SOME BOUNDS FOR ALTERNATING MATHIEU TYPE SERIES

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Abstract. Using recent investigated integral representations for the generalized alternating Mathieu series \( \tilde{S}(\alpha, \beta) \mu (r; \{a_n\}_{n=1}^{\infty}) \) \[9,14,18\] with \( a_n = n^\gamma, \ \gamma \in R^+ \) and Mellin-Laplace type integral transforms for the generalized hypergeometric functions and the Bessel function of first kind, some bounding inequalities for \( \tilde{S}(\alpha, \beta) \mu (r; \{n^\gamma\}_{n=1}^{\infty}) \) are presented. Namely, it is shown that the series \( \tilde{S}(\alpha, \beta) \mu (r; \{n^\gamma\}_{n=1}^{\infty}) \) under some conditions for parameters \( \alpha, \beta, \gamma \) and \( \mu \) are bounded with constants which do not depend on \( \alpha, \beta \) and \( \gamma \) but only depend on \( r \) and \( \mu \), i.e.

\[ \tilde{S}(\alpha, \beta) \mu (r; \{n^\gamma\}_{n=1}^{\infty}) \leq \frac{2}{(1 + r^2)^\mu}. \]

1. Introduction

The following familiar infinite series

\[ S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2} \quad (r \in R^+) \]  

(1.1)

is named after Emile Leonard Mathieu (1835-1890), who investigated it in his 1890 work [7] on elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two-dimensional rectangular domain (see [13], p.258, eq. (54)). An alternating version of (1.1)

\[ \tilde{S}(r) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2n}{(n^2 + r^2)^2} \quad (r \in R^+) \]  

(1.2)

was recently discussed by Pogany et.al [9].

Integral representations of (1.1) and (1.2) are given by (see [5] and [9])

\[ S(r) = \frac{1}{r} \int_0^{\infty} \frac{t \sin (rt)}{e^t - 1} dt, \]  

(1.3)


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Several interesting problems and solutions dealing with integral representations and bounds for the following slight generalization of the Mathieu series with a fractional power

\[
S_\mu (r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^\mu} \quad (r \in R^+; \mu > 1)
\]

(1.5)
can be found in the recent works by Diananda [2], Tomovski and Trenčevski [16] and Cerone and Lenard [1]. Motivated essentially by the works of Cerone and Lenard [1] (and Qi [12]) a family of generalized Mathieu series

\[
S^{(\alpha, \beta)}_\mu (r; a) = S^{(\alpha, \beta)}_\mu (r; \{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu} \quad (r, \alpha, \beta, \mu \in R^+)
\]

(1.6)
was defined in [14], where it is tacitly assumed that the positive sequence

\[a = \{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\} \quad \lim_{n \to \infty} a_n = \infty\]
is chosen such that the infinite series in definition (1.6) converges, that is, that the following auxiliary series

\[
\sum_{n=1}^{\infty} \frac{1}{a_n^{\alpha - \beta}}
\]
is convergent. Comparing the definitions (1.1), (1.5) and (1.6), we see that \(S_r (r) = S (r)\) and \(S_\mu (r) = S^{(2,1)}_\mu (r; \{n\}_{n=1}^{\infty})\). Furthermore, the special cases \(S^{(2,1)}_2 (r; \{a_n\}_{n=1}^{\infty})\), \(S^{(2,1)}_\mu (r; \{n\}_{n=1}^{\infty})\), \(S^{(2,1)}_\mu (r; \{n^\gamma\}_{n=1}^{\infty})\) and \(S^{(\alpha, \alpha/2)}_\mu (r; \{n\}_{n=1}^{\infty})\) were investigated by Qi [12]; Diananda [2]; Tomovski [16] and Cerone and Lenard [1].

