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ON A BOUNDED SUBCLASS OF CERTAIN ANALYTIC FUNCTIONS SATISFYING A DIFFERENTIAL INEQUALITY

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Abstract. In the present paper, using Jack's lemma, the authors investigate the differential inequality

$$\left|(1-\alpha)\frac{I_p(n,\lambda)f(z)}{z^p}+\alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}-1\right|<\mu,\ z\in E$$

regarding multivalent functions defined by multiplier transfomation in the open unit disk $E=\{z:|z|<1\}$. As consequences, sufficient conditions for univalence, starlikeness and strongly starlikeness of certain analytic functions are obtained.

1. Introduction

Let $\mathscr{A}_p$ denote the class of functions of the form $f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k$, $p\in\mathbb{N}=\{1,2,....\}$, which are analytic in the open unit disc $E=\{z:|z|<1\}$. We write $\mathscr{A}_1=\mathscr{A}$. A function $f\in\mathscr{A}_p$ is said to be p-valent starlike of order $\alpha(0\leqslant\alpha<p)$ in E if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\alpha,\quad z\in E.$$

We denote by $S_p^*(\alpha)$, the class of all such functions. A function $f\in\mathscr{A}_p$ is said to be p-valent convex of order $\alpha(0\leqslant\alpha<p)$ in E if

$$\operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\quad z\in E.$$

Let $K_p(\alpha)$ denote the class of all those functions $f\in\mathscr{A}_p$ which are multivalently convex of order α in E. Note that $S_1^*(\alpha)$ and $K_1(\alpha)$ are, respectively, the usual classes of univalent starlike functions of order α and univalent convex functions of order $\alpha, 0\leqslant\alpha<1$, and will be denoted here by $S^*(\alpha)$ and $K(\alpha)$, respectively. We shall use S^* and K to denote $S^*(0)$ and $K(0)$, respectively which are the classes of univalent starlike (w.r.t. the origin) and univalent convex functions.

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For $f \in \mathscr{A}_p$, we define the multiplier transformation $I_p(n,\lambda)$ as

$$I_p(n,\lambda)f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^n a_k z^k, \quad (\lambda \geqslant 0, n \in \mathbb{Z}). \tag{1}$$

The operator $I_p(n,\lambda)$ has been recently studied by Aghalary et.al. [1]. Earlier, the operator $I_1(n,\lambda)$ was investigated by Cho and Srivastava [3] and Cho and Kim [4], whereas the operator $I_1(n,1)$ was studied by Uralegaddi and Somanatha [13]. $I_1(n,0)$ is the well-known Sălăgean [8] derivative operator D^n, defined as: $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $f \in \mathscr{A}$.

A function $f \in \mathscr{A}_p$ is said to be in the class $S_n(p,\lambda,\alpha)$ for all z in E if it satisfies

$$\operatorname{Re}\left[\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)}\right] > \frac{\alpha}{p}, \tag{2}$$

for some $\alpha\,(0 \leqslant \alpha < p, p \in \mathbb{N})$. We note that $S_0(1,0,\alpha)$ and $S_1(1,0,\alpha)$ are the usual classes $S^*(\alpha)$ and $K(\alpha)$ of starlike functions of order α and convex functions of order α, respectively.

In 1989, Owa, Shen and Obradović [7] obtained a sufficient condition for a function $f \in \mathscr{A}$ to belong to the class $S_n(1,0,\alpha) = S_n(\alpha)$.

Recently, Li and Owa [6] studied the operator $I_1(n,0)$.

