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ON VAN DE LUNE – ALZER'S INEQUALITY

S. ABRAMOVICH, J. BARIĆ, M. MATIĆ AND J. PEČARIĆ

(*communicated by N. Elezović*)

Abstract. In this paper it is shown that the inequality known in the literature as Alzer's inequality (1993), has already been known since 1975. and is due to Jan van de Lune. A review of different methods in proving Van de Lune - Alzer's inequality and generalizations in a several directions, is given. It is shown how some results and proofs can be corrected, refined and extended. New results, inspired by the generalization of Van de Lune - Alzer's inequality for increasing convex sequences presented by N. Elezović and J. Pečarić, are obtained.

1. Introduction

In 1964. H. Minc and L. Sathre in [24] proved that, for $n \in \mathbb{N}$ the inequality

$$\frac{n}{n+1} < \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}. \tag{1.1}$$

holds.

In 1988. J. S. Martins, in [23], gave another lower bound for the ratio $(n!)^{\frac{1}{n}} / ((n+1)!)^{\frac{1}{n+1}}$ from (1.1):

Let r be a positive real number and let n be a natural number. Then

$$\left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}} \leqslant \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}. \tag{1.2}$$

So, H. Alzer came to the idea to compare the left-hand sides of (1.1) and (1.2) and, in 1993 in [2], he proved the next theorem.

THEOREM 1. *If r is a positive real number and if n is a positive integer, then*

$$\frac{n}{n+1} \leqslant \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}} \leqslant \frac{(n!)^{\frac{1}{n}}}{((n+1)!)^{\frac{1}{n+1}}}. \tag{1.3}$$

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Since then, the left-hand side of the inequality (1.3) is called Alzer's inequality.

Alzer's proof uses interesting techniques, but its complexity have invoked the interest of several mathematicians. The first easy proof of Alzer's inequality is due to J. Sándor who used in his proof Cauchy mean value theorem and mathematical induction, see [33]. Also, J. Sándor used in [36] the method of Lagrange mean value theorem and mathematical induction. The second elementary proof is given by J. S. Ume in [38] using differentiation and induction. And, finally, C.-P. Chen and F. Qi, in [6], presented two other simple proofs of Alzer's inequality using Lagrange's mean value theorem, monotonicity and convexity of functions, and mathematical induction.

However, in 1975, in [1], J. van de Lune stated *Problem 399.*, which was solved by several mathematicians, and which easily implicates Alzer's inequality.

The purpose of this paper is to show that what is called since 1993 "Alzer's inequality" is the result of Jan van de Lune's work, known already since 1975. According to that, in this paper the left-hand side of the inequality (1.3) will be called Van de Lune - Alzer's inequality.

We also give some generalizations of Alzer's inequality as well as corrections and refinements of alterative proofs of one of the inequalities in an article by H. Alzer ([2]).

2. Jan van de Lune's results

The inequality, which we recognize as Alzer's inequality, has already been known at least since 1975 and is due to J. van de Lune. In this section we present J. van de Lune's *Problem 399*, (see [1, p. 254]), it's connection with Alzer's inequality and our conclusions in the COMMENTS.

PROBLEM 399. *For* $n \in \mathbb{N}$ and $s \in \mathbb{R}$ *let*

$$\sigma_n(s) := \sum_{k=1}^{n} k^s, \qquad U_n(s) := n^{-s-1}\sigma_n(s), \qquad L_n(s) := n^{-s-1}\sigma_{n-1}(s),$$

where $\sigma_0(s) = 0$. *Prove that if* s *is positive*, $U_n(s)$ *is decreasing in* n *and* $L_n(s)$ *is increasing in* n.

The mathematicians F. J. Barning, R. Doornbos, A. A. Jagers, J. H. van Lint, J. van de Lune and G. R. Veldkamp gave solutions of Jan van de Lune's *Problem 399*. F. J. M. Barning, J. van de Lune, G. R. Veldkamp and R. Doornbos (this proof can be seen in [1]) used mathematical induction and J. H. van Lint showed that the problem is a special case of a more general situation in the following way:

Let f be increasing and convex on $[0,1]$. Let us consider

$$S_n = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \tag{2.1}$$

and

$$s_n = \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right). \tag{2.2}$$

Using convexity of f on $[0,1]$ J. van de Lint proved that $(S_n)_{n\in\mathbb{N}}$ is a decreasing and $(s_n)_{n\in\mathbb{N}}$ is an increasing sequence, i.e.

$$S_{n+1} \leqslant S_n, \tag{2.3}$$

and

$$s_n \leqslant s_{n+1}. \tag{2.4}$$

By applying the result to the function g defined by $g(x) = -f(1-x)$ it is easy to see that (2.3) and (2.4) also hold if f is increasing and concave. By applying the result to $f(x) = x^s$, $(s > 0)$, the assertion of the *Problem 399* follows.

COMMENTS. If the considered function f is strictly increasing and convex or strictly increasing and concave on $[0,1]$ then S_n is strictly decreasing and s_n is strictly increasing sequence. Now applying J. H. van Lint's results on function $f(x) = x^s$, $s > 0$, we obtain that U_n is strictly increasing and L_n is strictly decreasing function in $n \in \mathbb{N}$.

The fact that $U_n(s)$ is strictly decreasing in n is equivalent to

$$\begin{gathered}
U_n(s) > U_{n+1}(s),\\
n^{-s-1}(1^s + 2^s + \cdots + n^s) > (n+1)^{-s-1}(1^s + 2^s + \cdots + (n+1)^s),\\
\frac{1^s + 2^s + \cdots + n^s}{n^s \cdot n} > \frac{1^s + 2^s + \cdots + (n+1)^s}{(n+1)^s \cdot (n+1)},\\
\frac{n^s}{(n+1)^s} < \frac{(n+1)\sum_{i=1}^{n} i^s}{n\sum_{i=1}^{n+1} i^s},
\end{gathered}$$

i.e.

$$\frac{n}{n+1} < \left[\frac{(n+1)\sum_{i=1}^{n} i^s}{n\sum_{i=1}^{n+1} i^s}\right]^{\frac{1}{s}}, \tag{2.5}$$

where s is positive real number.

In 1993. Horst Alzer proved

$$\frac{n}{n+1} \leqslant \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}}, \tag{2.6}$$

where $n \in \mathbb{N}$ and r is positive real number.

It is obvious that Jan van de Lune's inequality (2.5) directly implies inequality (2.6) i.e. (2.5) holds with strict inequality " $<$ " in place of " $\leqslant$ ". Proving inequality (2.6), J. Sándor, ([33]), and J. S. Ume, ([38]), also came to conclusion that (2.6) is true for strict inequality.

We formulate J. H. van Lint's results in the following theorem:

THEOREM 2. *Let f be an increasing and convex or an increasing and concave function on $[0,1]$. Let S_n and s_n be defined by (2.1) and (2.2), respectively. Then S_n is a decreasing sequence and s_n is an increasing sequence i.e.*

$$\frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{k}{n+1}\right) \leqslant \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right) \tag{2.7}$$

and

$$\frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k}{n}\right) \leqslant \frac{1}{n+1}\sum_{k=0}^{n} f\left(\frac{k}{n+1}\right). \tag{2.8}$$

If f is strictly increasing and convex function or strictly increasing and concave function on $[0,1]$ then S_n is strictly decreasing and s_n is strictly increasing sequence and (2.7) and (2.8) holds with strict inequality.

