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FIXED POINTS AND GENERALIZED HYERS–ULAM STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS

CHOONKIL PARK AND THEMISTOCLES M. RASSIAS

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Abstract. Let X, Y be complex vector spaces. It is shown that if a mapping $f : X \to Y$ satisfies

$$f(x+iy)+f(x-iy)=2f(x)-2f(y) \tag{0.1}$$

or

$$f(x+iy)-f(ix+y)=2f(x)-2f(y) \tag{0.2}$$

for all $x, y \in X$, then the mapping $f : X \to Y$ satisfies

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all $x, y \in X$.

Furthermore, we prove the generalized Hyers-Ulam stability of the functional equations (0.1) and (0.2) in complex Banach spaces.

1. Introduction

The stability problem of functional equations originated from a question of Ulam [42] concerning the stability of group homomorphisms: Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(x*y), h(x)\diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$? If the answer is affirmative, we would say that the equation of homomorphism $H(x*y) = H(x)\diamond H(y)$ is stable. The concept of stability for a functional equation arises when we replace the functional equation by an inequality which acts as a perturbation of the equation. Thus the stability question of functional equations is that how do the solutions of the inequality differ from those of the given functional equation?

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Hyers [9] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces. Assume that $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leqslant \varepsilon$$

for all $x, y \in X$ and some $\varepsilon \geqslant 0$. Then there exists a unique additive mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leqslant \varepsilon$$

for all $x \in X$.

Th. M. Rassias [32] provided a generalization of Hyers' Theorem which allows the *Cauchy difference to be unbounded.*

THEOREM 1.1. (Th. M. Rassias). *Let $f : E \to E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leqslant \epsilon(\|x\|^p + \|y\|^p) \tag{1.1}$$

for all $x, y \in E$, where ϵ and p are constants with $\epsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \to E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leqslant \frac{2\epsilon}{2 - 2^p}\|x\|^p$$

for all $x \in E$. Also, if for each $x \in E$ the function $f(tx)$ is continuous in $t \in \mathbb{R}$, then L is $\mathbb{R}$-linear.

The above inequality (1.1) that was introduced for the first time by Th. M. Rassias [32] for the proof of the stability of the linear mapping between Banach spaces has provided a lot of influence in the development of what is now known as a *generalized Hyers-Ulam stability* or as *Hyers-Ulam-Rassias stability* of functional equations. Beginning around the year 1980 the topic of approximate homomorphisms, or the stability of the equation of homomorphism, was studied by a number of mathematicians. Găvruta [8] extended the Hyers-Ulam stability by proving the following theorem in the spirit of Th. M. Rassias' approach.

THEOREM 1.2. [8] *Let $f : E \to E'$ be a mapping for which there exists a function $\varphi : E \times E' \to [0, \infty)$ such that*

$$\begin{aligned} \widetilde{\varphi}(x,y) := \sum_{j=0}^{\infty} 2^{-j}\varphi(2^j x, 2^j y) &< \infty, \\ \|f(x+y) - f(x) - f(y)\| &\leqslant \varphi(x,y) \end{aligned}$$

for all $x, y \in E$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leqslant \frac{1}{2}\widetilde{\varphi}(x,x)$$

for all $x \in E$.

A square norm on an inner product space satisfies the important parallelogram equality
$$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2.$$
The functional equation
$$f(x+y)+f(x-y)=2f(x)+2f(y)$$
is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic function*. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [41] for mappings $f:X\to Y$, where X is a normed space and Y is a Banach space. Cholewa [5] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [6] proved the generalized Hyers-Ulam stability of the quadratic functional equation. The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [1], [12], [14]–[30], [34]–[40], [43]).

We recall two fundamental results in fixed point theory. The reader is referred to the book of D. H. Hyers, G. Isac and Th. M. Rassias [10] for an extensive account of fixed point theory with several applications.

THEOREM 1.3. [3, 4, 31] *Let (X,d) be a complete metric space and let $J:X\to X$ be strictly contractive, i.e.,*
$$d(Jx,Jy)\leqslant Lf(x,y),\qquad \forall x,y\in X$$
for some Lipschitz constant $L<1$. Then
(1) the mapping J has a unique fixed point $x^*=Jx^*$;
(2) the fixed point x^* is globally attractive, i.e.,
$$\lim_{n\to\infty}J^nx=x^*$$
for any starting point $x\in X$;
(3) one has the following estimation inequalities:
$$\begin{aligned}d(J^nx,x^*)&\leqslant L^nd(x,x^*),\\ d(J^nx,x^*)&\leqslant \frac{1}{1-L}d(J^nx,J^{n+1}x),\\ d(x,x^*)&\leqslant \frac{1}{1-L}d(x,Jx)\end{aligned}$$
for all nonnegative integers n and all $x\in X$.