Let $\mathscr{H}_\alpha(\beta)$ denote the class of functions $f \in \mathscr{A}$ which satisfy the condition

$$\Re\left[(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > \beta, \quad z \in E,$$

where α and β are pre-assigned real numbers. Al-Amiri and Reade [2], in 1975, have shown that for $\alpha \leqslant 0$ and also for $\alpha = 1$, the functions in $\mathscr{H}_\alpha(0)$ are univalent in E. In 2005, Singh, Singh and Gupta [12] proved that for $0 < \alpha < 1$, functions in $\mathscr{H}_\alpha(\alpha)$ are also univalent. In 2007, Singh, Gupta and Singh [11] proved that functions in $\mathscr{H}_\alpha(\beta)$ satisfy the differential inequality $\Re\, f'(z) > 0,\ z \in E$ and hence are univalent for all real numbers α and β satisfying $\alpha \leqslant \beta < 1$ and that the result is sharp in the sense that the constant β cannot be replaced by any real number less than α. The starlikeness of the class $\mathscr{H}_\alpha(\beta)$ is still, an open problem.

In the present paper, the particular case where $p = 1, n = 1$ and $\lambda = 0$ of our main results reduces to

$$\left|(1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1\right| < \mu, \quad z \in E,$$

for $f \in \mathscr{A}$. In this particular case, we study the univalence, starlikeness and strongly starlikeness of this bounded form of $\mathscr{H}_\alpha(0)$ for $0 < \alpha < 2$.

2. Preliminaries

To prove our results, we shall make use of the following definition and lemmas.

DEFINITION 2.1. A function $f \in \mathscr{A}$ is said to be strongly starlike of order α, $0 < \alpha \leqslant 1$, if

$$\left|\arg \frac{zf^{'}(z)}{f(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in E$$

or, equivalently

$$\frac{zf^{'}(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\alpha}, \quad z \in E.$$

LEMMA 2.1. (Jack [5]) *Suppose $w(z)$ be a nonconstant analytic function in E with $w(0) = 0$. If $|w(z)|$ attains its maximum value at a point $z_0 \in E$ on the circle $|z| = r < 1$, then $z_0 w'(z_0) = mw(z_0)$, where $m, m \geqslant 1$ is some real number.*

LEMMA 2.2. [9] *Suppose $f \in \mathscr{A}$ be such that $f^{'}(z) \prec 1 + az$ in E, where $0 < a \leqslant 1$, then*

$$\frac{zf^{'}(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\mu}, \quad z \in E$$

where $0 < a \leqslant \frac{2\sin\left(\frac{\pi\mu}{2}\right)}{\sqrt{5+4\cos\left(\frac{\pi\mu}{2}\right)}}, \quad 0 < \mu < 1.$

LEMMA 2.3. [10] *Suppose $f \in \mathscr{A}$ be such that $f^{'}(z) \prec 1 + az$ in E, where $0 < a \leqslant \frac{1}{2}$, then*

$$\frac{zf^{'}(z)}{f(z)} \prec 1 + \left(\frac{3a}{2-a}\right)z, \quad z \in E$$

We, now, state and prove our main results.

3. Main Results

LEMMA 3.1. *If $u(z) = 1 + u_1 z + u_2 z^2 + \ldots$ be an analytic function in E and satisfies the condition*

$$(1-\alpha)u(z) + \alpha\left(1 + \beta\frac{zu^{'}(z)}{u(z)}\right) \prec 1 + \mu z, \quad z \in E, \tag{3}$$

for some $\alpha, \beta \in R$ with $\alpha \leqslant 1, 0 < \alpha\beta < 1$ and $0 < \mu \leqslant \frac{\alpha\beta}{2}$, then

$$u(z) \prec 1 + \frac{2\mu}{\alpha\beta}z, \quad z \in E.$$

Proof. Let us write
$$u(z) = 1 + \frac{2\mu}{\alpha\beta} w(z)$$
where $w(z)$ be analytic in E with $w(0) = 0$.

