Journal of
Mathematical
Inequalities
Volume 2, Number 1 (2008), 9–15

CONSOLIDATIONS OF EXTENDED QI'S INEQUALITY AND BOUGOFFA'S INEQUALITY

WENJUN LIU, CHUNCHENG LI AND JIANWEI DONG

(*communicated by J. Pečarić*)

Abstract. In this paper, an extension of Bougoffa's inequality is given, which gives a complete answer to an open problem posed by Bougoffa in [3]. Moreover, consolidations of extended Qi's inequality and Bougoffa's inequality are obtained.

1. Introduction

In the paper [9] Qi proposed the following open problem, which has attracted much attention from some mathematicians (cf. [1, 2, 7, 8, 10, 11]).

Open Problem 1. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \geqslant \left(\int_a^b f(x)dx\right)^{t-1} \tag{1}$$

hold for $t > 1$?

Similar to Open Problem 1, in the paper [3] Bougoffa proposed the following

Open Problem 2. Under what conditions does the inequality

$$\int_a^b [f(x)]^t dx \leqslant \left(\int_a^b f(x)dx\right)^{1-t} \tag{2}$$

hold for $t < 1$?

By using Hölder's inequality, Bougoffa obtained an answer to Open Problem 2 for $0 < t \leqslant 1/2$ and $\min\limits_{[a,b]} f(x) > 0$. In the previous paper [6], we gave an answer to Open Problem 2 for $0 < t < 1/2$ and obtained an consolidation of Qi's inequality and Bougoffa's inequality.

Mathematics subject classification (2000): 26D15.

Key words and phrases: Qi's inequality, Bougoffa's inequality, integral inequality, Cauchy's mean value theorem, reversed Hölder inequality.

This work was supported by the Science Research Foundation of Nanjing University of Information Science and Technology and the Natural Science Foundation of Jiangsu Province Education Department under Grant No. 07KJD510133.

In the present paper, we first give an extension of Bougoffa's inequality from which we can give a complete answer to Problem 2. Then, we obtain consolidations of extended Qi's inequality and Bougoffa's inequality.

2. An extension of Bougoffa's inequality

In [10], the authors obtained the following result, which was an extension of Qi's inequality.

PROPOSITION 1. *Let $f(x)$ be continuous and not identically zero on $[a,b]$, differentiable in (a,b) with $f(a)=0$, and let α,β be positive real numbers such that $\alpha>\beta>1$. If*

$$[f^{(\alpha-\beta)/(\beta-1)}]'(x) \gtreqless \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{3}$$

for all $x\in(a,b)$, then

$$\int_a^b [f(x)]^{\alpha} dx \gtreqless \left(\int_a^b f(x)dx\right)^{\beta}. \tag{4}$$

Based on Proposition 1, we can obtain an extension of Bougoffa's inequality as follow, in which setting $\alpha=1/t$ and $\beta=1/(1-t)$ for $t\in(0,\frac{1}{2})\cup(\frac{1}{2},1)$, and $\alpha=\beta=2$ for $t=\frac{1}{2}$, we can give a direct answer to Open Problem 2.

THEOREM 1. *Let $f(x)$ be continuous and not identically zero on $[a,b]$, differentiable in (a,b) with $f(a)=0$, and let α,β be positive real numbers.*

(1) *In case of $\alpha>\beta>1$. If*

$$[f^{(\alpha-\beta)/\alpha(\beta-1)}]'(x)\geqslant 0 \ \text{and} \ [f^{(\alpha-\beta)/\alpha(\beta-1)}]'(x) \gtreqless \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{5}$$

for all $x\in(a,b)$, then

$$\int_a^b [f(x)]^{\frac{1}{\alpha}} dx \lesseqgtr \left(\int_a^b f(x)dx\right)^{\frac{1}{\beta}}. \tag{6}$$

(2) *In case of $\alpha=\beta>1$. If*

$$[\ln f(x)]'\geqslant 0 \ \text{and} \ [\ln f(x)]' \gtreqless \alpha^{\alpha/(\alpha-1)} \tag{7}$$

for all $x\in(a,b)$, then (6) *holds.*

(3) *In case of $\beta>\alpha>1$. If*

$$[f^{(\alpha-\beta)/\alpha(\beta-1)}]'(x)\leqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{8}$$

for all $x\in(a,b)$, then

$$\int_a^b [f(x)]^{\frac{1}{\alpha}} dx \leqslant \left(\int_a^b f(x)dx\right)^{\frac{1}{\beta}}. \tag{9}$$

Proof. (1) Substitute $f(x)$ in Proposition 1 with $[f(x)]^{\frac{1}{\alpha}}$.