Applying Theorem 2 to $f(x)=x^s$, $s>0$, we get following corollaries:

COROLLARY 1. *Let $\sigma_n(s)=\sum_{k=1}^{n} k^s$, for $n\in\mathbb{N}$ and $\sigma_0(s)=0$. Then*

$$U_n(s)=n^{-s-1}\sigma_n(s),$$

is strictly decreasing function in n and

$$L_n(s)=n^{-s-1}\sigma_{n-1}(s),$$

is strictly increasing function in n.

COROLLARY 2. *Let $f(x)=x^s$, where $s\in\mathbb{R}$, $s>0$. Then for $n\in\mathbb{N}$ it holds*

$$\frac{n}{n+1} < \left[\frac{(n+1)\sum_{k=1}^{n} k^s}{n\sum_{k=1}^{n+1} k^s}\right]^{\frac{1}{s}}.$$

3. Alternative proofs

In 1995. J. Sándor, in a short paper [33], gave an alternative proof of H. Alzer's inequality

$$\frac{n}{n+1} \leqslant \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}}, \tag{3.1}$$

based on mathematical induction and Cauchy's mean value theorem of differential calculus. In fact, he proved even sharper statement, discovered by Jan van de Lune in 1975, that (3.1) holds with strict inequality.

In 1996 J.S. Ume gave another elementary proof of inequality (3.1), (see [38]), using induction and differentiation. However, his proof can be modified, in the following way, to get strict inequality in (3.1). We give J.S. Ume's Lemma and it's proof with our corrections.

LEMMA 1. *If r is a positive real number, then*

$$1 < (1+x)^r \left[x + (1-x)^{r+1}\right], \qquad 0 < x \leqslant 1. \tag{3.2}$$

Proof. For $x \in [0,1]$, define

$$f(x) = (1+x)^r \left[x + (1-x)^{r+1}\right] - 1.$$

The function f is continuous on $[0,1]$ and $f(0) = 0$. To prove inequality (3.2) it suffices to show $f'(x) > 0$, for $0 < x < 1$. Differentiation of f yields

$$f'(x) = (1+x)^{r-1} \left\{r\left[x + (1-x)^{r+1}\right] + (1+x)\left[1 - (r+1)(1-x)^r\right]\right\}.$$

For $x \in [0,1]$, let's set

$$g(x) = r\left[x + (1-x)^{r+1}\right] + (1+x)\left[1 - (r+1)(1-x)^r\right].$$

The function g is continuous on $[0,1]$ and $g(0) = 0$. Now we have

$$g'(x) = (r+1)\{1 - (1-x)^r + r\left[(1+x)(1-x)^{r-1} - (1-x)^r\right]\}.$$

Since $(1-x)^r < 1$ and $(1-x)^r < (1+x)(1-x)^{r-1}$ for $0 < x < 1$, it follows $g'(x) > 0$, for $r > 0$ and $0 < x < 1$. Therefore $g(x) > g(0)$ which implies $f'(x) > 0$, for $0 < x < 1$. □

Now, (3.1) can be easily proved for strict inequality using mathematical induction and Lemma 1 (see [38]).

In 2003. C.-P. Chen and F. Qi showed, in [6], that J. Sándor's and J. S. Ume's proofs of (3.1) can be completed in other ways using Lagrange's mean value theorem, monotonicity and convexity of function's, and mathematical induction.

4. Generalizations of Van de Lune - Alzer's inequality

In 1999 F. Qi proved the next theorem.

THEOREM 3. *Let n and m be natural numbers, k a nonnegative integer. Then*

$$\frac{n+k}{n+m+k} < \left[\frac{\frac{1}{n}\sum\limits_{i=k+1}^{n+k} i^r}{\frac{1}{n+m}\sum\limits_{i=k+1}^{n+m+k} i^r}\right]^{\frac{1}{r}}, \tag{4.1}$$

where r is any given positive real number. The lower bound is best possible.

Theorem 3 is proved in [26] applying mathematical induction and Cauchy mean value theorem. Some further results related to this can be found in [32] and [19].

In 2002. J. S. Ume, in [39], showed how the result of H. Alzer can be extended using suitable mapping. His main result (contained in the following theorem) is proved using two lemmas in which we made some improvements.

Let us quote the first lemma.

LEMMA 2. *Let a, b, c and d be real numbers satisfying*

$$1 < a, c, \qquad 0 < b, d < 1, \qquad 0 < ab \leqslant 1,$$

$$1 < \frac{1}{2}(c+d) \quad \textit{and} \quad 1 \leqslant a^{c-1}(ab)^d.$$

If $f(x) := (c-1)a^x + d(ab)^x$, then

$$1 < f(x), \quad \textit{for all} \quad x \in [0, \infty).$$

COMMENT. By careful inspection of Ume's proof for the above Lemma we see that the sign " $<$ " in $1 < \frac{1}{2}(c+d)$ can be replaced with " $\leqslant$ " and still it can be proved that $f(x) > f(0) = (c-1) + d > 1$ for all $x \in (0, +\infty)$, which is crucial for the the rest of Ume's results in [39] .

From Lemma 2 follows the next lemma.

LEMMA 3. *Let $\varphi : (0, \infty) \to (0, \infty)$ be a function such that*

$$\varphi \quad \textit{is strictly increasing on} \quad (0, \infty), \tag{4.2}$$

$$\varphi' \quad \textit{exists on} \quad (0, \infty), \tag{4.3}$$

$$\varphi' \quad \textit{is strictly increasing on} \quad (0, \infty), \tag{4.4}$$

$$\frac{\varphi(x)}{\varphi(x+1)} \leqslant \frac{\varphi(x+1)}{\varphi(x+2)}, \quad \textit{for all} \quad x \in (0, \infty), \tag{4.5}$$

and

$$1 \leqslant \left[\frac{\varphi(u+2)}{\varphi(u+1)}\right]^{\frac{\varphi(v+2)}{\varphi(v+1)}-1} \cdot \left[\frac{\varphi(u)}{\varphi(u+1)} \cdot \frac{\varphi(u+2)}{\varphi(u+1)}\right]^{\frac{\varphi(v)}{\varphi(v+1)}}, \tag{4.6}$$

for all $u, v \in (0, \infty)$. Then

$$\varphi(v+1) < [\varphi(v+2)-\varphi(v+1)] \left\{\frac{\varphi(u+2)}{\varphi(u+1)}\right\}^r + \varphi(v) \left\{\frac{\varphi(u)}{\varphi(u+1)} \cdot \frac{\varphi(u+2)}{\varphi(u+1)}\right\}^r, \tag{4.7}$$

for all $u, v \in (0, \infty)$ and $r > 0$.

COMMENT. Considering the changes we made in Lemma 2 assertion (4.4) is changed to

$$\varphi' \quad \text{is increasing on} \quad (0, \infty),$$

i.e., the function φ is convex on $(0, \infty)$.

THEOREM 4. *Let* $\varphi:(0,\infty)\to(0,\infty)$ *be a function satisfying conditions (4.2), (4.3), (4.4), (4.5) and*

$$1\leqslant\left\{\frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)}\right\}^{\frac{\varphi(n+m+2)}{\varphi(n+m+1)}-1}\cdot\left\{\frac{\varphi(n+m+k)}{\varphi(n+m+k+1)}\cdot\frac{\varphi(n+m+k+2)}{\varphi(n+m+k+1)}\right\}^{\frac{\varphi(n+m)}{\varphi(n+m+1)}},\tag{4.8}$$

for all $n\in\mathbb{N},\ m,k\in\mathbb{N}\cup\{0\}$ *and* $1\leqslant\varphi(1)\leqslant\frac{1}{2}\varphi(2)$. *Then*

$$\frac{\varphi(n+k)}{\varphi(n+m+k)}<\left\{\frac{\frac{1}{\varphi(n)}\sum\limits_{i=k+1}^{n+k}[\varphi(i)]^r}{\frac{1}{\varphi(n+m)}\sum\limits_{i=k+1}^{n+m+k}[\varphi(i)]^r}\right\}^{\frac{1}{r}},\tag{4.9}$$

for all $n,m\in\mathbb{N},\ k\in\mathbb{N}\cup 0$ *and* $r>0$.