Now we will show that $|w(z)| < 1$, $z \in E$. If $|w(z)| \not< 1$, by lemma 2.1, there exists z_0, $|z_0| < 1$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$ where $k \geqslant 1$. When we put $w(z_0) = e^{i\theta}$, we have

$$\begin{aligned}
&\left|(1-\alpha)u(z_0) + \alpha\left(1 + \frac{\beta z_0 u'(z_0)}{u(z_0)}\right) - 1\right| \\
&= \left|(1-\alpha)\left(1 + \frac{2\mu}{\alpha\beta} w(z_0)\right) + \alpha\left(1 + \frac{\beta \frac{2\mu}{\alpha\beta} z_0 w'(z_0)}{1 + \frac{2\mu}{\alpha\beta} w(z_0)}\right) - 1\right| \\
&= \left|\frac{2(1-\alpha)\mu}{\alpha\beta} e^{i\theta} + \frac{2\mu k e^{i\theta}}{1 + \frac{2\mu}{\alpha\beta} e^{i\theta}}\right| \\
&\geqslant 2\mu \left|\frac{1-\alpha}{\alpha\beta} + \frac{1}{1 + \frac{2\mu}{\alpha\beta} e^{i\theta}}\right| \\
&\geqslant \frac{2\mu}{\alpha\beta + 2\mu}\left|(1-\alpha)\left(1 + \frac{2\mu}{\alpha\beta} e^{i\theta}\right) + \alpha\beta\right| \\
&= \frac{2\mu}{\alpha\beta + 2\mu}\left|1 - \alpha + \alpha\beta + \frac{2\mu(1-\alpha)}{\alpha\beta} e^{i\theta}\right| \\
&\geqslant \frac{2\mu}{\alpha\beta + 2\mu}\left|1 - \alpha + \alpha\beta - \frac{2\mu(1-\alpha)}{\alpha\beta}\right| \\
&\geqslant \mu
\end{aligned}$$

which is a contradiction to (3). Therefore, we must have $|w(z)| < 1, z \in E$.

Hence
$$u(z) \prec 1 + \frac{2\mu}{\alpha\beta} z, \quad z \in E. \qquad \square$$

LEMMA 3.2. *If $u(z) = 1 + u_1 z + u_2 z^2 + \ldots$ be an analytic function in E and satisfies the condition*
$$(1-\alpha)u(z) + \alpha\left(1 + \beta \frac{zu'(z)}{u(z)}\right) \prec 1 + \mu z, \quad z \in E, \tag{4}$$
for some $\alpha, \beta \in R$ with $\alpha \geqslant 1, 0 < \alpha\beta - 2(\alpha - 1) < 2$ and $0 < \mu \leqslant \frac{\alpha\beta - 2(\alpha-1)}{2}$, then
$$u(z) \prec 1 + \frac{2\mu}{\alpha\beta - 2(\alpha - 1)} z, \quad z \in E.$$

Proof. Let us write

$$u(z) = 1 + \frac{2\mu}{\alpha\beta - 2(\alpha - 1)} w(z)$$

where $w(z)$ be analytic in E with $w(0) = 0$.

Now we will show that $|w(z)| < 1, z \in E$. If $|w(z)| \nless 1$, by lemma 2.1, there exists z_0, $|z_0| < 1$ such that $|w(z_0)| = 1$ and $z_0 w'(z_0) = kw(z_0)$ where $k \geqslant 1$. When we put $w(z_0) = e^{i\theta}$, we have

$$\begin{aligned}
&\left|(1-\alpha)u(z_0) + \alpha\left(1 + \frac{\beta z_0 u'(z_0)}{u(z_0)}\right) - 1\right| \\
&= \left|(1-\alpha)\left(1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)} w(z_0)\right) + \alpha\left(1 + \frac{\beta \frac{2\mu}{\alpha\beta - 2(\alpha-1)} z_0 w'(z_0)}{1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)} w(z_0)}\right) - 1\right| \\
&= \left|\frac{2(1-\alpha)\mu}{\alpha\beta - 2(\alpha-1)} e^{i\theta} + \frac{\frac{2\alpha\beta\mu k e^{i\theta}}{\alpha\beta - 2(\alpha-1)}}{1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)} e^{i\theta}}\right| \\
&\geqslant \frac{2\mu}{\alpha\beta - 2(\alpha-1)} \left|\frac{(1-\alpha)\left(1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)} e^{i\theta}\right) + \alpha\beta}{1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)} e^{i\theta}}\right| \\
&\geqslant \frac{2\mu}{\alpha\beta - 2(\alpha-1)} \frac{\left|1 - \alpha + \alpha\beta - \frac{2\mu(\alpha-1)}{\alpha\beta - 2(\alpha-1)} e^{i\theta}\right|}{1 + \frac{2\mu}{\alpha\beta - 2(\alpha-1)}} \\
&\geqslant \mu
\end{aligned}$$

which is a contradiction to (4). Therefore, we must have $|w(z)| < 1, z \in E$.

