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CHARACTERIZATIONS OF CONVEX FUNCTIONS OF A VECTOR VARIABLE VIA HERMITE–HADAMARD'S INEQUALITY

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(*communicated by J. Pečarić*)

Abstract. The classical Hermite-Hadamard inequality characterizes the continuous convex functions of one real variable. The aim of the present paper is to give an analogous characterization for functions of a vector variable.

1. The Hermite-Hadamard inequality

In a letter sent on November 22, 1881, to the journal *Mathesis* (and published there two years later), Ch. Hermite [10] noted that every convex function $f : [a,b] \to \mathbf{R}$ satisfies the inequalities

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2}. \tag{1}$$

The left-hand side inequality was rediscovered ten years later by J. Hadamard [7]. Nowadays, the double inequality (1) is called *the Hermite-Hadamard inequality*. The interested reader can find its complete story in the historical note by D. S. Mitrinović and I. B. Lacković [12].

The Hermite-Hadamard inequality has evoked the interest of many mathematicians. Especially in the last three decades it has been intensively investigated and generalized in several directions. For instance, a dual Hermite-Hadamard inequality was discussed by C. P. Niculescu [13], while a complete extension of (1) to the class of n-convex functions was obtained by M. Bessenyei and Zs. Páles [1]. Likewise, M. Bessenyei and Zs. Páles [2] proved a generalization of (1) for real-valued functions defined on an open interval $I \subseteq \mathbf{R}$, which are convex with respect to a so-called positive regular pair over I. For an account on various results dealing with the Hermite-Hadamard inequality, the reader is referred to the monograph by S. S. Dragomir and C. E. M. Pearce [5].

In what follows, we are concerned with Hermite-Hadamard-type inequalities for functions of a vector variable. As pointed out by C. P. Niculescu [15], the Choquet theory (see [17]) provides the framework for a natural extension of (1) to such functions. For the reader's convenience, we briefly present here this extension. Let E be a real locally

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convex Hausdorff space, let K be a compact convex subset of E, and let μ be a positive Borel measure on K. Then there exists a unique point $x_\mu \in K$, satisfying

$$f(x_\mu) = \frac{1}{\mu(K)} \int_K f(x) d\mu(x)$$

for every continuous linear functional $f : E \to \mathbf{R}$ (see [16, Lemma 4.1.8]). The unique point in K with this property is called the μ-*barycenter* of K. Given another positive Borel measure λ on K, one says that μ *is majorized by* λ (abbreviated $\mu \prec \lambda$) if

$$\frac{1}{\mu(K)} \int_K f(x) d\mu(x) \leqslant \frac{1}{\lambda(K)} \int_K f(x) d\lambda(x)$$

holds for every continuous convex function $f : K \to \mathbf{R}$.

THE CHOQUET THEOREM. *Let E be a real locally convex Hausdorff space, let K be a compact convex subset of E, and let μ be a positive Borel measure on K. Then there exists a probability Borel measure λ on K satisfying the following conditions:*

(i) *$\mu \prec \lambda$ and μ and λ have the same barycenter;*

(ii) *λ is concentrated on* $\operatorname{Ext} K$*, the set of all extreme points of K (i.e., $\lambda(K \setminus \operatorname{Ext} K) = 0$).*

It should be remarked that, according to a well known result, $\operatorname{Ext} K$ is a G_δ-subset of K, hence a Borel set. Under the hypotheses of the above theorem, it holds that

$$f(x_\mu) \leqslant \frac{1}{\mu(K)} \int_K f(x) d\mu(x) \leqslant \int_{\operatorname{Ext} K} f(x) d\lambda(x) \tag{2}$$

for every continuous convex function $f : K \to \mathbf{R}$. Inequality (2) is the natural generalization of (1) for functions of a vector variable.