Applying the above theorem to the function $\varphi(x)=a^x$, for all $x\in(0,\infty)$, J.S. Ume proved the following corollary.

COROLLARY 3. *Let* $n,m\in\mathbb{N},\ k\in\mathbb{N}\cup\{0\},\ r>0$ *and* $a\geqslant 2$. *Then*

$$\frac{1}{a^m}<\left\{\frac{\frac{1}{a^n}\sum\limits_{i=k+1}^{n+k}a^{ir}}{\frac{1}{a^{n+m}}\sum\limits_{i=k+1}^{n+m+k}a^{ir}}\right\}^{\frac{1}{r}}.\tag{4.10}$$

In the next corollary J.S. Ume gave a generalization of H. Alzer's inequality.

COROLLARY 4. *If* $p=1$ *or* $p\geqslant 2$ *then*

$$\left(\frac{n+k}{n+m+k}\right)^p<\left\{\frac{\frac{1}{n^p}\sum\limits_{i=k+1}^{n+k}i^{pr}}{\frac{1}{(n+m)^p}\sum\limits_{i=k+1}^{n+m+k}i^{pr}}\right\}^{\frac{1}{r}},\tag{4.11}$$

where $n,m\in\mathbb{N},\ k\in\mathbb{N}\cup\{0\}$ *and* $r>0$.

In 2000. F. Qi, in [27], presented the inequality which generalizes Alzer's result, as well as one result proved by J.-C. Kuang and inequality (4.1). Namely, in 1999, J.-C. Kuang, in [20], proved the following inequality

$$\frac{1}{n}\sum_{k=1}^{n}f\left(\frac{k}{n}\right)>\frac{1}{n+1}\sum_{k=1}^{n+1}f\left(\frac{k}{n+1}\right)>\int_0^1 f(x)dx,\tag{4.12}$$

for a strictly increasing convex (or concave) function f on $(0,1]$.

Motivated by inequalities in (4.1) and (4.12), considering convexity, F. Qi proved the next theorem.

THEOREM 5. *Let f be a strictly increasing convex (or concave) function in $(0,1]$. Then the sequence $\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right)$ is decreasing in n and k and has a lower bound $\int_0^1 f(t)dt$, that is*

$$\frac{1}{n}\sum_{i=k+1}^{n+k} f\left(\frac{i}{n+k}\right) > \frac{1}{n+1}\sum_{i=k+1}^{n+k+1} f\left(\frac{i}{n+k+1}\right) > \int_0^1 f(t)dt, \tag{4.13}$$

where k is a nonnegative integer and n is a natural number.

Applying Theorem 5 to $f(x) = x^r$, for $r > 0$ and $k = 0$ it follows

$$\frac{1}{n+1}\sum_{i=1}^{n+1} \frac{i^r}{(n+1)^r} < \frac{1}{n}\sum_{i=1}^{n} \frac{i^r}{n^r},$$

$$\frac{n^r}{(n+1)^r} < \frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r},$$

i.e.

$$\frac{n}{n+1} < \left[\frac{(n+1)\sum_{i=1}^{n} i^r}{n\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}},$$

and that is Alzer's inequality with " $<$ " instead of " $\leqslant$ ".

Furthermore, applying Theorem 5 on $f(x) = x^r$, for $r > 0$ it follows

$$\frac{\sum_{i=k+1}^{n+k} i^r}{n(n+k)^r} > \frac{\sum_{i=k+1}^{n+k+1} i^r}{(n+1)(n+k+1)^r}, \tag{4.14}$$

i.e.

$$\frac{\sum_{i=k+1}^{n+k} i^r}{n(n+k)^r} > \frac{\sum_{i=k+1}^{n+k+m} i^r}{(n+m)(n+k+m)^r},$$

which is equivalent to inequality (4.1). For $k = 0$ inequality (4.13) becomes equivalent to (4.12).

COMMENT. Notice that the left-hand inequality in (4.12) is equivalent to J. H. van Lint's result i.e. to inequality (2.7) in Theorem 2 for strictly increasing convex (or concave) function f.

In 2001 F. Qi, in [28], proved an algebraic inequality which is an integral analogue of the following inequality

$$\frac{n+k}{n+m+k} < \left[\frac{\frac{1}{n} \sum\limits_{i=k+1}^{n+k} i^r}{\frac{1}{n+m} \sum\limits_{i=k+1}^{n+m+k} i^r} \right]^{\frac{1}{r}}, \tag{4.15}$$

proved in [26]. (An extension of this Qi's result can be found in paper [41]).

THEOREM 6. *Let $b > a > 0$ and $\delta > 0$ be real numbers. Then for any given positive $r \in \mathbb{R}$ we have*

$$\left[\frac{b+\delta-a}{b-a} \cdot \frac{b^{r+1}-a^{r+1}}{(b+\delta)^{r+1}-a^{r+1}} \right]^{\frac{1}{r}} > \frac{b}{b+\delta}. \tag{4.16}$$

The lower bound in (4.16) is best possible.

The inequality (4.16) can be rewritten as

$$\frac{b}{b+\delta} < \left[\frac{\frac{1}{b-a} \int\limits_a^b x^r dx}{\frac{1}{b+\delta-a} \int\limits_a^{b+\delta} x^r dx} \right]^{\frac{1}{r}}. \tag{4.17}$$

For $a = k$, $b = n+k$ and $\delta = m$, inequality(4.17) is integral analogue of the (4.15).

Inequality (4.17) was generalized by B. Gavrea and I. Gavrea in [18], to an inequality for linear positive functionals.

Using a completely different unexpected approach, I. Gavrea improved Van de Lune - Alzar's inequality and some related inequalities proved in [4]. Gavrea used Bernstein and Bernstein - Stancu operators to get his many new results [17]. By using Bernstein polynomials of degree n he improved inequality (2.3) for increasing convex functions. He got the following inequalities:

$$\begin{aligned} \frac{1}{6n(n+1)} \min_{k=0,n-1} \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f \right] &\leqslant \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) - \frac{1}{n+2} \sum_{k=0}^{n+1} f\left(\frac{k}{n+1}\right) \\ &\leqslant \frac{1}{6n(n+1)} \max_{k=0,n-1} \left[\frac{k}{n}, \frac{k+1}{n+1}, \frac{k+1}{n}; f \right]. \end{aligned}$$

Using Bernstein-Stancu type operators Gavrea proved some general inequalities from which he got

$$\begin{aligned} 0 &\leqslant \frac{1}{n} \sum_{k=0}^{n-1} f\left(\frac{k+1}{n+1}\right) - \frac{1}{n-1} \sum_{k=0}^{n-2} f\left(\frac{k+1}{n}\right) \\ &\leqslant \frac{1}{6} \left(\left[\frac{n-1}{n}, \frac{n}{n+1}; f \right] - \left[\frac{1}{n}, \frac{1}{n+1}; f \right] \right) \end{aligned}$$

as well as

$$\frac{\beta+1}{2(n+\beta)(n+\beta+1)}\left[\frac{\beta}{n+\beta},\frac{\beta+1}{n+\beta+1};f\right]$$
$$\leqslant \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i+\beta}{n+\beta}\right)-\frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i+\beta}{n+\beta+1}\right)$$
$$\leqslant \frac{\beta+1}{2(n+\beta)(n+\beta+1)}\left[\frac{n+\beta}{n+\beta+1},1;f\right]$$

for $\beta \geqslant 0$, where

$$[x_1,x_2;f]:=\frac{f(x_2)-f(x_1)}{x_2-x_1}; \quad [x_1,x_2,x_{3;};f]:=\frac{[x_2,x_3;f]-[x_1,x_2;f]}{x_3-x_1}.$$

