION 1. *Let Ω be a domain in $\mathbb{R}^n$. Then, for all $u \in C^2(\mathbb{R}^n)$*

$$\int_{\mathbb{S}^{n-1}} \left|\partial_v^2 u(\mathbf{x})\right|^2 d\omega(v) = \frac{1}{n(n+2)} \left[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \left|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\right|^2\right], \quad (5.1)$$

and for all $u \in C_0^2(\Omega)$

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \left|\partial_v^2 u(\mathbf{x})\right|^2 d\omega(v) d\mathbf{x} = \frac{3}{n(n+2)} \int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x}. \quad (5.2)$$

Proof.
For $v = (v_1, \ldots, v_n)$ we have

$$\partial_v^2 u = (v \cdot \nabla)^2 u = \sum_{\ell,m=1}^n v_\ell v_m u_{\ell m}$$

$$= \sum_{\ell=1}^n v_\ell^2 u_{\ell\ell} + 2\sum_{1 \leqslant \ell < m \leqslant n} v_\ell v_m u_{\ell m}$$

in which $u_{pq}(\mathbf{x}) := \frac{\partial^2 u(\mathbf{x})}{\partial x_p \partial x_q}$. Consequently,

$$\int_{\mathbb{S}^{n-1}} \left|\partial_v^2 u\right|^2 d\omega(v) = \sum_{\ell,m=1}^n u_{\ell\ell}\overline{u_{mm}} \int_{\mathbb{S}^{n-1}} (v_\ell)^2 (v_m)^2 d\omega(v)$$

$$+ 4\sum_{m=1}^n \sum_{1 \leqslant p < q \leqslant n} \Re\mathfrak{e}\,(u_{mm}\overline{u_{pq}}) \int_{\mathbb{S}^{n-1}} (v_m)^2 v_p v_q d\omega(v)$$

$$+ 4\sum_{1 \leqslant j < k \leqslant n} \sum_{1 \leqslant p < q \leqslant n} \Re\mathfrak{e}\,(u_{pq}\overline{u_{jk}}) \int_{\mathbb{S}^{n-1}} v_p v_q v_j v_k d\omega(v).$$
$$(5.3)$$

Let $\theta_j \in [0, \pi]$ for $j = 1, \ldots, n - 2$, and $\theta_{n-1} \in [0, 2\pi]$. Using the convention that $\Pi_{j=q}^{p} = 1$ for $p < q$ and $\theta_n = 0$, we have

$$v_j = \Pi_{k=1}^{j-1} \sin \theta_k \cos \theta_j, \qquad j = 1, \ldots, n,$$

$$d\omega(v) := \frac{(n-2)!!}{\gamma_n} \Pi_{k=1}^{n-2} (\sin \theta_k)^{n-1-k} d\theta_k d\theta_{n-1}, \tag{5.4}$$

for $n!! := n \cdot (n - 2) \cdot (n - 4) \cdots 1$ and

$$\gamma_n = \begin{cases} 2(2\pi)^{(n-1)/2} & \text{for } n \text{ odd,} \\ (2\pi)^{n/2} & \text{for } n \text{ even.} \end{cases}$$

Calculations show that

$$\int_{\mathbb{S}^{n-1}} (v_m)^2 v_p v_q d\omega(v) = 0, \qquad m = 1, \ldots, n, \quad 1 \leqslant p < q \leqslant n$$

implying that the second term on the right-hand side of (5.3) vanishes.