Let X be a set. A function $d:X\times X\to[0,\infty]$ is called a *generalized metric* on X if d satisfies
(1) $d(x,y)=0$ if and only if $x=y$;
(2) $d(x,y)=d(y,x)$ for all $x,y\in X$;
(3) $d(x,z)\leqslant d(x,y)+d(y,z)$ for all $x,y,z\in X$.

THEOREM 1.4. [7] *Let* (X,d) *be a complete generalized metric space and let* $J : X \to X$ *be a strictly contractive mapping with Lipschitz constant* $L < 1$. *Then for each given element* $x \in X$, *either*

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n *or there exists a positive integer* n_0 *such that*
(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geqslant n_0$ *;*
(2) the sequence $\{J^n x\}$ *converges to a fixed point* y^* *of* J *;*
(3) y^* *is the unique fixed point of* J *in the set* $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$ *;*
(4) $d(y,y^*) \leqslant \frac{1}{1-L} d(y, Jy)$ *for all* $y \in Y$.

In this paper, we solve the functional equations (0.1) and (0.2) and by using the fixed point method, we prove the generalized Hyers-Ulam stability of the functional equations (0.1) and (0.2) in complex Banach spaces.

In 1996, G. Isac and Th. M. Rassias [13] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications.

2. Quadratic functional equations

Throughout this section, assume that X and Y are complex vector spaces.

PROPOSITION 2.1. *If a mapping* $f : X \to Y$ *satisfies*

$$f(x+iy) + f(x-iy) = 2f(x) - 2f(y) \tag{2.1}$$

for all $x, y \in X$, *then the mapping* $f : X \to Y$ *is quadratic, i.e.,*

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

holds for all $x, y \in X$. *If a mapping* $f : X \to Y$ *is quadratic and* $f(ix) = -f(x)$ *holds for all* $x \in X$, *then the mapping* $f : X \to Y$ *satisfies* (2.1).

Proof. Assume that $f : X \to Y$ satisfies the equation (2.1).

Letting $x = y$ in (2.1), we get $f((1+i)x) + f((1-i)x) = 0$ for all $x \in X$. So $f(2ix) + f(2x) = 0$ for all $x \in X$. Hence $f(ix) = -f(x)$ for all $x \in X$. Thus

$$f(x+iy) + f(x-iy) = 2f(x) - 2f(y) = 2f(x) + 2f(iy) \tag{2.2}$$

for all $x, y \in X$. Letting $z = iy$ in (2.2), we get

$$f(x+z) + f(x-z) = 2f(x) + 2f(z)$$

for all $x, z \in X$.

Assume that a quadratic mapping $f : X \to Y$ satisfies $f(ix) = -f(x)$ for all $x \in X$.

$$f(x+iy) + f(x-iy) = 2f(x) + 2f(iy) = 2f(x) - 2f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ satisfies (2.1). □

PROPOSITION 2.2. *If a mapping $f : X \to Y$ satisfies $f(0) = 0$ and*

$$f(x+iy) - f(ix+y) = 2f(x) - 2f(y) \tag{2.3}$$

for all $x, y \in X$, then the mapping $f : X \to Y$ is quadratic. If a mapping $f : X \to Y$ is quadratic and $f(ix) = -f(x)$ holds for all $x \in X$, then the mapping $f : X \to Y$ satisfies (2.3).

Proof. Assume that $f : X \to Y$ satisfies the equation (2.3).

Letting $y = 0$ in (2.3), we get $f(x) - f(ix) = 2f(x)$ for all $x \in X$. So $f(ix) = -f(x)$ for all $x \in X$. Thus

$$\begin{aligned} f(x+iy) + f(x-iy) &= f(x+iy) - f(ix+y) \\ &= 2f(x) - 2f(y) = 2f(x) + 2f(iy) \end{aligned} \tag{2.4}$$

for all $x, y \in X$. Letting $z = iy$ in (2.4), we get

$$f(x+z) + f(x-z) = 2f(x) + 2f(z)$$

for all $x, z \in X$.

Assume that a quadratic mapping $f : X \to Y$ satisfies $f(ix) = -f(x)$ for all $x \in X$.

$$f(x+iy) - f(ix+y) = f(x+iy) + f(x-iy) = 2f(x) + 2f(iy) = 2f(x) - 2f(y)$$

for all $x, y \in X$. So the mapping $f : X \to Y$ satisfies (2.3). $\square$

3. Generalized Hyers-Ulam stability of quadratic functional equations

Throughout this section, assume that X is a normed vector space with norm $||\cdot||$ and that Y is a Banach space with norm $\|\cdot\|$.