(2) If the condition (7) is satisfied, we may assume $f(x) > 0$ for $x \in (a,b]$. Thus both sides of (6) are not 0. By using Cauchy's Mean Value Theorem twice, we have

$$\frac{\int_a^b [f(x)]^{\frac{1}{\alpha}} dx}{\left(\int_a^b f(x)dx\right)^{\frac{1}{\alpha}}} = \frac{[f(b_1)]^{\frac{1}{\alpha}-1}}{\frac{1}{\alpha}\left(\int_a^{b_1} f(x)dx\right)^{\frac{1}{\alpha}-1}} \quad (a < b_1 < b) \tag{10}$$

$$= \left(\frac{\int_a^{b_1} f(x)dx}{\alpha^{\frac{\alpha}{1-\alpha}} f(b_1)}\right)^{1-\frac{1}{\alpha}} \tag{11}$$

$$= \left(\frac{1}{\alpha^{\frac{\alpha}{1-\alpha}} [\ln f(x)]'_{x=b_2}}\right)^{1-\frac{1}{\alpha}} \quad (a < b_2 < b_1) \tag{12}$$

$$\lesseqgtr 1. \tag{13}$$

So the inequality (6) holds.

(3) Suppose that $[f^{(\alpha-\beta)/\alpha(\beta-1)}]'(x) \leqslant 0$ for $x \in (a,b)$, then $f^{(\alpha-\beta)/\alpha(\beta-1)}(x)$ is non-increasing function. It follows that $f(x) \geqslant 0$ for $x \in (a,b]$. If the condition (8) satisfied, we may assume $f(x) > 0$ for $x \in (a,b]$. Thus, we have

$$\frac{\int_a^b [f(x)]^{\frac{1}{\alpha}} dx}{\left(\int_a^b f(x)dx\right)^{\frac{1}{\beta}}} = \frac{[f(c_1)]^{\frac{1}{\alpha}-1}}{\frac{1}{\beta}\left(\int_a^{c_1} f(x)dx\right)^{\frac{1}{\beta}-1}} \quad (a < c_1 < b) \tag{14}$$

$$= \left(\frac{\int_a^{c_1} f(x)dx}{\beta^{\frac{\beta}{1-\beta}} [f(c_1)]^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}}}\right)^{1-\frac{1}{\beta}} \tag{15}$$

$$= \left(\frac{1}{\beta^{\frac{\beta}{1-\beta}} \frac{(\alpha-1)\beta}{\alpha(\beta-1)} [f(c_2)]^{\frac{(\alpha-1)\beta}{\alpha(\beta-1)}-2} f'(c_2)}\right)^{1-\frac{1}{\beta}} \quad (a < c_2 < c_1) \tag{16}$$

$$= \left(\frac{1}{\beta^{\frac{1}{1-\beta}} \frac{\alpha-1}{\alpha-\beta} [f^{\frac{\alpha-\beta}{\alpha(\beta-1)}}]'(c_2)}\right)^{1-\frac{1}{\beta}} \tag{17}$$

$$\leqslant 1. \tag{18}$$

So the inequality (9) holds. □

In case of $\beta > \alpha > 1$, it's easy to see that there is no function $f(x)$ to satisfy the conditions of Theorem 1 and $\frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \leqslant [f^{(\alpha-\beta)/\alpha(\beta-1)}]'(x) \leqslant 0$ for all $x \in (a,b)$ simultaneously. Therefore, we now seek another method to obtain the reverse inequality of (9). For this purpose, we need the reversed Hölder inequality of Nehari's (see [8]).