Let

\[
\bar{S}^{(\alpha, \beta)}_\mu (r; a) = \bar{S}^{(\alpha, \beta)}_\mu (r; \{a_n\}_{n=1}^{\infty}) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2a_n^\beta}{(a_n^\alpha + r^2)^\mu} \quad (r, \alpha, \beta, \mu \in R^+)
\]

(1.7)
be an alternating variant of (1.6), where the positive sequence \(\{a_n\}_{n=1}^{\infty}\) satisfied the same conditions of the definition (1.6). In [9, 14, 18] several integral representations of (1.6) and (1.7) in terms of the generalized hypergeometric functions and the Bessel function of first kind were obtained. Here we present some of them:

\[
\bar{S}^{(\alpha, \beta)}_\mu (r; \{n^\gamma\}_{n=1}^{\infty}) = \frac{2}{\Gamma (\mu)} \int_0^{\infty} x^{\nu (\mu \alpha - \beta) - 1} e^x + 1 \Psi_1 [(\mu, 1); (\gamma (\mu \alpha - \beta), \gamma \alpha); -r^2 x^{\nu \alpha}] \, dx
\]

(1.8)
\[
\tilde{S}_\mu^{(\alpha, \beta)} \left( r; \left\{ n^{q/\alpha} \right\}_{n=1}^{\infty} \right) = \frac{2}{\Gamma \left( q \left[ \mu - \frac{\beta}{\alpha} \right] \right)} \\
\times \int_0^\infty \frac{x^{q[\mu - \beta/\alpha] - 1}}{e^x + 1} \, {}_1F_q \left( \mu; \Delta (q; q [\mu - \beta/\alpha]); -r^2 \left( \frac{x}{q} \right)^q \right) dx
\]
\[
= \frac{\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu + \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx, \quad (r, \mu \in R^+)
\]
\[
\tilde{S}_\mu^{(\alpha, \alpha/2)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) = \tilde{S}_\mu^{(2, 1)} \left( r; \left\{ n \right\}_{n=1}^{\infty} \right) = \tilde{S}_{\mu + 1} (r)
\]
\[
= \frac{\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu + \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
\[
\tilde{S}_\mu^{(\alpha, 0)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(n^2 + r^2)^\mu} = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
\[
\tilde{S}_\mu^{(\alpha, 0)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(n^2 + r^2)^\mu} = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
\[
\tilde{S}_\mu^{(\alpha, 0)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(n^2 + r^2)^\mu} = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
\[
\tilde{S}_\mu^{(\alpha, 0)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(n^2 + r^2)^\mu} = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
where \( \Delta (q; \lambda) \) is the \( q \)-tuple \( \left( \frac{\lambda}{q}, \frac{\lambda + 1}{q}, \ldots, \frac{\lambda + q - 1}{q} \right) \);
\[
pFq \left( \alpha, \beta \right) \left( q \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2}{(n^2 + r^2)^\mu} = \frac{2\sqrt{\pi}}{(2r)^{\mu - \frac{1}{2}} \Gamma (\mu + 1)} \int_0^\infty \frac{\chi^{\mu - \frac{1}{2}}}{e^x + 1} J_{\mu - \frac{1}{2}} (rx) \, dx
\]
Here \( p \Psi_q \) denotes the Fox-Wright generalization of the hypergeometric \( pFq \) function with \( p \) numerator and \( q \) denominator parameters (see for example [15, Eq.1.5 (21), p.50])
\[
pFq \left( \alpha, \beta \right) \left( q \right) = p \Psi_q \left[ (a_l, \alpha_l)_{1,p}; (b_j, \beta_j)_{1,q}; x \right] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma (a_l + \alpha_lk)}{\prod_{j=1}^{q} \Gamma (b_j + \beta_jk)} \frac{x^k}{k!}
\]
\[
\left( a_l, b_j, \alpha_l, \beta_j \in R; \ l = 1, 2, \ldots, p; \ j = 1, 2, \ldots, q; \ 1 + \sum_{j=1}^{q} \beta_j - \sum_{l=1}^{p} \alpha_l > 0 \right)
\]
The generalized hypergeometric function is defined by
\[
pFq \left[ (a_l)_{1,p}; (b_j)_{1,q}; x \right] = \sum_{m=0}^{\infty} \frac{\prod_{l=1}^{p} (a_l)_m x^m}{\prod_{j=1}^{q} (b_j)_m m!}
\]
where \( (\delta)_m \) is the Pochhammer symbol, defined by
\[
(\delta)_0 = 1, \quad (\delta)_m = \delta (\delta + 1) \cdots (\delta + m - 1) = \frac{\Gamma (\delta + m)}{\Gamma (\delta)} \quad (m \in N),
\]
so that, obviously

\[ p \Psi_q \left[ (a_1, 1)_{1,p}; (b_j, 1)_{1,q}; x \right] = \frac{p \prod_{i=1}^{p} \Gamma (a_i)}{q \prod_{j=1}^{q} \Gamma (b_j)} pFq \left[ (a_i)_{1,p}; (b_j)_{1,q}; x \right] \]  

(1.14) 

\( (a_i > 0, b_j \notin \mathbb{Z}^-) \).