A similar consideration for the third term on the right-hand side of (5.3) shows that

$$\int_{\mathbb{S}^{n-1}} v_p v_q v_j v_k d\omega(v) \neq 0, \qquad 1 \leqslant p < q \leqslant n, \quad 1 \leqslant j < k \leqslant n,$$

only if $j = p$ and $k = q$. Therefore, (5.3) reduces to

$$\int_{\mathbb{S}^{n-1}} \left| \partial_v^2 u(\mathbf{x}) \right|^2 d\omega(v) = \sum_{\ell,m=1}^{n} u_{\ell\ell} \overline{u_{mm}} \int_{\mathbb{S}^{n-1}} (v_\ell)^2 (v_m)^2 d\omega(v)$$

$$+ 4 \sum_{1 \leqslant p < q \leqslant n} |u_{pq}|^2 \int_{\mathbb{S}^{n-1}} (v_p)^2 (v_q)^2 d\omega(v). \tag{5.5}$$

However, further calculations show that

$$\int_{\mathbb{S}^{n-1}} v_p^2 v_q^2 d\omega(v) = \begin{cases} \frac{1}{n(n+2)} & 1 \leqslant p < q \leqslant n, \\ \frac{3}{n(n+2)} & p = q = 1, \ldots, n \end{cases} \tag{5.6}$$

implying that

$$\int_{\mathbb{S}^{n-1}} \left| \partial_v^2 u \right|^2 d\omega(v) = \frac{3}{n(n+2)} \sum_{m=1}^{n} |u_{mm}|^2 + \frac{1}{n(n+2)} \sum_{1 \leqslant p < q \leqslant n} \left[4|u_{pq}|^2 + 2\Re\mathfrak{e}(u_{pp}\overline{u_{qq}}) \right]$$

$$= \frac{1}{n(n+2)} \left[|\Delta u(\mathbf{x})|^2 + 2 \sum_{i,j=1}^{n} \left| \frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 \right] \tag{5.7}$$

which is (5.1). Equality (5.2) now follows since

$$\sum_{i,j=1}^{n} \int_{\Omega} \left| \frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j} \right|^2 d\mathbf{x} = \int_{\Omega} |\Delta u(\mathbf{x})|^2 d\mathbf{x}.$$

$\square$

Define

$$d(\mathbf{x}; \sigma) := \begin{cases} \delta(\mathbf{x})^\sigma, & \sigma < 0, \\ \left(\frac{D(\Omega)}{2}\right)^\sigma, & \sigma \in [0, 1]; \end{cases}$$

$$\beta(n, \sigma) := \frac{(1 - \sigma)^2 (3 - \sigma)^2 n(n + 2)}{16};$$

and

$$C_R(n, \sigma) := 2^{4-\sigma} k(\sigma - 2) \left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \left(1 + 2^{2-\sigma} k(\sigma - 2)\right) \qquad (5.8)$$

for $\sigma \leqslant 1$ and $k(\sigma)$ defined in Lemma 2.

THEOREM 3.   *For $\sigma \leqslant 1$ and $u \in C_0^2(\Omega)$,*

$$\int_\Omega d(\mathbf{x}; \sigma) \left[|\Delta u(\mathbf{x})|^2 + 2 \sum_{i,j=1}^n \left|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\right|^2\right] d\mathbf{x}$$

$$\geqslant \beta(n, \sigma) \left\{\int_\Omega \rho(\mathbf{x}; \sigma - 4) |u(\mathbf{x})|^2 d\mathbf{x}\right.$$

$$\left. + 2^{4-\sigma} k(\sigma - 2) \left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x}\right\} \qquad (5.9)$$

holds when $n \geqslant 4 - \sigma$ and

$$\int_\Omega d(\mathbf{x}; \sigma) \left[|\Delta u(\mathbf{x})|^2 + 2 \sum_{i,j=1}^n \left|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\right|^2\right] d\mathbf{x}$$

$$\geqslant \beta(n, \sigma) \left\{\int_\Omega \rho(\mathbf{x}; \sigma - 4) |u(\mathbf{x})|^2 d\mathbf{x}\right.$$

$$+ 2^{4-\sigma} k(\sigma - 2) \left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x}$$

$$\left. + 2^{2(3-\sigma)} k(\sigma - 2)^2 \left[\frac{s_{n-1}}{n}\right]^{\frac{4+t-\sigma}{n}} \int_\Omega \frac{\delta(\mathbf{x})^t |u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4+t-\sigma}{n}}} d\mathbf{x}\right\} \qquad (5.10)$$

holds when $n \geqslant 4 + t - \sigma$ and $t \geqslant 2 - \sigma$.