For a given mapping $f : X \to Y$, we define

$$Cf(x,y) : \;=\; f(x+iy) + f(x-iy) - 2f(x) + 2f(y)$$

for all $x, y \in X$.

We prove the generalized Hyers-Ulam stability of the quadratic functional equation $Cf(x,y) = 0$.

THEOREM 3.1. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(ix) = -f(x)$ and*

$$\|Cf(x,y)\| \leqslant \theta(||x||^p + ||y||^p) \tag{3.1}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{2\theta}{4-2^p}||x||^p \tag{3.2}$$

for all $x \in X$.

Proof. Since $f(ix) = -f(x)$ for all $x \in X$, $f(0) = 0$.

$$f(-x) = f(i^2 x) = -f(ix) = f(x)$$

for all $x \in X$.

Letting $y = -ix$ in (3.1), we get

$$\|f(2x) - 4f(x)\| \leqslant 2\theta ||x||^p \tag{3.3}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \leqslant \frac{\theta}{2}||x||^p$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \leqslant \sum_{j=l}^{m-1} \frac{2^{pj}\theta}{2^{2j+1}}||x||^p \tag{3.4}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.4) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n}f(2^n x)$$

for all $x \in X$.

By (3.1),

$$\|CQ(x,y)\| = \lim_{n\to\infty} \frac{1}{4^n}\|Cf(2^n x, 2^n y)\| \leqslant \lim_{n\to\infty} \frac{2^{pn}\theta}{4^n}(||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing to the limit as m approaches infinity in (3.4), we get (3.2).

Now, let $T : X \to Y$ be another quadratic mapping satisfying (2.1) and (3.2). Then we have

$$\begin{aligned}
\|Q(x) - T(x)\| &= \frac{1}{4^n}\|Q(2^n x) - T(2^n x)\| \\
&\leqslant \frac{1}{4^n}(\|Q(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|) \\
&\leqslant \frac{4\theta}{4-2^p} \cdot \frac{2^{pn}}{4^n}||x||^p,
\end{aligned}$$

which tends to zero as $n \to \infty$ for all $x \in X$. So we can conclude that $Q(x) = T(x)$ for all $x \in X$. This proves the uniqueness of Q. So there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.1) and (3.2). □

THEOREM 3.2. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.1) and $f(ix) = -f(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leqslant \frac{2\theta}{2^p - 4}||x||^p \tag{3.5}$$

for all $x \in X$.

Proof. It follows from (3.3) that

$$\|f(x) - 4f(\frac{x}{2})\| \leqslant \frac{2\theta}{2^p}||x||^p$$

for all $x \in X$. Hence

$$\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\| \leqslant \sum_{j=l}^{m-1} \frac{2 \cdot 4^j \theta}{2^{pj+p}}||x||^p \tag{3.6}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.6) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (3.1),

$$\|CQ(x,y)\| = \lim_{n\to\infty} 4^n \|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \leqslant \lim_{n\to\infty} \frac{4^n\theta}{2^{pn}}(||x||^p + ||y||^p) = 0$$

for all $x, y \in X$. So $CQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.6), we get (3.5).

The rest of the proof is similar to the proof of Theorem 3.1. $\square$

THEOREM 3.3. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(ix) = -f(x)$ and*

$$\|Cf(x,y)\| \leqslant \theta \cdot ||x||^p \cdot ||y||^p \tag{3.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leqslant \frac{\theta}{4 - 4^p}||x||^{2p} \tag{3.8}$$

for all $x \in X$.

Proof. Letting $y = -ix$ in (3.7), we get

$$\|f(2x) - 4f(x)\| \leqslant \theta||x||^{2p} \tag{3.9}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \leqslant \frac{\theta}{4}||x||^{2p}$$

for all $x \in X$. Hence

$$\|\frac{1}{4^l}f(2^l x) - \frac{1}{4^m}f(2^m x)\| \leqslant \sum_{j=l}^{m-1} \frac{4^{pj}\theta}{4^{j+1}}||x||^{2p} \tag{3.10}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.10) that the sequence $\{\frac{1}{4^n}f(2^n x)\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n}f(2^n x)\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$.

By (3.7),

$$\|CQ(x,y)\| = \lim_{n\to\infty} \frac{1}{4^n}\|Cf(2^n x, 2^n y)\| \leqslant \lim_{n\to\infty} \frac{4^{pn}\theta}{4^n} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.10), we get (3.8).