When $E = \mathbf{R}^n$, the Euclidean n-dimensional space, then we have

$$x_\mu = \frac{1}{\mu(K)} \int_K x d\mu(x),$$

i.e., the barycenter coincides with the first order moment of μ. If, moreover, $K = [a_0, a_1, \ldots, a_n]$ is an arbitrary n-dimensional simplex in $\mathbf{R}^n$, then (2) becomes

$$\begin{aligned} f(x_\mu) &\leqslant \frac{1}{\mu(K)} \int_K f(x) d\mu(x) \\ &\leqslant \frac{1}{m(K)} \sum_{k=0}^{n} m([a_0, \ldots, \widehat{a_k}, \ldots, a_n]) f(a_k) \end{aligned} \tag{3}$$

for every continuous convex function $f : K \to \mathbf{R}$. Here $[a_0, \ldots, \widehat{a_k}, \ldots, a_n]$ represents the simplex obtained by replacing a_k by x_μ (the subsimplex of K opposite to a_k when adding x_μ as a new vertex), while m denotes the Lebesgue measure in $\mathbf{R}^n$.

The inequality (3) can be further specialized by choosing $E = \mathbf{R}$ and $K = [a, b]$. In this case (3) becomes

$$f(x_\mu) \leqslant \frac{1}{\mu([a,b])} \int_a^b f(x) d\mu(x) \leqslant \frac{b - x_\mu}{b - a} f(a) + \frac{x_\mu - a}{b - a} f(b) \tag{4}$$

for every continuous convex function $f : [a,b] \to \mathbf{R}$, which is also a generalization of (1).

We note that the left-hand side inequality in (2), (3) and (4), respectively, works not only for positive Borel measures, but also for some signed Borel measures satisfying an additional condition – the so-called Steffensen-Popoviciu measures (see [14] and [16, Chapter 4]). Likewise, the right-hand side inequality in (4) works also for some signed Borel measures (see A. M. Fink [6]). Finally, we mention that an extension of (1) for functions f which are not convex on the whole interval $[a,b]$ was recently obtained by P. Czinder [4].

2. Characterizations of convex functions via Hermite-Hadamard's inequality

It is interesting to note that the Hermite-Hadamard inequality (1) is not merely a consequence of convexity, but also characterizes it. More precisely, the following result holds:

THEOREM 1. *Given an open interval $I \subseteq \mathbf{R}$ and a continuous function $f : I \to \mathbf{R}$, the following assertions are equivalent:*

1° *f is convex.*

2° *For all elements $x < y$ of I, it holds that*

$$f\left(\frac{x+y}{2}\right) \leqslant \frac{1}{y-x}\int_x^y f(t)dt.$$

3° *For all elements $x < y$ of I, it holds that*

$$\frac{1}{y-x}\int_x^y f(t)dt \leqslant \frac{f(x)+f(y)}{2}.$$

As noted by D. S. Mitrinović and I. B. Lacković [12] "it remains unclear who and when made the transition from the inequality (1) to the convexity criterion given in Theorem 1." For instance, the implication $2^\circ \Rightarrow 1^\circ$ appears in the classical book by G. H. Hardy, J. E. Littlewood and G. Pólya [8, p. 98], while the implication $3^\circ \Rightarrow 1^\circ$ appears in the book by A. W. Roberts and D. E. Varberg [20, Problem Q, p. 15] (see also M. Kuczma [11, Exercise 8, p. 205]). A similar characterization of convexity with respect to a positive regular pair over I was recently obtained by M. Bessenyei and Zs. Páles [3]. Likewise, an analogue of Theorem 1, giving a characterization of convex set-valued functions, was obtained by B. Piatek [18] For other generalizations of Theorem 1 the reader is referred to [19] and [9].

It should be remarked that all the above mentioned characterizations of convexity by means of the Hermite-Hadamard inequality deal with functions of a single variable. Similar characterizations for functions of a vector variable are lacking in the literature. The main purpose of the present paper is to fill this gap by proving a counterpart of Theorem 1 for functions f defined on an open convex set $C \subseteq \mathbf{R}^n$. More precisely, we deal with the following question: when the validity of the left-hand side or of the right-hand side inequality in (2) implies the convexity of f ? First of all, we note that, in general, the Borel probability measure λ whose existence is ensured by the Choquet theorem is not unique, except for the case of simplices (see [17, Chapter 9]). On the other hand, if one looks for characterizations of convexity by means of inequality (2), then it is natural to keep at minimum the family of compact convex sets $K \subset C$, as well as the family of measures μ for which (2) holds. As we will see in the next section, the validity of (2) for every n-dimensional simplex $K \subset C$ (i.e., the validity of (3)) implies the convexity of f.