Proof. For $\sigma \leqslant 1$, it follows that

$$\int_0^{2b} \rho(t)^\sigma |u''(t)|^2 dt \geqslant \int_0^{2b} \rho(t)^\sigma \left[1 - \left(\frac{\rho(t)}{\mu(t)}\right)^{1-\sigma}\right]^2 |u''(t)|^2 dt$$

$$\geqslant \left(\frac{1 - \sigma}{2}\right)^2 \int_0^{2b} \rho(t)^{\sigma-2} |u'(t)|^2 dt$$

by (2.4). Therefore, for $\sigma \leqslant 1$ and $u \in C_0^2(0, 2b)$,

$$\int_0^{2b} \rho(t)^\sigma |u''(t)|^2 \, dt$$

$$\geqslant \left(\frac{(1-\sigma)(3-\sigma)}{4}\right)^2 \int_0^{2b} \rho(t)^{\sigma-4} \left[1 + k(\sigma-2) \left(\frac{2\rho(t)}{\mu(t)}\right)^{3-\sigma}\right]^2 |u(t)|^2 dt \tag{5.11}$$

by (2.3).

From (5.11) we have for $u \in C_0^2(\Omega)$

$$\int_\Omega \rho_v(\mathbf{x})^\sigma \left|\partial_v^2 u(\mathbf{x})\right|^2 d\mathbf{x}$$

$$\geqslant \left(\frac{(1-\sigma)(3-\sigma)}{4}\right)^2 \int_\Omega \rho_v(\mathbf{x})^{\sigma-4} \left\{1 + k(\sigma-2) \left(\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})}\right)^{3-\sigma}\right\}^2 |u(\mathbf{x})|^2 d\mathbf{x} \tag{5.12}$$

for $\sigma \leqslant 1$. As in (3.8) we write

$$\rho_v(\mathbf{x})^{\sigma-4} \left\{1 + k(\sigma-2) \left(\frac{2\rho_v(\mathbf{x})}{\mu_v(\mathbf{x})}\right)^{3-\sigma}\right\}^2$$

$$= \rho_v(\mathbf{x})^{\sigma-4} + 2^{4-\sigma} k(\sigma-2) \frac{\rho_v^{-1}(\mathbf{x})}{\mu_v(\mathbf{x})^{3-\sigma}} + 2^{2(3-\sigma)} k(\sigma-2)^2 \frac{\rho_v^{-\sigma+2}(\mathbf{x})}{\mu_v(\mathbf{x})^{2(3-\sigma)}}. \tag{5.13}$$

Since $\rho_v(\mathbf{x})\mu_v(\mathbf{x}) = \tau_v(\mathbf{x})\tau_{-v}(\mathbf{x})$, in the second term on the right-hand side of (5.13) we may write

$$\frac{\rho_v^{-1}(\mathbf{x})}{\mu_v(\mathbf{x})^{3-\sigma}} = \frac{1}{[\tau_v(\mathbf{x})\tau_{-v}(\mathbf{x})]\mu_v(\mathbf{x})^{2-\sigma}} =: I(v; \mathbf{x}).$$

Thus

$$\int_{\mathbb{S}^{n-1}} I(v; \mathbf{x}) d\omega(v) = \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{\sigma-3}(\mathbf{x}) \tau_{-v}(\mathbf{x})^{-1} d\omega(v)$$

$$+ \int_{\tau_v(\mathbf{x}) \leqslant \tau_{-v}(\mathbf{x})} \tau_{-v}(\mathbf{x})^{\sigma-3}(\mathbf{x}) \tau_v(\mathbf{x})^{-1} d\omega(v)$$

$$\geqslant \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{\sigma-4}(\mathbf{x}) d\omega(v)$$

