ON THE QUASI–MONOTONE AND ALMOST INCREASING SEQUENCES

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Abstract. In this paper, a theorem of Bor and Özarslan [3] dealing with \( |C, \alpha; \beta|_k \) summability factors has been generalized for \( |C, \alpha, \gamma; \beta|_k \) summability methods.

1. Introduction

We will use the following notations and notions in our paper:

If \( g > 0 \), then \( f = O(g) \) means that \( |f| < K g \), for some constant \( K > 0 \) (see [7]). Let \( (u_n) \) be a sequence. We write that \( \Delta u_n = u_n - u_{n+1} \), \( \Delta^0 u_n = u_n \) and \( \Delta^k u_n = \Delta \Delta^{k-1} u_n \), for \( k = 1, 2, \ldots \) (see [7]).

Abel’s transformation ([8]): Let \( (a_k), (b_k) \) be complex sequences, and write

\[
S_n = a_1 + a_2 + \ldots + a_n .
\]

Then

\[
\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n-1} S_k \Delta b_k + S_n b_n .
\] (1)

Hölder’s inequality ([8]): If \( p > 1 \), \( \frac{1}{p} + \frac{1}{q} = 1 \) and \( a_1, a_2, a_3, \ldots, a_n \geq 0 \); \( b_1, b_2, b_3, \ldots, b_n \geq 0 \) , then

\[
\sum_{k=1}^{n} a_k b_k \leq \left( \sum_{k=1}^{n} a_k^p \right)^{1/p} \left( \sum_{k=1}^{n} b_k^q \right)^{1/q} .
\] (2)

A sequence \( (b_n) \) of positive numbers is said to be \( \delta \)-quasi-monotone, if \( b_n \to 0 \), \( b_n > 0 \) ultimately and \( \Delta b_n \geq -\delta_n \), where \( (\delta_n) \) is a sequence of positive numbers (see [2]). A positive sequence \( (b_n) \) is said to be almost increasing if there exists a positive increasing sequence \( c_n \) and two positive constants \( A \) and \( B \) such that \( A c_n \leq b_n \leq B c_n \) (see [1]). Let \( \sum a_n \) be a given infinite series with partial sums \( (s_n) \). We denote by \( \sigma_n^\alpha \) and \( t_n^\alpha \) the \( n \)-th Cesàro means of order \( \alpha \), with \( \alpha > -1 \), of the sequence \( (s_n) \) and \( (na_n) \), respectively, i.e.,

\[
\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_n^{\alpha-1} s_v
\] (3)

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\[ t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \quad (4) \]

where
\[ A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_n^\alpha = 0 \quad \text{for} \quad n > 0. \quad (5) \]

The series \( \sum a_n \) is said to be summable \( | C, \alpha |_k \), \( k \geq 1 \) and \( \alpha > -1 \), if (see [5])
\[ \sum_{n=1}^{\infty} n^{k-1} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right|^k = \sum_{n=1}^{\infty} \frac{1}{n} \left| t_n^\alpha \right|^k < \infty \quad (6) \]
and it is said to be summable \( | C, \alpha; \beta |_k \), \( k \geq 1 \), \( \alpha > -1 \) and \( \beta \geq 0 \), if (see [6])
\[ \sum_{n=1}^{\infty} n^{\beta k+k-1} \left| \sigma_n^\alpha - \sigma_{n-1}^\alpha \right|^k = \sum_{n=1}^{\infty} n^{\beta k-1} \left| t_n^\alpha \right|^k < \infty. \quad (7) \]

The series \( \sum a_n \) is said to be summable \( | C, \alpha, \gamma; \beta |_k \), \( k \geq 1 \) and \( \alpha > -1 \), \( \delta \geq 0 \) and \( \gamma \) is a real number, if (see [9])
\[ \sum_{n=1}^{\infty} n^{(\beta k+k-1)-k} \left| t_n^\alpha \right|^k < \infty. \quad (8) \]

If we take \( \gamma = 1 \), then \( | C, \alpha, \gamma; \beta |_k \) summability reduces to \( | C, \alpha; \beta |_k \) summability.

Bor and Özarslan [3] have proved the following theorem for \( | C, \alpha; \beta |_k \) summability factors.