Also by using inequalities resulting from Bernstein-Stancu operators he got as a special case the known inequality proved by F. Qi and B-N. Guo in [30]:

$$\frac{1}{n}\sum_{k=1}^{n} f\left(\frac{a_k}{a_n}\right) \geqslant \frac{1}{n+1}\sum_{k=1}^{n+1} f\left(\frac{a_k}{a_{n+1}}\right)$$

which holds for an increasing sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in [0,1]$, such that $\left(n\left(1-\frac{a_n}{a_{n+1}}\right)\right)_{n\in\mathbb{N}}$ is also an increasing sequence, and for increasing convex function f on $[0,1]$.

5. Applications to sequences

In 1998 N. Elezović and J. Pečarić, in [16], showed that Alzer's inequality is satisfied for a large class of increasing convex sequences. They gave a generalization of (3.1) contained in the next theorem.

THEOREM 7. *If the sequence $(a_n)_{n\geqslant 1}$, of positive real numbers satisfies*

$$1 \leqslant \left(\frac{a_{n+2}}{a_{n+1}}\right)^r \left[\frac{a_{n+2}}{a_{n+1}}-1+\left(\frac{a_n}{a_{n+1}}\right)^{r+1}\right], \tag{5.1}$$

for $n \geqslant 0$, $a_0=0$, $r \in \mathbb{R}^+$, then

$$\frac{a_n}{a_{n+1}} \leqslant \left(\frac{a_{n+1}\sum\limits_{i=1}^{n} a_i^r}{a_n\sum\limits_{i=1}^{n+1} a_i^r}\right)^{\frac{1}{r}}. \tag{5.2}$$

N. Elezović and J. Pečarić proved Theorem 7 using mathematical induction and Lemma 1. In [16] they also gave a simplification of proof of Lemma 1 as well as the following examples and corollaries of Theorem 7 for sequences of positive real numbers.

COROLLARY 5. *Let the sequence* (a_n) *of positive real numbers satisfy*

$$\frac{a_2}{a_1} \geqslant \left(\frac{a_1}{a_2}\right)^r + 1, \tag{5.3}$$

$$a_n - 2a_{n+1} + a_{n+2} \geqslant 0, \quad n \geqslant 1. \tag{5.4}$$

Then (5.2) holds.

EXAMPLE 1. The sequence $a_n = n$ satisfies (5.3) and (5.4). Hence, Theorem 7 generalizes Alzer's inequality.

COROLLARY 6. *For each strictly increasing sequence* (a_n) *of positive real numbers there exist an* $r > 0$ *such that (5.2) holds.*

EXAMPLE 2. The sequence $a_n = 2n - 1$ satisfies (5.3) and (5.4). Therefore we have

$$\frac{2n-1}{2n+1} \leqslant \left(\frac{(2n+1)\sum\limits_{i=1}^{n}(2i-1)^r}{(2n-1)\sum\limits_{i=1}^{n+1}(2i-1)^r}\right)^{\frac{1}{r}}.$$

EXAMPLE 3. The sequence $a_n = k(n-1)+1$, $k > 0$, satisfies (5.4). Further, (5.3) is equivalent to

$$k(k+1)^r \geqslant 1. \tag{5.5}$$

Therefore, (5.2) holds for this sequence whenever (5.5) is valid.

EXAMPLE 4. The sequence $a_n = a^n$, $a > 1$, satisfies (5.4). Further, (5.3) is equivalent to

$$a \geqslant \frac{1}{a^r} + 1. \tag{5.6}$$

As in previous example, for each $r > 0$ there exist $a > 1$ for which (5.6) is valid.

COMMENTS. Inequality (5.2) is equivalent to

$$\frac{\sum\limits_{i=1}^{n+1} a_i^r}{a_{n+1}^r \cdot a_{n+1}} \leqslant \frac{\sum\limits_{i=1}^{n} a_i^r}{a_n \cdot a_n^r}$$

If we regard the left and right sides of the last inequality as members of a real sequence (A_n) we can conclude that sequence (A_n) is decreasing i.e.

$$(\forall n \in \mathbb{N}) \quad (\forall p \in \mathbb{N}) \quad A_{n+p} \leqslant A_n.$$

Hence, inequality (5.2) is equivalent to

$$\frac{\sum\limits_{i=1}^{n+m} a_i^r}{a_{n+m}^r \cdot a_{n+m}} \leqslant \frac{\sum\limits_{i=1}^{n} a_i^r}{a_n \cdot a_n^r},$$

i.e.

$$\frac{a_n}{a_{n+m}} \leqslant \left[\frac{a_{n+m} \sum_{i=1}^{n} a_i^r}{a_n \sum_{i=1}^{n+m} a_i^r} \right]^{\frac{1}{r}}.$$

NEW RESULTS. Following the reasoning of N. Elezović and J. Pečarić in results presented above ([16]) we extend the results obtained there for $f(x) = x^r, \quad x \geqslant 0$, $r \geqslant 0$ to an increasing $f(x)$ where $xf(x)$ is convex.

Instead of dealing with

$$\sum_{i=1}^{n} \frac{a_i^r}{a_n^{r+1}} \geqslant \sum_{i=1}^{n+1} \frac{a_i^r}{a_{n+1}^{r+1}}, \quad r > 0,\ a_n > 0,\ n = 1, 2, \ldots \tag{5.7}$$

we deal with

$$\sum_{i=1}^{n} \frac{1}{a_n} \left(\frac{f(a_i)}{f(a_n)} \right) \geqslant \sum_{i=1}^{n+1} \frac{1}{a_{n+1}} \left(\frac{f(a_i)}{f(a_{n+1})} \right). \tag{5.8}$$

REMARK 1. Inequality (5.8) is equivalent to

$$\sum_{i=1}^{n+1} f(a_i) \left(1 - \frac{a_n f(a_n)}{a_{n+1} f(a_{n+1})} \right) \geqslant f(a_{n+1}). \tag{5.9}$$

Therefore, if $a_i > 0$, $f(a_i) > 0$ i=1,... the inequality

$$1 - \frac{a_n f(a_n)}{a_{n+1} f(a_{n+1})} \geqslant 0 \tag{5.10}$$

is a necessary condition for (5.8) to hold.