2. Bounds derivable from the integral representations of \( S_{\mu}^{(\alpha, \beta)} (r; \{ n^\nu \}_{n=1}^{\infty}) \)

2.1. The Landau estimates (see [6])

\[ |J_\nu (x)| \leq b_L v^{-1/3} \text{ with } b_L = \sqrt[3]{2} \sup_{x \in \mathbb{R}^+} \{ A_i (x) \} = 0.674885 \ldots, \text{ uniformly in } x, \]

(2.1)

\[ |J_\nu (x)| \leq c_L x^{-1/3} \text{ with } c_L = \sup_{x \in \mathbb{R}^+} \left\{ x^{1/3} J_0 (x) \right\} = 0.78574687 \ldots, \text{ uniformly in } \nu, \]

(2.2)

where \( A_i (z) \) denotes the known Airy function, were used in [9] to prove the following bounds:

\[ \left| S_{\mu}^{(2,1)} (r; \{ n^\nu \}_{n=1}^{\infty}) \right| \leq \frac{b_L \tilde{C}_\mu (r) \Gamma (\mu + \frac{1}{2})}{(\mu - \frac{3}{2})^{1/3}} \]  

(2.3)

\[ \left| S_{\mu}^{(2,1)} (r; \{ n^\nu \}_{n=1}^{\infty}) \right| \leq \frac{c_L \tilde{C}_\mu (r) \Gamma (\mu + \frac{3}{2})}{\sqrt{r}} \]  

(2.4)

where \( \tilde{C}_\mu (r) = \frac{\sqrt{\pi}}{(2r)^{\mu - \frac{3}{2}} \Gamma (\mu)} \). Moreover, if \( \mu \geq \frac{1}{2} (1 + r^2) \), then

\[ 0 < S_{\mu}^{(2,1)} (r; \{ n^\nu \}_{n=1}^{\infty}) \leq \frac{b_L \tilde{C}_\mu (r) \Gamma (\mu + \frac{1}{2})}{(\mu - \frac{3}{2})^{1/3}} = N_b (r, \mu) \]  

(2.5)

\[ 0 < S_{\mu}^{(2,1)} (r; \{ n^\nu \}_{n=1}^{\infty}) \leq \frac{c_L \tilde{C}_\mu (r) \Gamma (\mu + \frac{3}{2})}{\sqrt{r}} = N_c (r, \mu). \]  

(2.6)

2.2. Now we shall improve the right sided bounding inequalities (2.5)–(2.6) by showing that \( S_{\mu}^{(2,1)} (r; \{ n^\nu \}_{n=1}^{\infty}) \) is bounded with the constant

\[ M (r, \mu) = \frac{2}{(1 + r^2)^{\mu}} \]

(2.7)

under the conditions \( \gamma \in \mathbb{R}^+, \mu \geq \frac{1}{2} (1 + r^2) \). Let \( \varphi (x) = \frac{x^{\gamma - 1}}{(x^2 + r^2)^{\mu + 1}} \) with \( \mu \geq \frac{1}{2} (1 + r^2) \). Since

\[ \varphi' (x) = \frac{\gamma x^{\gamma - 1} (r^2 - (2\mu - 1) x^2)}{(x^2 + r^2)^{\mu + 1}} < 0 \]
for all \( x \) constrained by the inequality \( x > \left( \frac{r^2}{2\mu - 1} \right)^{\frac{1}{2}} \) it follows that \( \varphi (x) \) is a decreasing function of \( x \). So we have

\[
S^{(2,1)}_{\mu} (r; \{n^{\gamma}\}_{n=1}^{\infty}) = M (r, \mu) - 2 \sum_{n=1}^{\infty} [\varphi (2n) - \varphi (2n + 1)] < M (r, \mu) \quad (2.8)
\]

when \( \left( \frac{r^2}{2\mu - 1} \right)^{1/2} \gamma \) \( \leq 1 \). But, this condition holds, since \( \mu \geq \frac{1}{2} (1 + r^2) \).