LEMMA 1. *Let f, g be nonnegative concave functions on $[a,b]$. Then, for $p, q > 0$ such that $p^{-1} + q^{-1} = 1$, we have*

$$\left(\int_a^b f^p\right)^{1/p} \left(\int_a^b g^q\right)^{1/q} \leqslant N(p,q) \int_a^b f g, \tag{19}$$

where

$$N(p,q) = \frac{6}{(1+p)^{1/p}(1+q)^{1/q}}. \tag{20}$$

THEOREM 2. *Let $f(x)$ be nonnegative, concave and integrable on $[a,b]$ and $\beta > \alpha > 1$. Suppose*

$$f(x) \geqslant \left(\frac{(1+\alpha)(2\alpha-1)^{\alpha-1}}{6^\alpha(\alpha-1)^{\alpha-1}(b-a)^{1-\beta}}\right)^{\frac{\alpha}{\alpha-\beta}} \tag{21}$$

for $x \in [a,b]$. Then the inequality (9) reverses.

Proof. Put $f \equiv 1,\ q = \alpha$ into Lemma 1. Then we clearly obtain

$$\int_a^b g^\alpha(x)dx \leqslant N^\alpha\left(\frac{\alpha}{\alpha-1}, \alpha\right)(b-a)^{1-\alpha}\left(\int_a^b g(x)dx\right)^{\alpha-\beta}\left(\int_a^b g(x)dx\right)^\beta \tag{22}$$

$$\leqslant N^\alpha\left(\frac{\alpha}{\alpha-1}, \alpha\right)(b-a)^{1-\beta}(\min_{[a,b]} g(x))^{\alpha-\beta}\left(\int_a^b g(x)dx\right)^\beta. \tag{23}$$

Here $N(\cdot,\cdot)$ is the Nehari constant given by (20). Set $g^\alpha = f$, we have

$$\int_a^b f(x)dx \leqslant N^\alpha\left(\frac{\alpha}{\alpha-1}, \alpha\right)(b-a)^{1-\beta}(\min_{[a,b]} f(x))^{\frac{\alpha-\beta}{\alpha}}\left(\int_a^b f^{\frac{1}{\alpha}}(x)dx\right)^\beta, \tag{24}$$

By (24) the proof is complete. □

3. Consolidations of two extended inequalities

In this section, by combining Theorem 1 and Proposition 1, we obtain another result of this paper which gives consolidations of extended Qi's inequality and Bougoffa's inequality. To our best knowledge, this result is not found in other works.

THEOREM 3. *Let $f(x)$ be continuous and not identically zero on $[a,b]$, differentiable in (a,b) with $f(a) = 0$, and let α, β be positive real numbers such that $\alpha > \beta > 1$.*

(1) *If*

$$[f^{\frac{\alpha-\beta}{\alpha(\beta-1)}}]'(x) \geqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \quad \text{and} \quad [f^{\frac{\alpha-\beta}{\beta-1}}]'(x) \geqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{25}$$

for $x \in (a,b)$, *then*

$$\left(\int_a^b [f(x)]^{\frac{1}{\alpha}} dx\right)^{\beta} \leqslant \int_a^b f(x)dx \leqslant \left(\int_a^b [f(x)]^{\alpha} dx\right)^{\frac{1}{\beta}}. \tag{26}$$

(2) *If*

$$0 \leqslant [f^{\frac{\alpha-\beta}{\alpha(\beta-1)}}]'(x) \leqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \text{ and } 0 \leqslant [f^{\frac{\alpha-\beta}{\beta-1}}]'(x) \leqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{27}$$

for $x \in (a,b)$, *then the inequality* (26) *reverses.*

(3) *If*

$$[f^{\frac{\beta-\alpha}{\beta(\alpha-1)}}]'(x) \leqslant \frac{(\beta-\alpha)\alpha^{1/(\alpha-1)}}{\beta-1} \quad \text{and} \quad [f^{\frac{\alpha-\beta}{\beta-1}}]'(x) \geqslant \frac{(\alpha-\beta)\beta^{1/(\beta-1)}}{\alpha-1} \tag{28}$$

for $x \in (a,b)$, *then*

$$\left(\int_a^b [f(x)]^{\frac{1}{\beta}} dx\right)^{\alpha} \leqslant \int_a^b f(x)dx \leqslant \left(\int_a^b [f(x)]^{\alpha} dx\right)^{\frac{1}{\beta}}. \tag{29}$$