THEOREM 8. *Let* $a_n > 0, \quad n = 1, \ldots$ *and let the function* $f(x) > 0$ *be defined on* $[0, \infty)$, *and satisfies (5.10). If*

$$\frac{f(a_{n+2})}{f(a_{n+1})} \left[\frac{a_{n+2}}{a_{n+1}} - 1 + \frac{a_n f(a_n)}{a_{n+1} f(a_{n+1})} \right] \geqslant 1 \tag{5.11}$$

for $n \geqslant 0, \quad a_0 = 0$, *then (5.8) holds.*

Proof. The proof is by induction and follows the steps of the proof of Theorem 7. For $n = 1$ inequality (5.8) is equal to (5.11) for $n = 0$. For a general n inequality (5.8) is equivalent to (5.9) and therefore the induction hypothesis is equivalent to

$$\sum_{i=1}^{n+1} f(a_i) \geqslant \frac{a_{n+1} f^2(a_{n+1})}{a_{n+1} f(a_{n+1}) - a_n f(a_n)}$$

which means that

$$\sum_{i=1}^{n+2} f(a_i) \geqslant \frac{a_{n+1} f^2(a_{n+1})}{a_{n+1} f(a_{n+1}) - a_n f(a_n)} + f(a_{n+2})$$

and hence it is sufficient to prove that

$$\frac{a_{n+1}f^2(a_{n+1})}{a_{n+1}f(a_{n+1})-a_nf(a_n)}+f(a_{n+2})\geqslant\frac{a_{n+2}f^2(a_{n+2})}{a_{n+2}f(a_{n+2})-a_{n+1}f(a_{n+1})}\tag{5.12}$$

which is equivalent to (5.11). □

THEOREM 9. *Let $f(x)$ be a positive increasing function on $[0,\infty)$ such that $xf(x)$ is convex and*

$$f(xA)f\left(\frac{A}{x}\right)\geqslant f^2(A),\quad 1\leqslant x\leqslant 2.\tag{5.13}$$

Let the sequence $a_n>0,\ n=1,\ldots,\ a_0=0$ satisfy

$$a_{n+2}-a_{n+1}\geqslant a_{n+1}-a_n,\tag{5.14}$$

and

$$\frac{a_2}{a_1}\geqslant\frac{f(a_1)}{f(a_2)}+1.\tag{5.15}$$

Then (5.8) holds.

Proof. Inequality (5.15) is equivalent to (5.11) for $n=0$. Let us denote

$$w=\frac{a_{n+2}}{a_{n+1}}-1.\tag{5.16}$$

Then it follows from (5.14) that $w>0$.

If $\frac{a_{n+2}}{a_{n+1}}\geqslant 2$ then $w\geqslant 1$ and then

$$\frac{f(a_{n+2})}{f(a_{n+1})}\left[\frac{a_{n+2}}{a_{n+1}}-1+\frac{a_nf(a_n)}{a_{n+1}f(a_{n+1})}\right]\geqslant\frac{f(a_{n+2})}{f(a_{n+1})}\geqslant 1.$$

Therefore, in this case (5.11) holds and from Theorem 8 we get that (5.8) holds.

If $0\leqslant w\leqslant 1,$ inequality (5.14) is equivalent to

$$a_n\geqslant(1-w)\,a_{n+1}.\tag{5.17}$$

As $f(x)\geqslant 0$ and is increasing we get that

$$\begin{aligned}&\frac{f(a_{n+2})}{f(a_{n+1})}\left[\frac{a_{n+2}}{a_{n+1}}-1+\frac{a_nf(a_n)}{a_{n+1}f(a_{n+1})}\right]\\&\quad\geqslant\frac{f(a_{n+1}(1+w))}{f(a_{n+1})}\left[w+\frac{(1-w)f(a_{n+1}(1-w))}{f(a_{n+1})}\right]\\&\quad=\left(\frac{(1+w)(a_{n+1})f((1+w)\,a_{n+1})}{((a_{n+1})f(a_{n+1}))^2}\right)\\&\qquad\times\left(\frac{w}{1+w}(a_{n+1}f(a_{n+1}))+\frac{1}{1+w}a_{n+1}(1-w)f(a_{n+1}(1-w))\right).\end{aligned}\tag{5.18}$$

As $xf(x)$ is convex, we get that

$$\begin{aligned}&\frac{w}{1+w}\left(a_{n+1}f\left(a_{n+1}\right)\right)+\frac{1}{1+w}\left(a_{n+1}\left(1-w\right)f\left(a_{n+1}\left(1-w\right)\right)\right)\\&\quad\geqslant a_{n+1}\left(\frac{w}{1+w}+\frac{1-w}{1+w}\right)f\left(a_{n+1}\left(\frac{w}{1+w}+\frac{1-w}{1+w}\right)\right)\\&\quad=\frac{a_{n+1}}{1+w}f\left(\frac{a_{n+1}}{1+w}\right).\end{aligned}\tag{5.19}$$

Inserting (5.19) in (5.18) we get that

$$\begin{aligned}&\frac{f\left(a_{n+2}\right)}{f\left(a_{n+1}\right)}\left[\frac{a_{n+2}}{a_{n+1}}-1+\frac{a_nf\left(a_n\right)}{a_{n+1}f\left(a_{n+1}\right)}\right]\\&\quad\geqslant\frac{\left(1+w\right)\left(a_{n+1}\right)f\left(\left(1+w\right)a_{n+1}\right)}{\left(\left(a_{n+1}\right)f\left(a_{n+1}\right)\right)^2}\frac{a_{n+1}}{\left(1+w\right)}f\left(\frac{a_{n+1}}{1+w}\right)\\&\quad=\frac{f\left(\left(1+w\right)a_{n+1}\right)f\left(\frac{a_{n+1}}{1+w}\right)}{f^2\left(a_{n+1}\right)}.\end{aligned}\tag{5.20}$$

As $0\leqslant w\leqslant 1,$ therefore $1\leqslant 1+w\leqslant 2$. Then, from (5.13) we get that

$$f\left(\left(1+w\right)a_{n+1}\right)f\left(\frac{a_{n+1}}{1+w}\right)\geqslant f^2\left(a_{n+1}\right).\tag{5.21}$$

Together with (5.20) we get that (5.11) holds. Hence (5.8) is proved. □

COROLLARY 7. *Let $f(x)=x^r,\ r\geqslant 0,\ x\geqslant 0$. Therefore $xf(x)=x^{r+1}$ is convex for $x\geqslant 0$, so (5.21) holds with equality and inequality (5.8) becomes equal to inequality (5.7).*

COROLLARY 8. *Let $\log f(x)$ be a convex function. Then (5.21) holds and as $f(x)$ is increasing, also $f(x)$ is convex and therefore (5.8) holds.*

In 2000 F. Qi and L. Debnath, in [29], using mathematical induction and Cauchy mean value theorem, proved following results.

THEOREM 10. *Let n and m be natural numbers. Suppose $\{a_1,a_2,\cdots\}$ is a positive and increasing sequence satisfying*

$$\frac{(k+2)a_{k+2}^r-(k+1)a_{k+1}^r}{(k+1)a_{k+1}^r-ka_k^r}\geqslant\left(\frac{a_{k+2}}{a_{k+1}}\right)^r,\tag{5.22}$$

for any given positive real number r and $k\in\mathbb{N}$. Then we have the inequality

$$\frac{a_n}{a_{n+m}}\leqslant\left[\frac{\frac{1}{n}\sum\limits_{i=1}^{n}a_i^r}{\frac{1}{n+m}\sum\limits_{i=1}^{n+m}a_i^r}\right]^{\frac{1}{r}}.\tag{5.23}$$

The lower bound of (5.23) is best possible.

COROLLARY 9. *Let n and m be natural numbers. Suppose $a = \{a_1, a_2, \cdots\}$ is a positive and increasing sequence satisfying*

$$a_{k+1}^2 \geqslant a_k a_{k+2}, \tag{5.24}$$

$$\frac{a_{k+1} - a_k}{a_{k+1}^2 - a_k a_{k+2}} \geqslant max\left\{\frac{k+1}{a_{k+1}}, \frac{k+2}{a_{k+2}}\right\}, \quad k \in \mathbb{N}. \tag{5.25}$$

Then, for any given positive real number r, we have inequality (5.23). The lower bound of (5.23) is best possible.