Hence

$$u(z) \prec 1 + \frac{2\mu}{\alpha\beta - 2(\alpha - 1)} z, \quad z \in E. \qquad \square$$

THEOREM 3.1. *If $f \in \mathscr{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)f(z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $\alpha \leqslant 1, 0 < \frac{\alpha}{\lambda + p} < 1$ and $0 < \mu \leqslant \frac{\alpha}{2(\lambda + p)}$, then

$$\frac{I_p(n,\lambda)f(z)}{z^p} \prec 1 + \frac{2(\lambda + p)\mu}{\alpha} z, \quad z \in E.$$

Proof. Let us write

$$\frac{I_p(n,\lambda)f(z)}{z^p} = u(z), \quad z \in E.$$

Differentiate logarithmetically, we obtain

$$\frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} - p = \frac{zu'(z)}{u(z)}, \quad z \in E. \tag{5}$$

Using the fact that

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z)$$

Thus (5) reduces to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 + \frac{zu'(z)}{(\lambda+p)u(z)}.$$

By taking $\beta = \frac{1}{\lambda+p}$, the proof, now follows by lemma 3.1. □

THEOREM 3.2. *If $f \in \mathscr{A}_p$ satisfies*

$$(1-\alpha)\frac{I_p(n,\lambda)f(z)}{z^p} + \alpha\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $\alpha \geqslant 1, 0 < \frac{\alpha}{\lambda+p} - 2(\alpha-1) < 2$ and $0 < \mu \leqslant \frac{\frac{\alpha}{\lambda+p} - 2(\alpha-1)}{2}$, then

$$\frac{I_p(n,\lambda)f(z)}{z^p} \prec 1 + \frac{2\mu}{\frac{\alpha}{\lambda+p} - 2(\alpha-1)}z, \quad z \in E.$$

Proof. Let us write

$$\frac{I_p(n,\lambda)f(z)}{z^p} = u(z), \quad z \in E.$$

Differentiate logarithmetically, we obtain

$$\frac{zI_p'(n,\lambda)f(z)}{I_p(n,\lambda)f(z)} - p = \frac{zu'(z)}{u(z)}, \quad z \in E. \tag{6}$$

Using the fact that

$$zI_p'(n,\lambda)f(z) = (p+\lambda)I_p(n+1,\lambda)f(z) - \lambda I_p(n,\lambda)f(z).$$

Thus (6) reduces to

$$\frac{I_p(n+1,\lambda)f(z)}{I_p(n,\lambda)f(z)} = 1 + \frac{zu'(z)}{(\lambda+p)u(z)}.$$

By taking $\beta = \frac{1}{\lambda+p}$, the proof, now follows by lemma 3.2. □

4. Corollaries

By taking $p = 1$ and $\lambda = 0$ in theorem 3.1. We have the following corollary.

COROLLARY 4.1. *If $f \in \mathscr{A}$ satisfies*

$$(1-\alpha)\frac{D^n f(z)}{z} + \alpha\frac{D^{n+1} f(z)}{D^n f(z)} \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $0 < \alpha < 1$ and $0 < \mu \leqslant \frac{\alpha}{2}$, then

$$\frac{D^n f(z)}{z} \prec 1 + \frac{2\mu}{\alpha} z, \quad z \in E.$$

By taking $p = 1$ and $\lambda = 0$ in theorem 3.2. We have the following corollary.