3. Main result

Throughout this section m will denote the Lebesgue measure on $\mathbf{R}^n$, while Σ_n will denote the standard n-dimensional simplex in $\mathbf{R}^n$, i.e.,

$$\Sigma_n = \{ (u_1, \dots, u_n) \mid u_1, \dots, u_n \in [0, \infty),\ u_1 + \cdots + u_n \leqslant 1 \}.$$

THEOREM 2. *Given a nonempty open convex set $C \subseteq \mathbf{R}^n$ and a continuous function $f : C \to \mathbf{R}$, the following assertions are equivalent:*

1° *f is convex.*

2° *For every n-dimensional simplex $K = [a_0, a_1, \dots, a_n] \subset C$ it holds that*

$$f\left(\frac{a_0 + a_1 + \cdots + a_n}{n+1}\right) \leqslant \frac{1}{m(K)} \int_K f(x)dm(x). \tag{1}$$

3° *For every n-dimensional simplex $K = [a_0, a_1, \dots, a_n] \subset C$ it holds that*

$$\frac{1}{m(K)} \int_K f(x)dm(x) \leqslant \frac{f(a_0) + f(a_1) + \cdots + f(a_n)}{n+1}. \tag{2}$$

Proof. We have to prove only that $2^\circ \Rightarrow 1^\circ$ and $3^\circ \Rightarrow 1^\circ$.

$2^\circ \Rightarrow 1^\circ$ We proceed by reductio ad absurdum. Assuming that f is not convex, it follows that there exist two points $\bar{a}_0, \bar{b}_0 \in C$, as well as a number $\alpha \in (0,1)$ such that

$$f(\bar{x}) > \alpha f(\bar{a}_0) + (1-\alpha) f(\bar{b}_0),$$

where $\bar{x} := \alpha \bar{a}_0 + (1-\alpha)\bar{b}_0$. Taking into account that C is open and that f is continuous, we can select n points $\bar{a}_1, \dots, \bar{a}_n \in C$, sufficiently close to $\bar{b}_0$, satisfying the following conditions:

(i) the vectors $\{\bar{a}_1 - \bar{a}_0, ..., \bar{a}_n - \bar{a}_0\}$ are linear independent, i.e., $\bar{K} := [\bar{a}_0, \bar{a}_1, ..., \bar{a}_n]$ is an n-dimensional simplex;

(ii) $\bar{b}_0 = \dfrac{\bar{a}_1 + \cdots + \bar{a}_n}{n}$;

(iii) $f(\bar{x}) > \alpha f(\bar{a}_0) + (1-\alpha) \dfrac{f(\bar{a}_1) + \cdots + f(\bar{a}_n)}{n}$.

Setting $\alpha_0 := \alpha$, $\alpha_1 := \cdots = \alpha_n := \dfrac{1-\alpha}{n}$, we have $\alpha_0, \alpha_1, \dots, \alpha_n \in (0,1)$,

$$\alpha_0 + \alpha_1 + \cdots + \alpha_n = 1, \qquad \bar{x} = \alpha_0 \bar{a}_0 + \alpha_1 \bar{a}_1 + \cdots + \alpha_n \bar{a}_n,$$

and

$$f(\bar{x}) > \alpha_0 f(\bar{a}_0) + \alpha_1 f(\bar{a}_1) + \cdots + \alpha_n f(\bar{a}_n). \tag{3}$$