Applying Corollary 9 to $a = (k+1, k+2, \cdots)$ inequality (4.1) follows.

In 2002 Z. Xu and D. Xu, in [40], gave some new results related to Alzer's and Martin's inequality. We will present the results related to Alzer's inequality.

THEOREM 11. *Let $(a_n)_{n\in\mathbb{N}}$ be a strictly increasing positive sequence, and let m be a natural number and r be a positive real number. If*

$$\left(\frac{a_n}{a_{n-1}}\right)^{\frac{n-1}{n}} \leqslant \frac{a_{n+1}}{a_n} \leqslant \frac{a_n}{a_{n-1}}, \qquad n \geqslant 2, \tag{5.26}$$

then

$$\frac{a_n}{a_{n+m}} < \left[\frac{\frac{1}{n}\sum\limits_{i=1}^{n} a_i^r}{\frac{1}{n+m}\sum\limits_{i=1}^{n+m} a_i^r}\right]^{\frac{1}{r}}, \qquad n \geqslant 1. \tag{5.27}$$

The lower bound in (5.27) is best possible.

COMMENTS. Notice that the inequality (5.27) is equal to (5.23) with different conditions. Condition (5.26) can be interpreted in the following way:

The inequality

$$\left(\frac{a_n}{a_{n-1}}\right)^{\frac{n-1}{n}} \leqslant \frac{a_{n+1}}{a_n}$$

is equivalent to

$$\left(\frac{a_n}{a_{n-1}}\right)^{n-1} \leqslant \left(\frac{a_{n+1}}{a_n}\right)^{n},$$

which means that the sequence $\left\{\left(\frac{a_{n+1}}{a_n}\right)^n\right\}_{n\in\mathbb{N}}$ is increasing in n.

Furthermore, inequality

$$\frac{a_{n+1}}{a_n} \leqslant \frac{a_n}{a_{n-1}}$$

is equivalent to

$$a_{n-1}a_{n+1} \leqslant a_n^2,$$

which means that the sequence $(a_n)_{n\in\mathbb{N}}$ is logarithmic concave for $n \geqslant 2$.

As a consequence of Theorem 11 Z. Xu and D. Xu easily proved that the inequality

$$\frac{n+k}{n+m+k} < \left[\frac{\frac{1}{n}\sum_{i=1}^{n}(i+k)^r}{\frac{1}{n+m}\sum_{i=1}^{n+m}(i+k)^r} \right]^{\frac{1}{r}}, \tag{5.28}$$

is valid for any nonnegative real number k and not only for k being nonnegative integer like it was presented in above cases.

In 2004. F. Qi, B.-N. Guo and L. Debnath, in [31], using mathematical induction, proved inequality (5.27), mentioned above, with different conditions. We quote their result and corollary.

THEOREM 12. *Let n and m be natural numbers. Suppose $(a_i)_{i=1}^{n+m}$ is an increasing, logarithmically convex, and positive sequence. Denote the power mean $P_n(r)$ for any given positive real number r by*

$$P_n(r) = \left(\frac{1}{n}\sum_{i=1}^{n} a_i^r \right)^{\frac{1}{r}}. \tag{5.29}$$

Then the sequence $\left\{ \frac{P_i(r)}{a_i} \right\}_{i=1}^{n+m}$ is decreasing for any given positive real number r, that is

$$\frac{P_n(r)}{P_{n+m}(r)} \geqslant \frac{a_n}{a_{n+m}} \tag{5.30}$$

The lower bound in (5.30) is the best possible.

Considering that the exponential functions a^{x^α} and a^{α^x}, for given constants $\alpha \geqslant 1$ and $a > 1$, are logarithmically convex on $[0,\infty)$, as a corollary of Theorem 12 it follows.

COROLLARY 10. *Let $\alpha \geqslant 1$ and $a > 1$ be two constants. For any given real number r the following inequalities hold*

$$\frac{a^{(n+k)^\alpha}}{a^{(n+m+k)^\alpha}} \leqslant \left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} a^{i^\alpha r}}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} a^{i^\alpha r}} \right)^{\frac{1}{r}}, \tag{5.31}$$

$$\frac{a^{\alpha^{n+k}}}{a^{\alpha^{n+m+k}}} \leqslant \left(\frac{\frac{1}{n}\sum_{i=k+1}^{n+k} a^{\alpha^i r}}{\frac{1}{n+m}\sum_{i=k+1}^{n+m+k} a^{\alpha^i r}} \right)^{\frac{1}{r}}, \tag{5.32}$$

where n an m are natural numbers, and k is a nonnegative integer. The lower bounds in (5.31) and (5.32) are the best possible.

6. An inequality of Van de Lune - Alzer for negative powers

The inequality of Van de Lune - Alzer for negative powers was proved by H. Alzer in [3].

The results which we will present here offer new proofs and extensions.

In 2003. C.-P. Chen and F. Qi, in [7], proved that Van de Lune - Alzer's inequality is valid for all real numbers r (not only for $r > 0$). We now quote their result.

THEOREM 13. *Let n be a natural number. Then for all real numbers r it holds*

$$\frac{n}{n+1} < \left[\frac{\frac{1}{n}\sum_{i=1}^{n} i^r}{\frac{1}{n+1}\sum_{i=1}^{n+1} i^r}\right]^{\frac{1}{r}} < 1. \tag{6.1}$$

Both bounds are best possible.

Theorem 13 is proved by using mathematical induction and Jensen's inequality. J. Sándor, in [34], gave an elegant proof of inequality (6.1) using Cauchy's mean value theorem instead of Jensen's inequality. For some further results on this topic the reader is also referred to the papers [10], [11] and [12] written by C.-P. Chen and F. Qi.

In 2004. C.-P. Chen and F. Qi, in [9], studying monotonicity property of generalized logarithmic means defined by

$$L_p(a,b) = \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & p \neq -1, 0; \\ \frac{b-a}{\ln b - \ln a}, & p = -1; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p = 0, \end{cases}$$

proved the following theorem.

THEOREM 14. *Let $c > b > a > 0$ be real numbers. Then the function*

$$f(r) = \frac{L_r(a,b)}{L_r(a,c)} \tag{6.2}$$

is strictly decreasing with $r \in (-\infty, \infty)$.

The following corollary is straightforward.

COROLLARY 11. *Let $c > b > a > 0$ be real numbers. Then for any real number $r \in \mathbb{R}$*

$$\frac{b}{c} < \frac{L_r(a,b)}{L_r(a,c)}. \tag{6.3}$$

The lower bound in (6.3) is best possible.

For $c = b+\delta$, $\delta > 0$, Corollary 11 gives an extension of integral version of Van de Lune - Alzer's inequality i.e. inequality

$$\frac{b}{b+\delta} < \left[\frac{\frac{1}{b-a}\int_a^b x^r dx}{\frac{1}{b+\delta-a}\int_a^{b+\delta} x^r dx}\right]^{\frac{1}{r}},$$

which is valid for all real numbers r.

In 2005. C.-P. Chen and F. Qi, in [8], proved that Theorem 14 can be generalized as follows.

THEOREM 15. *Let $c > b > a$ and r be real numbers, and let f be a positive, twice differentiable function and satisfy $f'(t) > 0$ and $(\ln f(t))'' \geqslant 0$ on $(a,+\infty)$. Then*

$$\frac{\sup_{x\in[a,b]} f(x)}{\sup_{x\in[a,c]} f(x)} < \left(\frac{\frac{1}{b-a}\int_a^b f^r(x)dx}{\frac{1}{c-a}\int_a^c f^r(x)dx}\right)^{\frac{1}{r}} < 1, \tag{6.4}$$

for all real r. Both bounds in (6.4) are best possible.