**THEOREM A.** Let \( (X_n) \) be an almost increasing sequence such that \( | \Delta X_n | = O(\frac{X_n}{n}) \) and \( \lambda_n \to 0 \) as \( n \to \infty \). Suppose that there exists a sequence of numbers \( (B_n) \) such that it is \( \delta \)-quasi-monotone with \( \sum nX_n \delta_n < \infty \), \( \sum B_nX_n \) is convergent and \( \left| \Delta \lambda_n \right| \leq B_n \) for all \( n \). If the sequence \( (u_n^\alpha) \), defined by (see [10])
\[ u_n^\alpha = \begin{cases} \left| t_n^\alpha \right|, & \alpha = 1 \\ \max_{1 \leq v \leq n} \left| t_v^\alpha \right|, & 0 < \alpha < 1 \end{cases} \quad (9) \]
satisfies the condition
\[ \sum_{n=1}^{m} n^{\beta k-1} (u_n^\alpha)^k = O(X_m) \quad \text{as} \quad m \to \infty, \quad (10) \]
then the series \( \sum a_n \lambda_n \) is summable \( | C, \alpha; \beta |_k \), \( k \geq 1 \) and \( 0 \leq \beta < \alpha \leq 1 \).
2. The main result

The aim of this paper is to generalize Theorem A for \(| C, \alpha, \gamma; \beta |_k \) summability factors. We shall prove the following theorem.

THEOREM. Let \((X_n)\) be an almost increasing sequence such that \(| \Delta X_n | = O(\frac{X_n}{n})\) and \(\lambda_n \to 0\) as \(n \to \infty\). Suppose that there exists a sequence of numbers \((B_n)\) such that it is \(\delta\)-quasi-monotone with \(\sum n X_n \delta_n < \infty\), \(\sum B_n X_n\) is convergent and \(| \Delta \lambda_n | \leq | B_n |\) for all \(n\). If the sequence \((u_n^\alpha)\), defined by (9) satisfies the condition

\[
\sum_{n=1}^{m} n^{\gamma (\beta k + k - 1) - k} (u_n^\alpha)^k = O(X_m) \text{ as } m \to \infty, \tag{11}
\]

then the series \(\sum a_n \lambda_n\) is summable \(| C, \alpha, \gamma; \beta |_k\), where \(k \geq 1\), \(\beta \geq 0\), \(0 < \alpha \leq 1\) and \(\gamma\) is a real number such that \(k + \alpha k - \gamma (\beta k + k - 1) > 1\).

We need the following lemmas for the proof of our theorem.

**Lemma 1.** ([4]) If \(0 < \alpha \leq 1\) and \(1 \leq v \leq n\), then

\[
| \sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p | \leq \max_{1 \leq m \leq v} | \sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p |. \tag{12}
\]

**Lemma 2.** ([3]) Under the conditions regarding \((\lambda_n)\) and \((X_n)\) of the Theorem, we have

\[
| \lambda_n | X_n = O(1) \text{ as } n \to \infty. \tag{13}
\]

**Lemma 3.** ([3]) Under the conditions pertaining to \((X_n)\) and \((B_n)\) of the Theorem, we have that

\[
\sum_{n=1}^{\infty} n X_n | \Delta B_n | < \infty. \tag{15}
\]

3. Proof of the Theorem

Let \((T_n^\alpha)\) be the \(n\)-th \((C, \alpha)\) mean of the sequence \((n a_n \lambda_n)\). Then, by (4) we have

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v \lambda_v. \tag{16}
\]

Using Abel’s transformation, we get that

\[
T_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} \Delta \lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p a_p + \frac{\lambda_n}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v,
\]
so that making use of Lemma 1, we have

\[
| T_n^\alpha | \leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} | \Delta \lambda_v | \sum_{p=1}^{v} A_{n-p}^{\alpha-1} p |a_p| + \frac{| \lambda_n |}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v |
\]

\[
\leq \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha \Delta \lambda_v | + | \lambda_n | w_n^\alpha
\]

\[= T_{n,1}^\alpha + T_{n,2}^\alpha, \text{ say.} \]

Since

\[
| T_{n,1}^\alpha + T_{n,2}^\alpha |^k \leq 2^k (| T_{n,1}^\alpha |^k + | T_{n,2}^\alpha |^k),
\]

to complete the proof of the Theorem, it is sufficient to show that

\[
\sum_{n=1}^{\infty} n^{\gamma(\beta k + k - 1) - k} | T_{n,r}^\alpha |^k < \infty \quad \text{for} \quad r = 1, 2, \quad \text{by} \quad (8).
\]