Proof. The case (1) and (2) are from Proposition 1 and (1) of Theorem 1. For the proof of (3) we need to substitute α,β in (3) of Theorem 1 with β,α respectively and combine with Proposition 1. □

In [6], the following result was obtained.

COROLLARY 1. *Let* $p > 2$ *be a positive number and* $f(x)$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *such that* $f(a) = 0$.

(1) *If* $[f^{p-2}]'(x) \geqslant p^p(p-2)/(p-1)^{p+1}$ *and* $[f^{\frac{1}{p-2}}]'(x) \geqslant (p-1)^{\frac{1}{p-2}-1}$ *for* $x \in (a,b)$, *then*

$$\left(\int_a^b [f(x)]^{\frac{1}{p}} dx\right)^{p} \leqslant \left(\int_a^b f(x)dx\right)^{p-1} \leqslant \int_a^b [f(x)]^{p} dx. \tag{30}$$

(2) *If* $0 \leqslant [f^{p-2}]'(x) \leqslant p^p(p-2)/(p-1)^{p+1}$ *and* $0 \leqslant [f^{\frac{1}{p-2}}]'(x) \leqslant (p-1)^{\frac{1}{p-2}-1}$ *for* $x \in (a,b)$, *then the inequality* (30) *reverses.*

Note that when selecting appropriate α,β in (1) and (2) of Theorem 3 we can obtain Corollary 1. So Corollary 1 is just a special case of Theorem 3.

COROLLARY 2. *Let* $f(x)$ *be continuous and not identically zero on* $[a,b]$ *and differentiable on* (a,b) *such that* $f(a) = 0$.

(1) *If* $[f^{\frac{1}{3}}]'(x) \geqslant 1$ *and* $f'(x) \geqslant 1$ *for* $x \in (a,b)$, *then*

$$\left(\int_a^b [f(x)]^{\frac{1}{3}} dx\right)^{2} \leqslant \int_a^b f(x)dx \leqslant \left(\int_a^b [f(x)]^{3} dx\right)^{\frac{1}{2}}. \tag{31}$$

(2) *If* $0 \leqslant [f^{\frac{1}{3}}]'(x) \leqslant 1$ *and* $0 \leqslant f'(x) \leqslant 1$ *for* $x \in (a,b)$*, then the inequality* (31) *reverses.*

(3) *If* $[f^{-\frac{1}{4}}]'(x) \leqslant -\sqrt{3}$ *and* $f'(x) \geqslant 1$ *for* $x \in (a,b)$*, then*

$$\left(\int_a^b [f(x)]^{\frac{1}{2}} dx\right)^3 \leqslant \int_a^b f(x)dx \leqslant \left(\int_a^b [f(x)]^3 dx\right)^{\frac{1}{2}}. \tag{32}$$

Proof. Set $\alpha = 3$ and $\beta = 2$ in Theorem 3. □

In order to illustrate a possible practical use of Corollary 2, we shall give three simple examples in which we can apply the inequalities.

EXAMPLE 1. Let $f(x) = 8(e^x - e)$ on $[1,2]$, we see that $[f^{\frac{1}{3}}]'(x) = \frac{2}{3}e^x(e^x - e)^{-\frac{2}{3}} > 1$ and $f'(x) > 8e > 1$ for $x \in (1,2)$, other conditions of Corollary 2 are fulfilled and straightforward computation yields

$$\left(\int_1^2 [8(e^x - e)]^{\frac{1}{3}} dx\right)^2 \approx 5.4 < \int_1^2 8(e^x - e)dx \approx 15.6 < \left(\int_1^2 [8(e^x - e)]^3 dx\right)^{\frac{1}{2}} \approx 98.0.$$