$$+ \int_{\tau_v(x) \leqslant \tau_{-v}(\mathbf{x})} \tau_{-v}(\mathbf{x})^{\sigma-4}(\mathbf{x}) d\omega(v)$$

and

$$\left\{ \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{\sigma-4} d\omega(v) \right\}^{-1} \leqslant \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{-\sigma+4} d\omega(v)$$

$$\leqslant \left\{ \int_{\mathbb{S}^{n-1}} \tau_v^n(\mathbf{x}) d\omega(v) \right\}^{(4-\sigma)/n}$$

$$= \left(\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right)^{(4-\sigma)/n}$$

for $n \geqslant 4 - \sigma$. Therefore for the second term on the right-hand side of (5.13), for $\sigma \leqslant 1$ and $n \geqslant 4 - \sigma$, it follows that

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \frac{\rho_v^{-1}(\mathbf{x}) d\omega(v)}{\mu_v(\mathbf{x})^{3-\sigma}} |u(\mathbf{x})|^2 d\mathbf{x} \geqslant \left(\frac{s_{n-1}}{n} \right)^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4-\sigma}{n}}} d\mathbf{x}. \qquad (5.14)$$

For any $t \in (-\infty, \infty)$, we may write the third term in (5.13) as

$$\frac{\rho_v^{-\sigma+2}(\mathbf{x})}{\mu_v(\mathbf{x})^{2(3-\sigma)}} = \rho_v(\mathbf{x})^t \left(\tau_v(\mathbf{x}) \tau_{-v}(\mathbf{x}) \right)^{2-\sigma-t} \mu(\mathbf{x})^{-8+3\sigma+t} =: \rho_v(\mathbf{x})^t J(v, \mathbf{x}).$$

If $t \geqslant 2 - \sigma$

$$\int_{\mathbb{S}^{n-1}} J(v; \mathbf{x}) d\omega(v) \geqslant \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{-4+\sigma-t} d\omega(v) + \int_{\tau_v(\mathbf{x}) \leqslant \tau_{-v}(\mathbf{x})} \tau_{-v}(\mathbf{x})^{-4+\sigma-t} d\omega(v).$$

As before

$$\left\{ \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{-4+\sigma-t} d\omega(v) \right\}^{-1} \leqslant \int_{\tau_v(\mathbf{x}) \geqslant \tau_{-v}(\mathbf{x})} \tau_v(\mathbf{x})^{4-\sigma+t} d\omega(v)$$

$$\leqslant \left\{ \int_{\mathbb{S}^{n-1}} \tau_v^n(\mathbf{x}) d\omega(v) \right\}^{(4-\sigma+t)/n}$$

$$= \left(\frac{n}{s_{n-1}} |\Omega_{\mathbf{x}}| \right)^{(4-\sigma+t)/n}$$

if $n \geqslant 4 - \sigma + t$. Associated with the third term on the right-hand side of (5.13), we have for $\sigma \leqslant 1$, $t \geqslant 2 - \sigma > 0$, and $n \geqslant 4 - \sigma + t$

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \frac{\rho_v^{-\sigma+2}(\mathbf{x}) d\omega(v)}{\mu_v(\mathbf{x})^{2(3-\sigma)}} |u(\mathbf{x})|^2 d\mathbf{x} \geqslant \left(\frac{s_{n-1}}{n} \right)^{\frac{4+t-\sigma}{n}} \int_\Omega \frac{\delta(\mathbf{x})^t |u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4+t-\sigma}{n}}} d\mathbf{x}. \qquad (5.15)$$