Next, it would be of interest to research the efficiency and the sharpness of \( M (r, \mu) \). In this goal, the \( r^- \) domains in which \( M (r, \mu) \) is superior to \( N_b \) and \( N_c \) have to be obtained. We shall prove that \( M (r, \mu) < N_b (r, \mu) \) for all \( r \in R^+ \) and all \( \mu > \frac{3}{2} \).

Let

\[
P = P (r, \mu) = \frac{M (r, \mu)}{N_b (r, \mu)}. \quad (2.9)
\]

Then

\[
P = \frac{(\mu - 3/2)^{1/3}}{b_L \sqrt{2\pi}} \frac{\Gamma (\mu)}{\Gamma (\mu + 1/2)} \left( \frac{2r}{1 + r^2} \right)^{\mu - 3/2} (1 + r^2)^{-3/2}. \quad (2.10)
\]

We consider the function

\[
f (r) = \frac{r^{\mu - 3/2}}{(1 + r^2)^{\mu}} \quad \left( r \in R^+, \; \mu > \frac{3}{2} \right) \quad (2.11)
\]

It is easy to show that

\[
\max_{r \in R^+} f (r) = f \left( \sqrt{\frac{\mu - 3/2}{\mu + 3/2}} \right) \quad \left( \mu > \frac{3}{2} \right). \quad (2.12)
\]

Using the elementary inequality

\[
\left( \frac{2r}{1 + r^2} \right)^{\mu - 3/2} \leq 1 \quad \left( r \in R^+, \; \mu > \frac{3}{2} \right) \quad (2.13)
\]

and Gautschi’s inequality (see [4])

\[
\frac{\Gamma (\mu)}{\Gamma (\mu + 1/2)} \leq \frac{1}{\sqrt{\mu - 1/4}} \quad \left( \mu > \frac{3}{2} \right) \quad (2.14)
\]

we get

\[
P \leq \frac{(\mu - 3/2)^{1/3}}{b_L \sqrt{2\pi}} \frac{1}{\sqrt{\mu - 1/4}} \left( 1 + \left( \sqrt{\frac{\mu - 3/2}{\mu + 3/2}} \right)^2 \right)^{-3/2}
\]

\[
= \frac{(\mu - 3/2)^{1/3}}{b_L \sqrt{2\pi}} \frac{1}{\sqrt{\mu - 1/4}} \frac{(\mu + 3/2)^{3/2}}{(2\mu)^{3/2}}
\]
the integral formula

Because of

Next, we shall present some elegant bounds for the alternating Mathieu series

Using the well-known formula

we get

\[ S(0, \infty) = 2 \int_0^\infty e^{-x} \sin(\pi \zeta) \, d\zeta, \]

we get

\[ S(r) \leq \frac{1}{r} \int_0^\infty xe^{-x} \sin(\pi \zeta) \, d\zeta = M(r, 2). \]

2.3.2. In the theory of Bessel functions, it is fairly well-known that (cf; e.g. [3, p.49, Eq. 7.7.3 (16)]

\[ \int_0^\infty e^{-st}\lambda^{-1}J_v(\rho t)dt = \left( \frac{\rho}{2s} \right)^v s^{-\lambda} \frac{\Gamma(\nu+\lambda)}{\Gamma(\nu+1)} \, _2F_1 \left[ \frac{1}{2} (\nu+\lambda), \frac{1}{2} (\nu+\lambda+1); \nu+1; -\frac{\rho^2}{s^2} \right] \left( \Re(s) > |\Im(\rho)|, \Re(\nu+\lambda) > 0 \right). \]

Because of

\[ _1F_0(\lambda; -; z) = (1-z)^{-\lambda} \quad (|z| < 1; \lambda \in C) \]

the integral formula (2.18) would simplify considerably when \( \lambda = \nu + 1 \) and when \( \lambda = \nu + 2 \), giving us [see also (1.10) and (1.11) above]

\[ \int_0^\infty e^{-st}\nu^1J_v(\rho t)dt = \frac{(2\rho)^v}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{1}{2})}{(s^2+\rho^2)^{v+\frac{1}{2}}} \left( \Re(s) > |\Im(\rho)|, \Re(\nu) > -\frac{1}{2} \right); \]