EXAMPLE 2. Let $f(x) = \frac{1}{8}x^3$ on $[0,1]$, then $0 \leqslant [f^{\frac{1}{3}}]'(x) < 1$ and $0 \leqslant f'(x) < 1$ for $x \in (0,1)$, other conditions of Corollary 2 are fulfilled and direct calculation produces that

$$\left(\int_0^1 \left[\frac{1}{8}x^3\right]^{\frac{1}{3}} dx\right)^2 = \frac{1}{16} > \int_0^1 \frac{1}{8}x^3 dx = \frac{1}{32} > \left(\int_0^1 \left[\frac{1}{8}x^3\right]^3 dx\right)^{\frac{1}{2}} = \frac{1}{\sqrt{5120}}.$$

EXAMPLE 3. Let $f(x) = \frac{1}{3}(x^3 - 1)$ on $[1, 1.1]$, we see that $[f^{-\frac{1}{4}}]'(x) = -3^{\frac{5}{4}}x^2(x^3 - 1)^{-\frac{5}{4}}/4 < -\sqrt{3}$ and $f'(x) > 1$ for $x \in (1, 1.1)$, other conditions of Corollary 2 are fulfilled and straightforward computation yields

$$\left(\int_1^{1.1} \left[\frac{1}{3}(x^3 - 1)\right]^{\frac{1}{2}} dx\right)^3 < \int_1^{1.1} \frac{1}{3}(x^3 - 1)dx \approx 0.0053 < \left(\int_1^{1.1} \left[\frac{1}{3}(x^3 - 1)\right]^3 dx\right)^{\frac{1}{2}}.$$

Acknowledgements. The authors wish to express their gratitude to the anonymous referee for a number of valuable comments and suggestions.

REFERENCES

[1] M. AKKOUCHI, *On an integral inequality of Feng Qi*, Divulgaciones Math., **13**(1) (2005), 11–19.
[2] L. BOUGOFFA, *Notes on Qi type integral inequalities*, J. Inequal. Pure Appl. Math., **4**(4) (2003), Art. 77.
[3] L. BOUGOFFA, *An integral inequality similar to Qi's inequality*, J. Inequal. Pure Appl. Math., **6**(1) (2005), Art. 27.
[4] Y. CHEN AND J. KIMBALL, *Note on an open problem of Feng Qi*, J. Inequal. Pure Appl. Math., **7**(1) (2006), Art. 4.
[5] J. C. KUANG, *Applied Inequalities*, 3nd edition, Shandong Science and Technology Press, Jinan, China, 2004. (In Chinese)
[6] W. J. LIU, C. C. LI AND J. W. DONG, *Note on Qi's inequality and Bougoffa's inequality*, J. Inequal. Pure Appl. Math., **7**(4) (2006), Art. 129.
[7] S. MAZOUZI AND F. QI, *On an open problem regarding an integral inequality*, J. Inequal. Pure Appl. Math., **4**(2) (2003), Art. 31.
[8] T. K. POGANY, *On an open problem of F. Qi*, J. Inequal. Pure Appl. Math., **3**(4) (2002), Art. 54.
[9] F. QI, *Several integral inequalities*, J. Inequal. Pure Appl. Math., **1**(2) (2000), Art. 19.
[10] F. QI, A. J. LI, W. Z. ZHAO, D. W. NIU AND J. CAO, *Extensions of Several integral inequalities*, J. Inequal. Pure Appl. Math. ,**7**(3) (2006), Art. 107.
[11] K. W. YU AND F. QI, *A short note on an integral inequality*, RGMIA Res. Rep. Coll., **4**(1) (2001), Art. 4, 23–25.

(Received September 7, 2006)

Wenjun Liu
College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
e-mail: wjliu@nuist.edu.cn

Chuncheng Li
College of Mathematics and Physics
Nanjing University of Information Science and Technology
Nanjing 210044, China
e-mail: lichunchengcxy@126.com

Jianwei Dong
Department of Mathematics and Physics
Zhengzhou Institute of Aeronautical Industry Management
Zhengzhou 450015, China
e-mail: dongjianweiccm@163.com