On the other hand, in 1994., C. E. M. Pearce and J. Pečarić, in [25], proved the following theorem which generalizes Theorem 14.

THEOREM 16. *Let a_i, b_i, $(i = 1,2)$, be positive numbers satisfying*

$$max\left(\frac{a_1}{a_2},\frac{a_2}{a_1}\right) \geqslant max\left(\frac{b_1}{b_2},\frac{b_2}{b_1}\right).$$

Then the function G defined by

$$G(r) = \frac{L_r(a_1,a_2)}{L_r(b_1,b_2)} \tag{6.5}$$

is nondecreasing.

We quote the proof:

Proof. Power integral means of order p are defined by

$$M_p(f;a,b) = \begin{cases} \left[\frac{1}{b-a}\int_a^b f(t)^p dt\right]^{\frac{1}{p}}, & p \neq 0, \\ exp\left[\frac{1}{b-a}\int_a^b \log f(t)dt\right], & p = 0. \end{cases}$$

If $e(t) = t$, $e_{x,y}(t) = xt + y(1-t)$, then the generalized logarithmic means have the two integral representations

$$L_p(x,y) = M_p(e;x,y), \qquad L_p(x,y) = M_p(e_{x,y};1,0).$$

Thus

$$G(r)=\frac{M_r(e_{a_1,a_2};0,1)}{M_r(e_{b_1,b_2};0,1)}$$

and our result is a simple consequence of the following result (see [22]).

Let f and g be positive and integrable functions on $[a,b]$. If the maps $x\to g(x)$, $x\to\frac{f(x)}{g(x)}$ are monotonic in the same sense, then the function F defined by $F(r)=\frac{M_r(f;a,b)}{M_r(g;a,b)}$ is nondecreasing.

In our case $f(t)=e_{a_1,a_2}(t)$ and $g(t)=e_{b_1,b_2}(t)$. If $b_1=b_2$ then the denominator in (6.5) is independent of r and the claim reduces to the well-known result on the nondecreasing character of L_r. If $b_1\neq b_2$, then since the denominator in the definition for G is invariant under the interchange of b_1 and b_2, we may without loss of generality suppose that $b_1>b_2$. Similar symmetry in the numerator of (6.5) allows us to assume $a_1\geqslant a_2$, so that we can suppose that $\frac{a_1}{a_2}\geqslant\frac{b_1}{b_2}>1$.

If $b_1>b_2$ the function g is increasing and since

$$\left(\frac{a}{b}\right)'=\frac{b_2a_1-a_2b_1}{(e_{b_1,b_2}(t))^2},$$

(6.5) tells us that $\frac{f}{g}$ is nondecreasing, concluding the proof. □

The same assertion as one that C. E. M. Pearce and J. Pečarić gave in Theorem 16 was obtained by A.-J. Li, X.-M. Wang and C.-P. Chen, in [21], in 2006.

Extending the Ky Fan inequality to several general integral forms, they obtained the following theorems on monotonic properties of the function $\frac{L_s(a,b)}{L_s(\alpha-a,\alpha-b)}$ with $\alpha,a,b\in(0,+\infty)$ and $s\in\mathbb{R}$.

THEOREM 17. *Let*

$$f_\alpha(s)=\left(\frac{\int\limits_a^b x^s dx}{\int\limits_a^b(\alpha-x)^s dx}\right)^{\frac{1}{s}}=\frac{L_s(a,b)}{L_s(\alpha-a,\alpha-b)},$$

$s\in(-\infty,+\infty)$ and α be a positive number. Then $f_\alpha(s)$ is a strictly increasing function for $[a,b]\subseteq\left(0,\frac{\alpha}{2}\right]$, and is a strictly decreasing function for $[a,b]\subseteq\left[\frac{\alpha}{2},\alpha\right)$.

THEOREM 18. *Let*

$$f(s)=\left(\frac{\frac{1}{b-a}\int\limits_a^b x^s dx}{\frac{1}{d-c}\int\limits_c^d x^s dx}\right)^{\frac{1}{s}}=\frac{L_s(a,b)}{L_s(c,d)},$$

$s\in(-\infty,+\infty)$ and a,b,c,d be positive numbers. Then $f(s)$ is a strictly increasing function for $ad<bc$, or a strictly decreasing function for $ad>bc$.

Proofs of Theorem 17 and Theorem 18 are done using analogous method as one that C.-P. Chen and F. Qi used in [9] to prove Theorem 14.

In 2006. J. Sándor, in [37], proved the following theorem.

THEOREM 19. *Suppose that* $f:(0,1]\to\mathbb{R}$ *is a strictly decreasing, convex (or concave) function. Then one has the inequality*

$$\frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) > \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right). \tag{6.6}$$

COMMENTS. Notice that the reversed sign inequality, when f is strictly increasing and concave or convex function, was proved by J.-C. Kuang, in [20], in 1999, and, moreover, was known since 1975. thanks to Jan van de Lune's work ([1, p. 254]).

J. Sándor's proof is based on the method of J.-C. Kuang, [20].

Applying Theorem 19 to the function $f(x)=\frac{1}{x^s}=x^{-s}$, which is convex and strictly decreasing, we get

$$\frac{1}{n+1}\sum_{i=1}^{n+1}\left(\frac{n+1}{i}\right)^s > \frac{1}{n}\sum_{i=1}^{n}\left(\frac{n}{i}\right)^s,$$

which is equivalent to left-hand side of (6.1) for $r=-s$, $s>0$.

7. Application in guessing theory

In 1998 S. S. Dragomir and J. van der Hoek, in [14], proved an analytic inequality which has important applications to the estimation for the moments of guessing mappings.

To prove their main result they start with sequences $S_p(n)$ and $G_p(n)$ defined in the following way:

$$S_p(n)=\sum_{j=1}^{n} j^p \quad \text{and} \quad G_p(n)=\frac{S_p(n)}{n^{p+1}},$$

where p is positive real number and $n\geqslant 1$ is natural number. Then the next theorem holds.

THEOREM 20. *Let* $p\geqslant 1$, $p\in\mathbb{R}$. *Then*
(1) The lower bound for $G_p(n)$ *is*

$$G_p(n)\geqslant \frac{(n+1)^p}{(n+1)^{p+1}-n^{p+1}}, \quad \textit{for all} \quad n\geqslant 1; \tag{7.1}$$

(2) The sequence $G_p(\cdot)$ *is nonincreasing, i.e.*

$$G_p(n+1)\leqslant G_p(n), \quad \textit{for all} \quad n\geqslant 1. \tag{7.2}$$

COMMENTS. Note that sequence $G_p(n)$ is defined in the same way as the function $U_n(s)$ in J. van de Lune's *Problem 399.* Moreover, inequality (7.1) is identical to Alzer's inequality (3.1) (replace p with r). Namely, according to the definition of $G_p(n)$ and the fact that

$$\sum_{i=1}^{n+1} i^p = \sum_{i=1}^{n} i^p + (n+1)^p,$$

inequality (7.1) is equivalent to

$$\frac{\sum\limits_{i=1}^{n} i^p}{n^{p+1}} \geqslant \frac{(n+1)^p}{(n+1)^{p+1} - n^{p+1}},$$

$$(n+1)^{p+1}\sum_{i=1}^{n} i^p - n^{p+1}\sum_{i=1}^{n} i^p \geqslant n^{p+1}(n+1)^p,$$

$$(n+1)^{p+1}\sum_{i=1}^{n} i^p \geqslant n^{p+1}\left(\sum_{i=1}^{n} i^p + (n+1)^p\right),$$

$$(n+1)^{p+1}\sum_{i=1}^{n} i^p \geqslant n^{p+1}\sum_{i=1}^{n+1} i^p,$$

$$\frac{n}{n+1} \leqslant \left[\frac{(n+1)\sum\limits_{i=1}^{n} i^p}{n\sum\limits_{i=1}^{n+1} i^p}\right]^{\frac{1}{p}},$$

for $p \in \mathbb{R}$, $p \geqslant 1$. Further, inequality (7.2) is equivalent to J. van de Lune's statement that function $U_n(s)$ is strictly decreasing in $n \in \mathbb{N}$.