COROLLARY 4.2. *If $f \in \mathscr{A}$ satisfies*

$$(1-\alpha)\frac{D^n f(z)}{z} + \alpha\frac{D^{n+1} f(z)}{D^n f(z)} \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $1 \leqslant \alpha < 2$ and $0 < \mu \leqslant \frac{2-\alpha}{2}$, then

$$\frac{D^n f(z)}{z} \prec 1 + \frac{2\mu}{2-\alpha} z, \quad z \in E.$$

By taking $p = 1, n = 1$ and $\lambda = 0$ in theorem 3.1. We have the following corollary.

COROLLARY 4.3. *If $f \in \mathscr{A}$ satisfies*

$$(1-\alpha) f'(z) + \alpha\left(1 + \frac{z f''(z)}{f'(z)}\right) \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $0 < \alpha < 1$ and $0 < \mu \leqslant \frac{\alpha}{2}$, then

$$f'(z) \prec 1 + \frac{2\mu}{\alpha} z, \quad z \in E$$

and therefore f is univalent in E.

By taking $p = 1, n = 1$ and $\lambda = 0$ in theorem 3.2. We have the following corollary.

COROLLARY 4.4. *If $f \in \mathscr{A}$ satisfies*

$$(1-\alpha) f'(z) + \alpha\left(1 + \frac{z f''(z)}{f'(z)}\right) \prec 1 + \mu z, \quad z \in E$$

for some $\alpha \in R$ with $1 \leqslant \alpha < 2$ and $0 < \mu \leqslant \frac{2-\alpha}{2}$, then

$$f'(z) \prec 1 + \frac{2\mu}{2-\alpha} z, \quad z \in E$$

and therefore f is univalent in E.

The corollary 4.3, together with lemma 2.2, gives the following result.

COROLLARY 4.5. *If* $f \in \mathscr{A}$ *satisfies*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\mu z, \quad z \in E$$

for some $\alpha \in R$ *with* $0<\alpha<1$ *and* $0<\mu \leqslant \frac{\alpha}{2}$, *then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \quad z \in E$$

where $0<\frac{2\mu}{\alpha} \leqslant \frac{2\sin\left(\frac{\pi\delta}{2}\right)}{\sqrt{5+4\cos\left(\frac{\pi\delta}{2}\right)}}$ $,0<\delta<1$ *and hence* f *is strongly starlike for* $0<\alpha<1$ *in* E.

The corollary 4.4, together with lemma 2.2, gives the following result.

COROLLARY 4.6. *If* $f \in \mathscr{A}$ *satisfies*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\mu z, \quad z \in E$$

for some $\alpha \in R$ *with* $1 \leqslant \alpha<2$ *and* $0<\mu \leqslant \frac{2-\alpha}{2}$, *then*

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \quad z \in E$$

where $0<\frac{2\mu}{2-\alpha} \leqslant \frac{2\sin\left(\frac{\pi\delta}{2}\right)}{\sqrt{5+4\cos\left(\frac{\pi\delta}{2}\right)}}$ $,0<\delta<1$ *and hence* f *is strongly starlike for* $1 \leqslant \alpha<2$ *in* E.

The corollary 4.3, together with lemma 2.3, gives the following result.

COROLLARY 4.7. *If* $f \in \mathscr{A}$ *satisfies*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\mu z, \quad z \in E$$

for some $\alpha \in R$ *with* $0<\alpha<1$ *and* $0<\mu \leqslant \frac{\alpha}{4}$, *then*

$$\frac{zf'(z)}{f(z)} \prec 1+\frac{3\mu}{\alpha-\mu}z, \quad z \in E$$

and hence f *is starlike for* $0<\alpha<1$ *in* E.

The corollary 4.4, together with lemma 2.3, gives the following result.

COROLLARY 4.8. *If $f \in \mathscr{A}$ satisfies*

$$(1-\alpha)f'(z)+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right) \prec 1+\mu z, \quad z \in E$$

for some $\alpha \in R$ with $1 \leqslant \alpha < 2$ and $0 < \mu \leqslant \frac{2-\alpha}{4}$, then

$$\frac{zf'(z)}{f(z)} \prec 1+\frac{3\mu}{2-\alpha-\mu}z, \quad z \in E$$

and hence f is starlike for $1 \leqslant \alpha < 2$ in E.

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