The rest of the proof is similar to the proof of Theorem 3.1. □

THEOREM 3.4. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (3.7) and $f(ix) = -f(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \leqslant \frac{\theta}{4^p - 4}||x||^{2p} \tag{3.11}$$

for all $x \in X$.

Proof. It follows from (3.9) that

$$\|f(x) - 4f(\frac{x}{2})\| \leqslant \frac{\theta}{4^p}||x||^{2p}$$

for all $x \in X$. Hence

$$\|4^l f(\frac{x}{2^l}) - 4^m f(\frac{x}{2^m})\| \leqslant \sum_{j=l}^{m-1} \frac{4^j\theta}{4^{pj+p}}||x||^{2p} \tag{3.12}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.12) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is Cauchy for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $Q : X \to Y$ by

$$Q(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$.

By (3.7),

$$\|CQ(x,y)\| = \lim_{n\to\infty} 4^n\|Cf(\frac{x}{2^n}, \frac{y}{2^n})\| \leqslant \lim_{n\to\infty} \frac{4^n\theta}{4^{pn}} \cdot ||x||^p \cdot ||y||^p = 0$$

for all $x, y \in X$. So $CQ(x,y) = 0$. By Proposition 2.1, the mapping $Q : X \to Y$ is quadratic. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.12), we get (3.11).

The rest of the proof is similar to the proof of Theorem 3.1. □

4. Fixed points and generalized Hyers-Ulam stability of quadratic functional equations

Throughout this section, assume that X is a normed vector space with norm $||\cdot||$ and that Y is a Banach space with norm $\|\cdot\|$.

For a given mapping $f: X \to Y$, we define

$$Df(x,y): \ = \ f(x+iy) - f(ix+y) - 2f(x) + 2f(y)$$

for all $x, y \in X$.

Using the fixed point method, we prove the generalized Hyers-Ulam stability of the quadratic functional equation $Df(x,y) = 0$.

THEOREM 4.1. *Let $f: X \to Y$ be a mapping with $f(ix) = -f(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ such that*

$$\sum_{j=0}^{\infty} 4^{-j}\varphi(2^j x, 2^j y) \ < \ \infty, \tag{4.1}$$

$$\|Df(x,y)\| \ \leqslant \ \varphi(x,y) \tag{4.2}$$

for all $x, y \in X$. If there exists an $L < 1$ such that $\varphi(x,-ix) \leqslant 4L\varphi(\frac{x}{2}, -\frac{ix}{2})$ for all $x \in X$, then there exists a unique quadratic mapping $Q: X \to Y$ satisfying (2.3) and

$$\|f(x) - Q(x)\| \leqslant \frac{1}{4-4L}\varphi(x,-ix) \tag{4.3}$$

for all $x \in X$.

Proof. Since $f(ix) = -f(x)$ for all $x \in X$, $f(0) = 0$.

$$f(-x) = f(i^2 x) = -f(ix) = f(x)$$

for all $x \in X$. So $Df(x,y) = Cf(x,y)$ for all $x, y \in X$.

Consider the set

$$S := \{g : X \to Y\}$$

and introduce the *generalized metric* on S:

$$d(g,h) = \inf\{K \in \mathbb{R}_+ : \|g(x) - h(x)\| \leqslant K\varphi(x,-ix), \quad \forall x \in X\}.$$

It is easy to show that (S,d) is complete.

Now we consider the linear mapping $J: S \to S$ such that

$$Jg(x) := \frac{1}{4}g(2x)$$

for all $x \in X$.

By Theorem 3.1 of [3],

$$d(Jg, Jh) \leqslant Ld(g,h)$$

for all $g, h \in S$.

Letting $y = -ix$ in (4.2), we get

$$\|f(2x) - 4f(x)\| \leqslant \varphi(x, -ix) \tag{4.4}$$

for all $x \in X$. So

$$\|f(x) - \frac{1}{4}f(2x)\| \leqslant \frac{1}{4}\varphi(x, -ix)$$

for all $x \in X$. Hence $d(f, Jf) \leqslant \frac{1}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, i.e.,

$$Q(2x) = 4Q(x) \tag{4.5}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (4.5) such that there exists $K \in (0, \infty)$ satisfying

$$\|f(x) - Q(x)\| \leqslant K\varphi(x, -ix)$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} \frac{f(2^n x)}{4^n} = Q(x) \tag{4.6}$$

for all $x \in X$.

(3) $d(f, Q) \leqslant \frac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leqslant \frac{1}{4 - 4L}.$$

This implies that the inequality (4.3) holds.