From (5.12) – (5.15) we obtain

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^\sigma \left| \partial_v^2 u(\mathbf{x}) \right|^2 d\omega(v) d\mathbf{x} \geqslant$$

$$\frac{(1-\sigma)^2(3-\sigma)^2}{16} \left\{ \int_\Omega \rho(\mathbf{x}; \sigma-4) |u(\mathbf{x})|^2 d\mathbf{x} + 2^{4-\sigma} k(\sigma-2) \left[\frac{s_{n-1}}{n} \right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4-\sigma}{n}}} d\mathbf{x} \right.$$

$$\left. + 2^{2(3-\sigma)} k(\sigma - 2)^2 \left[\frac{s_{n-1}}{n} \right]^{\frac{4+t-\sigma}{n}} \int_\Omega \frac{\delta(\mathbf{x})^t |u(\mathbf{x})|^2}{|\Omega_{\mathbf{x}}|^{\frac{4+t-\sigma}{n}}} d\mathbf{x} \right\}$$

provided $\sigma \leqslant 1$, $t \geqslant 2 - \sigma$, and $n \geqslant 4 + t - \sigma$.

Note, that we may simply choose zero as a lower bound for the third term on the right-hand side of (5.13) and conclude that

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^\sigma |\partial_v^2 u(\mathbf{x})|^2 d\omega(v) d\mathbf{x} \geqslant$$

$$\frac{(1-\sigma)^2(3-\sigma)^2}{16} \left\{ \int_\Omega \rho(\mathbf{x}; \sigma-4)|u(\mathbf{x})|^2 d\mathbf{x} + 2^{4-\sigma}k(\sigma-2)\left[\frac{s_{n-1}}{n}\right]^{\frac{4-\sigma}{n}} \int_\Omega \frac{|u(\mathbf{x})|^2}{|\Omega_\mathbf{x}|^{\frac{4-\sigma}{n}}} d\mathbf{x} \right\}$$

for $\sigma \leqslant 1$ and $n \geqslant 4 - \sigma$.

Now, it follows from Proposition 1 that

$$\int_\Omega \int_{\mathbb{S}^{n-1}} \rho_v(\mathbf{x})^\sigma \left|\partial_v^2 u(\mathbf{x})\right|^2 d\omega(v) d\mathbf{x} \leqslant \frac{1}{n(n+2)} \int_\Omega d(\mathbf{x}; \sigma) \left[|\Delta u(\mathbf{x})|^2 + 2\sum_{i,j=1}^n \left|\frac{\partial^2 u(\mathbf{x})}{\partial x_i \partial x_j}\right|^2\right] d\mathbf{x}.$$

Thus, (5.9) and (5.10) are proved. □

It follows from Theorem 1.2 of Barbatis [1] that for a convex bounded domain Ω and all $u \in C_0^\infty(\Omega)$

$$\int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x} \geqslant \frac{9}{16} \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^4} d\mathbf{x} + \frac{11}{48}n(n+2)\left[\frac{s_{n-1}}{n|\Omega|}\right]^{4/n} \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x}. \quad (5.16)$$

As in Theorem 2, for a convex domain $\Omega \subset \mathbb{R}^n$, we may replace $\rho(\mathbf{x}, \sigma - 4)$ in Theorem 3 by $B(n, 4 - \sigma)/\delta(\mathbf{x})^{4-\sigma}$ and $|\Omega_x|$ by $|\Omega|$ to conclude from (5.9) that for $n \geqslant 4$

$$\int_\Omega |\Delta u(\mathbf{x})|^2 d\mathbf{x} \geqslant \frac{9}{16} \int_\Omega \frac{|u(\mathbf{x})|^2}{\delta(\mathbf{x})^4} d\mathbf{x} + c_4 n(n+2)\left[\frac{s_{n-1}}{n|\Omega|}\right]^{4/n} \int_\Omega |u(\mathbf{x})|^2 d\mathbf{x} \quad (5.17)$$

for all $u \in C_0^\infty(\Omega)$ in which $c_4 = 3k(-2) \approx 1.25$. Therefore (5.17) improves the bound given by (5.16) for all $n \geqslant 4$.

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W.D.Evans
School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4AG
UK
e-mail: EvansWD@cardiff.ac.uk

R.T.Lewis
Department of Mathematics
University of Alabama at Birmingham
Birmingham
AL 35294-1170
USA
e-mail: lewis@math.uab.edu

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www.ele-math.com
jmi@ele-math.com