Now, when \( k > 1 \), applying Hölder’s inequality with indices \( k \) and \( k' \), where \( \frac{1}{k} + \frac{1}{k'} = 1 \), we get that

\[
\sum_{n=1}^{m+1} n^{\gamma(\beta k + k - 1) - k} | T_{n,1}^\alpha |^k \leq \sum_{n=2}^{m+1} n^{\gamma(\beta k + k - 1) - k} \left\{ \frac{1}{A_n^\alpha} \sum_{v=1}^{n-1} A_v^\alpha u_v^\alpha \Delta \lambda_v \right\}^k
\]

\[
= O(1) \sum_{n=2}^{m+1} n^{\gamma(\beta k + k - 1) - k - \alpha k} \left\{ \sum_{v=1}^{n-1} v^{\alpha k} (u_v^\alpha)^k | B_v | \right\}^k
\]

\[
\times \left\{ \sum_{v=1}^{n-1} | B_v | \right\}
\]

\[
= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k | B_v | \sum_{n=v+1}^{m+1} \frac{1}{n^{k + \alpha k - \gamma(\beta k + k - 1)}}
\]

\[
= O(1) \sum_{v=1}^{m} v^{\alpha k} (u_v^\alpha)^k | B_v | \int_{v}^{\infty} \frac{dx}{x^{\alpha k + k - \gamma(\beta k + k - 1)}}
\]

\[
= O(1) \sum_{v=1}^{m} v | B_v | v^{\gamma(\beta k + k - 1) - k} (u_v^\alpha)^k
\]

\[
= O(1) \sum_{v=1}^{m} \Delta(v | B_v |) \sum_{r=1}^{v} r^{\gamma(\beta k + k - 1) - k} (u_r^\alpha)^k
\]

\[
+ O(1) m | B_m | \sum_{v=1}^{m-1} v^{\gamma(\beta k + k - 1) - k} (u_v^\alpha)^k
\]

\[= O(1) \sum_{v=1}^{m-1} \Delta(v | B_v |) X_v + O(1) m | B_m | X_m
\]
\[= \mathcal{O}(1) \sum_{v=1}^{m-1} v \mid \Delta B_v \mid X_v + \mathcal{O}(1) \sum_{v=1}^{m-1} \mid B_{v+1} \mid X_{v+1} + \mathcal{O}(1) \mid B_m \mid X_m = \mathcal{O}(1) \text{ as } m \to \infty,\]

by virtue of the hypotheses of the Theorem and Lemma 3.

Again, since \(|\lambda_n| = \mathcal{O}(1/X_n) = \mathcal{O}(1)\) by (13), we have that
\[
\sum_{n=1}^{m} n^{\gamma(\beta k + k - 1) - k} \mid T_{n,2}^\alpha \mid^k = \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\gamma(\beta k + k - 1) - k} (u_n^\alpha)^k
\]
\[
= \mathcal{O}(1) \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{\gamma(\beta k + k - 1) - k} (u_n^\alpha)^k
\]
\[
= \mathcal{O}(1) \sum_{n=1}^{m-1} \Delta \mid \lambda_n \mid \sum_{v=1}^{n} v^{\gamma(\beta k + k - 1) - k} (u_v^\alpha)^k
\]
\[
+ \mathcal{O}(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{\gamma(\beta k + k - 1) - k} (u_n^\alpha)^k
\]
\[
= \mathcal{O}(1) \sum_{n=1}^{m-1} \mid \Delta \lambda_n \mid X_n + \mathcal{O}(1) \mid \lambda_m \mid X_m
\]
\[
= \mathcal{O}(1) \sum_{n=1}^{m-1} \mid B_n \mid X_n + \mathcal{O}(1) \mid \lambda_m \mid X_m = \mathcal{O}(1) \text{ as } m \to \infty,
\]

by virtue of the hypotheses of the Theorem, Lemma 2 and Lemma 3.

Therefore, we get that
\[
\sum_{n=1}^{m} n^{\gamma(\beta k + k - 1) - k} \mid T_{n,r}^\alpha \mid^k = \mathcal{O}(1) \text{ as } m \to \infty, \text{ for } r = 1, 2.
\]

This completes the proof of the Theorem.

If we take \(\gamma = 1\), then we get Theorem A. In this case condition (11) reduces to condition (10). Also if we take \(\gamma = 1\) and \(\beta = 0\), then we have a new result concerning \(|C, \alpha|_k\) summability factors. Finally, if we take \(\gamma = 1, \beta = 0\) and \(\alpha = 1\), then we obtain a new result related to \(|C, 1|_k\) summability factors.

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