It follows from (4.1), (4.2) and (4.6) that

$$\|DQ(x, y)\| = \lim_{n\to\infty} \frac{1}{4^n}\|Df(2^n x, 2^n y)\| \leqslant \lim_{n\to\infty} \frac{1}{4^n}\varphi(2^n x, 2^n y) = 0$$

for all $x, y \in X$. So $DQ(x, y) = 0$ for all $x, y \in X$.

Similarly, one can show that $Q(ix) = -Q(x)$ for all $x \in X$. By Proposition 2.2, the mapping $Q : X \to Y$ is quadratic, as desired. □

COROLLARY 4.2. *Let $p < 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(ix) = -f(x)$ and*

$$\|Df(x, y)\| \leqslant \theta(||x||^p + ||y||^p) \tag{4.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.3) and

$$\|f(x) - Q(x)\| \leqslant \frac{2\theta}{4 - 2^p}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking

$$\varphi(x,y) := \theta(||x||^p + ||y||^p)$$

for all $x,y \in X$. Then $L = 2^{p-2}$ and we get the desired result. □

COROLLARY 4.3. *Let $p < 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(ix) = -f(x)$ and*

$$\|Df(x,y)\| \leqslant \theta \cdot ||x||^p \cdot ||y||^p \tag{4.8}$$

for all $x,y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.3) *and*

$$\|f(x) - Q(x)\| \leqslant \frac{\theta}{4-4^p}||x||^{2p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.1 by taking

$$\varphi(x,y) := \theta \cdot ||x||^p \cdot ||y||^p$$

for all $x,y \in X$. Then $L = 4^{p-1}$ and we get the desired result. □

THEOREM 4.4. *Let $f : X \to Y$ be a mapping with $f(ix) = -f(x)$ for all $x \in X$ for which there exists a function $\varphi : X^2 \to [0,\infty)$ satisfying* (4.2) *such that*

$$\sum_{j=0}^{\infty} 4^j \varphi(\frac{x}{2^j}, \frac{y}{2^j}) < \infty \tag{4.9}$$

for all $x,y \in X$. If there exists an $L < 1$ such that $\varphi(x,-ix) \leqslant \frac{1}{4}L\varphi(2x,-2ix)$ for all $x \in X$, then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.3) *and*

$$\|f(x) - Q(x)\| \leqslant \frac{L}{4-4L}\varphi(x,-ix) \tag{4.10}$$

for all $x \in X$.

Proof. We consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g(\frac{x}{2})$$

for all $x \in X$.

It follows from (4.4) that

$$\|f(x) - 4f(\frac{x}{2})\| \leqslant \varphi(\frac{x}{2}, -\frac{ix}{2}) \leqslant \frac{L}{4}\varphi(x,-ix)$$

for all $x \in X$. Hence $d(f, Jf) \leqslant \frac{L}{4}$.

By Theorem 1.4, there exists a mapping $Q : X \to Y$ such that

(1) Q is a fixed point of J, i.e.,

$$Q(2x) = 4Q(x) \tag{4.11}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (4.11) such that there exists $K \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \leqslant K\varphi(x, -ix)$$

for all $x \in X$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$\lim_{n\to\infty} 4^n f(\frac{x}{2^n}) = Q(x)$$

for all $x \in X$.

(3) $d(f,Q) \leqslant \frac{1}{1-L} d(f,Jf)$, which implies the inequality

$$d(f,Q) \leqslant \frac{L}{4-4L},$$

which implies that the inequality (4.10) holds.

The rest of the proof is similar to the proof of Theorem 4.1. □

COROLLARY 4.5. *Let $p > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (4.7) and $f(ix) = -f(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.3) and*

$$\|f(x) - Q(x)\| \leqslant \frac{2\theta}{2^p - 4}||x||^p$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x,y) := \theta(||x||^p + ||y||^p)$$

for all $x, y \in X$. Then $L = 2^{2-p}$ and we get the desired result. □

COROLLARY 4.6. *Let $p > 1$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying (4.8) and $f(ix) = -f(x)$ for all $x \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ satisfying (2.3) and*

$$\|f(x) - Q(x)\| \leqslant \frac{\theta}{4^p - 4}||x||^{2p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 4.4 by taking

$$\varphi(x,y) := \theta \cdot ||x||^p \cdot ||y||^p$$

for all $x,y \in X$. Then $L = 4^{1-p}$ and we get the desired result. $\square$

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Choonkil Park
Department of Mathematics
Hanyang University
Seoul 133–791
Republic of Korea
e-mail: baak@hanyang.ac.kr

Themistocles M. Rassias
Department of Mathematics
National Technical University of Athens
Zografou Campus
15780 Athens
Greece
e-mail: trassias@math.ntua.gr