\[ \int_0^\infty e^{-st}\nu^1+1J_v(\rho t)dt = \frac{2s(2\rho)^v}{\sqrt{\pi}} \frac{\Gamma(\nu+\frac{3}{2})}{(s^2+\rho^2)^{v+\frac{3}{2}}} \left( \Re(s) > |\Im(\rho)|, \Re(\nu) > -1 \right). \]
Using the formulas (2.20), (2.21), and integral representations (1.11) and (1.10), we obtain the following bounds for $S^{(\alpha,0)}_\mu \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^\infty \right)$ and $S_{\mu+1}(r)$ respectively:

\begin{align*}
S^{(\alpha,0)}_\mu \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^\infty \right) &\leq \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu)} \int_0^\infty e^{-x} x^{\mu-\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) \, dx \\
&= \frac{2\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu)} \frac{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu)}{\sqrt{\pi}(1+r^2)^\mu} = M(r, \mu) \quad (r \in \mathbb{R}^+, \mu > \frac{1}{2})
\end{align*}

(2.22)

\begin{align*}
S^{(r)}_{\mu+1} &\leq \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)} \int_0^\infty e^{-x} x^{\mu+\frac{1}{2}} J_{\mu-\frac{1}{2}}(rx) \, dx \\
&= \frac{\sqrt{\pi}}{(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)} \frac{2(2r)^{\mu-\frac{1}{2}} \Gamma(\mu+1)}{\sqrt{\pi}(1+r^2)^{\mu+1}} = M(r, \mu+1) \quad (r, \mu \in \mathbb{R}^+)
\end{align*}

(2.23)

2.3.3. In the theory of generalized hypergeometric functions it is known that the following integral formula (see [11], p. 335):

\begin{align*}
\int_0^\infty x^{l\alpha-1} e^{-x} pF_q \left( (a_p); (b_q); -\omega x^l \right) \, dx \\
= \sigma^{-l\alpha} \Gamma(l\alpha) \frac{\Delta(l, l\alpha)}{l} \frac{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)}{\Gamma\left( q \left\{ \frac{\beta}{\alpha} \right\} \right)} \frac{\Gamma\left( q \left\{ \frac{\beta}{\alpha} \right\} \right)}{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)} \frac{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)}{\Gamma\left( q \right)} \frac{\Gamma\left( q \left\{ \frac{\beta}{\alpha} \right\} \right)}{\Gamma\left( q \right)}
\end{align*}

(2.24)

holds.

Using the formula (2.24) and integral representation (1.9) we obtain the following bound for $S^{(\alpha,\beta)}_\mu \left( r; \left\{ n^{q/\alpha} \right\}_{n=1}^\infty \right)$:

\begin{align*}
S^{(\alpha,\beta)}_\mu \left( r; \left\{ n^{q/\alpha} \right\}_{n=1}^\infty \right) &\leq \frac{2}{\Gamma\left( q \left\{ \frac{\beta-\mu}{\alpha} \right\} \right)} \int_0^\infty x^{\frac{\beta-\mu}{\alpha}} e^{-x} \frac{1}{x} \frac{\Delta\left( q; q; \frac{\beta-\mu}{\alpha} \right)}{\Gamma\left( \frac{\beta-\mu}{\alpha} \right)} \left( \frac{x}{q} \right)^{\frac{\beta}{\alpha}} \, dx \\
&= \frac{2}{\Gamma\left( q \left\{ \frac{\beta-\mu}{\alpha} \right\} \right)} \frac{\Gamma\left( q \left\{ \frac{\beta-\mu}{\alpha} \right\} \right)}{\Gamma\left( q \left\{ \frac{\beta}{\alpha} \right\} \right)} \frac{\Gamma\left( q \left\{ \frac{\beta}{\alpha} \right\} \right)}{\Gamma\left( q \right)} \frac{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)}{\Gamma\left( q \right)} \frac{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)}{\Gamma\left( q \right)} \frac{\Gamma\left( q \left\{ \frac{\beta(q-\mu)}{\alpha} \right\} \right)}{\Gamma\left( q \right)}
\end{align*}