J. Sándor in his paper [35] (1999.) also point out two things. First, that (7.1) is actually Alzer's inequality proved also in his paper [33] for $p > 0$ and for strict inequality and second, that (7.2) is equivalent to (7.1).

S. S. Dragomir and J. van der Hoek proved Theorem 20 using the following lemma,

LEMMA 4. *For $p \geqslant 1$, $p \in \mathbb{R}$ and $n \geqslant 1$ we have*

$$(n+2)^p\left[n^{p+1} + (n+1)^p\right] \geqslant (n+1)^{2p+1}, \tag{7.3}$$

and their main result is proved using Theorem 20.

In 2003. C.-P. Chen, F. Qi, P. Cerone and S. S. Dragomir, in [13], among other results, presented the following theorem.

THEOREM 21. *Let f be an increasing and convex (or concave) function defined on $[0,1]$. Then the sequence $\left\{\frac{1}{n}\sum\limits_{i=1}^{n} f\left(\frac{i}{n}\right)\right\}_{n\in\mathbb{N}}$ decreases and $\left\{\frac{1}{n}\sum\limits_{i=0}^{n-1} f\left(\frac{i}{n}\right)\right\}_{n\in\mathbb{N}}$*

increases, and

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) &\geqslant \frac{1}{n+1}\sum_{i=1}^{n+1} f\left(\frac{i}{n+1}\right) \geqslant \int_0^1 f(t)dt \\ &\geqslant \frac{1}{n+1}\sum_{i=0}^{n} f\left(\frac{i}{n+1}\right) \geqslant \frac{1}{n}\sum_{i=0}^{n-1} f\left(\frac{i}{n}\right).\end{aligned} \tag{7.4}$$

COMMENTS. The first inequality in (7.4) is equivalent to Kuang's inequality (4.12), moreover it is equivalent to inequality (2.7) which was proved by J. H. van Lint in Theorem 2. The last inequality in (7.4) is equivalent to inequality (2.8) which is also known since 1975.

Applying Theorem 21 to $f(x) = x^r$ for $x \in [0,1]$ and $r > 0$ the authors of [13] proved the following corollary.

COROLLARY 12. *Let $n \in \mathbb{N}$. Then for all real number $r > 0$, it follows*

$$\left(\frac{\frac{1}{n}\sum\limits_{i=1}^{n-1} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n} i^r}\right)^{\frac{1}{r}} \leqslant \frac{n}{n+1} \leqslant \left(\frac{\frac{1}{n}\sum\limits_{i=1}^{n} i^r}{\frac{1}{n+1}\sum\limits_{i=1}^{n+1} i^r}\right)^{\frac{1}{r}}. \tag{7.5}$$

The right hand inequality in (7.5) is Van de Lune - Alzer's inequality.

In 2003. I. Brnetić and J. Pečarić, in [5], generalized inequality (7.2) from Theorem 20. They used following function.

$$F(n,p,a) = \frac{\sum\limits_{i=1}^{n} f(i)}{nf(n)},$$

where $f(i) = (i+a)^p$. Obviously, $F(n,p,0) = G_p(n)$. By obtaining the same result as S. S. Dragomir and J. van der Hoek gave in [14] and [15], with F instead of G, I. Brnetić and J. Pečarić obtained the best estimates for some inequalities in mentioned papers.

Generalizing inequality (7.2), they presented the following theorem.

THEOREM 22. *Let $n \geqslant 2$ be an integer and $p \geqslant 1$, $a \geqslant -1$ be real numbers. Let's define*

$$F(n,p,a) = \frac{\sum\limits_{i=1}^{n} (i+a)^p}{n(n+a)^p}.$$

Then $F(n+1,p,a) \leqslant F(n,p,a)$ for each $p \geqslant 1$, $a \geqslant -1$ and for each integer $n \geqslant 2$.

COMMENTS. Inequality

$$F(n+1,p,a) \leqslant F(n,p,a)$$

is equivalent to

$$\frac{\sum_{i=1}^{n+1}(i+a)^p}{(n+1)(n+1+a)^p} \leqslant \frac{\sum_{i=1}^{n}(i+a)^p}{n(n+a)^p},$$

i.e.

$$\frac{n+a}{n+a+1} \leqslant \left[\frac{(n+1)\sum_{i=1}^{n}(i+a)^p}{n\sum_{i=1}^{n+1}(i+a)^p}\right]^{\frac{1}{p}}, \tag{7.6}$$

for each integer $n \geqslant 2$ and for real numbers $p \geqslant 1$, $a \geqslant -1$.

Notice that inequality (7.6) is the generalization of F. Qi's inequality (4.1) because now it is proved for all real numbers $k \geqslant -1$ not only for nonnegative integers k.

The outline of the proof of Theorem 22 goes as follows:

It need's to be shown that $F(n,p,a) - F(n+1,p,a) \geqslant 0$ for $p \geqslant 1$, $a \geqslant 1$ and $n \geqslant 2$. We have

$$\begin{aligned} &F(n,p,a) - F(n+1,p,a) \\ &\quad = \frac{\sum_{i=1}^{n}(i+a)^p}{n(n+a)^p} - \frac{\sum_{i=1}^{n+1}(i+a)^p}{(n+1)(n+1+a)^p} \\ &\quad = \sum_{i=1}^{n}(i+a)^p\left(\frac{1}{n(n+a)^p} - \frac{1}{(n+1)(n+1+a)^p}\right) - \frac{1}{n+1} \\ &\quad = \frac{1}{n+1}\left(F(n,p,a)\frac{(n+1)(n+1+a)^p - n(n+a)^p}{(n+1+a)^p} - 1\right). \end{aligned}$$

So, we have to prove that

$$F(n,p,a) \geqslant \frac{(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p},$$

which is, by definition of function $F(n,p,a)$, equivalent to

$$\sum_{i=1}^{n}(i+a)^p \geqslant \frac{n(n+a)^p(n+1+a)^p}{(n+1)(n+1+a)^p - n(n+a)^p}. \tag{7.7}$$

Inequality (7.7) was proved in [5], using mathematical induction, convexity of function $f(x) = (x+a)^p$, for $p \geqslant 1$ and $x \geqslant -a$ and applying Jensen's inequality.

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S. Abramovich
Department of Mathematics
University of Haifa
Haifa, 31905
Israel
e-mail: abramos@math.haifa.ac.il

J. Barić
FESB
University of Split
Ruđera Boškovića b.b., 21000 Split
Croatia
e-mail: jbaric@fesb.hr

M. Matić
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split Teslina 12, 21000 Split
Croatia
e-mail: mmatic@pmfst.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, 10000 Zagreb
e-mail: pecaric@element.hr