(2.25)
Specifically for \( p = 1, \ q = 2, \ l = 2, \) we get from (2.24)
\[
S_{\mu}^{(\alpha, \beta)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right)
\leq \frac{2}{\Gamma \left( 2 \left[ \frac{\mu - \beta}{\alpha} \right] \right)} \int_{0}^{\infty} x^{2} \left[ \frac{\mu - \beta}{\alpha} \right] -1 e^{-x} \Psi_{1} \left( \mu; \left[ \mu - \beta \right], \left[ \mu - \beta \right] + \frac{1}{2}; -\frac{r^{2}x^{2}}{4} \right) dx
\]
\[
= \frac{2}{\Gamma \left( 2 \left[ \frac{\mu - \beta}{\alpha} \right] \right)} \Gamma \left( 2 \left[ \frac{\mu - \beta}{\alpha} \right] \right)
\times \Psi_{1} \left( \mu, \left[ \mu - \beta \right], \left[ \mu - \beta \right] + \frac{1}{2}; -r^{2} \right)
\]
\[
= \frac{2}{(1 + r^{2})^{\mu}},
\]
i.e.
\[
S_{\mu}^{(\alpha, \beta)} \left( r; \left\{ n^{2/\alpha} \right\}_{n=1}^{\infty} \right) \leq M \left( r, \mu \right).
\] (2.26)

2.3.4. Now applying the formula (see [11, p.355])
\[
\int_{0}^{\infty} x^{\alpha-1} e^{-\sigma x} \Psi_{q} \left[ (\alpha, \beta) ; (b_{q}, \sigma \beta) ; -w \sigma \right] dx
\]
\[
= \frac{1}{\sigma^{\alpha+p+1}} \Psi_{q} \left[ \alpha, \beta ; (b_{q}, \sigma \beta) ; -\frac{w}{\sigma} \right]
\]
we obtain from the integral representation (1.8),
\[
S_{\mu}^{(\alpha, \beta)} \left( r; \left\{ n^{\gamma} \right\}_{n=1}^{\infty} \right)
\leq \frac{2}{\Gamma \left( \mu \right)} \int_{0}^{\infty} x^{\gamma(\mu \alpha - \beta)} e^{-x} \Psi_{1} \left( \mu; (1, \gamma) ; (\mu \alpha - \beta), \gamma \alpha ; -r^{2}x^{\gamma} \right) dx
\]
\[
= \frac{2}{\Gamma \left( \mu \right)} \Psi_{1} \left[ (\gamma \mu \alpha - \beta) ; (\mu, 1) ; (\gamma \mu \alpha - \beta), \gamma \alpha ; -r^{2} \right]
\]
\[
= \frac{2}{\Gamma \left( \mu \right)} \sum_{m=0}^{\infty} \frac{\Gamma \left( \gamma \mu \alpha - \beta + \gamma \alpha m \right) \Gamma \left( \mu + m \right)}{\Gamma \left( \gamma \mu \alpha - \beta + \gamma \alpha m \right) m!} \left( -r^{2} \right)^{m}
\]
\[
= 2 \Psi_{0} \left( \mu ; -; -r^{2} \right) = \frac{2}{(1 + r^{2})^{\mu}},
\]
i.e.
\[
S_{\mu}^{(\alpha, \beta)} \left( r; \left\{ n^{\gamma} \right\}_{n=1}^{\infty} \right) \leq M \left( r, \mu \right) \quad (r, \alpha, \beta, \gamma \in R^{+}, \gamma \left( \mu \alpha - \beta \right) > 1)
\] (2.27)

In Figure 1, we present some numerical results for three alternating series: \( \tilde{S} \left( r, \right), \tilde{S}_{2}^{(2,2)} \left( r; \left\{ n \right\}_{n=1}^{\infty} \right), \tilde{S}_{2}^{(2,5,2.1)} \left( r; \left\{ n^{3/2} \right\}_{n=1}^{\infty} \right) \) bounded by the constants \( M \left( r, 2 \right) \) with \( 0 < r < 3. \)
Figure 1. Alternating Mathieu type series $\tilde{S}(\alpha, \beta) \left( r; \{ n^\lambda \}_{n=1}^\infty \right)$ with $\alpha = 2$, $\beta = 1$, $\lambda = 1$, $\mu = 2$ (dashed), $\alpha = 2$, $\beta = 2$, $\lambda = 1$, $\mu = 2$ (dashdotted) and $\alpha = 2.5$, $\beta = 2.1$, $\lambda = 1.5$, $\mu = 2$ (dotted) as functions of $r$ with $0 < r < 3$ and their bound $M(r, 2)$ (solid line).

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