Further, let $a \in \mathbf{R}^n$ and $c \in \mathbf{R}$ be chosen such that the function $g : C \to \mathbf{R}$, defined by $g(x) := f(x) - \langle a, x \rangle - c$, satisfies $g(\bar{a}_0) = g(\bar{a}_1) = \cdots = g(\bar{a}_n) = 0$ (the existence of a and c is ensured by condition (i)). It is immediately seen that g is continuous and satisfies an inequality similar to (1), namely

$$g\left(\frac{a_0 + a_1 + \cdots + a_n}{n+1}\right) \leqslant \frac{1}{m(K)} \int_K g(x)dm(x) \tag{4}$$

for every n-dimensional simplex $K = [a_0, a_1, \ldots, a_n] \subset C$. In addition, due to (3) we have

$$\begin{aligned} g(\bar{x}) &= f(\bar{x}) - \langle a, \bar{x} \rangle - c \\ &> \sum_{j=0}^{n} \alpha_j f(\bar{a}_j) - \left\langle a, \sum_{j=0}^{n} \alpha_j \bar{a}_j \right\rangle - c \\ &= \sum_{j=0}^{n} \alpha_j [f(\bar{a}_j) - \langle a, \bar{a}_j \rangle - c] \\ &= \sum_{j=0}^{n} \alpha_j g(\bar{a}_j) = 0. \end{aligned}$$

Let $x_0 \in \bar{K}$ be the point at which g attains its maximum on $\bar{K}$. Since

$$g(\bar{x}) > 0 = g(\bar{a}_0) = g(\bar{a}_1) = \cdots = g(\bar{a}_n),$$

it follows that $g(x_0) > 0$ and $x_0 \notin \{ \bar{a}_0, \bar{a}_1, \ldots, \bar{a}_n \}$. On the other hand, x_0 is a convex combination of $r+1$ vertices of $\bar{K}$ (with $r \leqslant n$). Without loss of generality, we may assume that these vertices are $\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_r$. Therefore, there exist real numbers $\lambda_0, \lambda_1, \ldots, \lambda_r \in (0,1)$ such that

$$\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1 \qquad \text{and} \qquad x_0 = \lambda_0 \bar{a}_0 + \lambda_1 \bar{a}_1 + \cdots + \lambda_r \bar{a}_r.$$

Also without losing the generality we may assume that

$$\lambda_0 = \max \{ \lambda_0, \lambda_1, \ldots, \lambda_r \}.$$

Then we can select the points $a_0 := \bar{a}_0$, $a_1 \in [\bar{a}_0, \bar{a}_1]$, $\ldots$, $a_r \in [\bar{a}_0, \bar{a}_r]$ such that $x_0 = \dfrac{a_0 + a_1 + \cdots + a_r}{r+1}$. Obviously we have

$$[a_0, a_1, \ldots, a_r] \subseteq [\bar{a}_0, \bar{a}_1, \ldots, \bar{a}_r].$$

Now we construct for each $j \in \{r+1, \ldots, n\}$ a sequence $(a_j^p)_{p \geqslant 1}$ of points in $\mathbf{R}^n$ with the following properties:

- $\lim_{p\to\infty} a_j^p = x_0$ for every $j \in \{r+1, \ldots, n\}$;
- $K_p := [a_0, a_1, \ldots, a_r, a_{r+1}^p, \ldots, a_n^p]$ is an n-dimensional simplex contained in $\bar{K}$.

According to (4), we have

$$g\left(\frac{a_0+a_1+\cdots+a_r+a_{r+1}^p+\cdots+a_n^p}{n+1}\right)\leqslant\frac{1}{m(K_p)}\int_{K_p}g(x)dm(x) \tag{5}$$

for every positive integer p.

Given any positive integer p, let A_p be the $n\times n$ matrix whose columns are the vectors $a_1-a_0,\ldots,a_r-a_0,a_{r+1}^p-a_0,\ldots,a_n^p-a_0$. Then the transform $\varphi_p:\mathbf{R}^n\to\mathbf{R}^n$, defined by $\varphi_p(u):=a_0+A_pu$, maps Σ_n onto K_p. Its Jacobian equals

$$\det A_p=n!\,m(K_p).$$

Making the change of variables $x=\varphi_p(u)=a_0+A_pu$ in the integral in (5), we deduce that

$$g\left(\frac{a_0+a_1+\cdots+a_r+a_{r+1}^p+\cdots+a_n^p}{n+1}\right)\leqslant n!\int_{\Sigma_n}g(a_0+A_pu)dm(u)$$

for every positive integer p. Letting $p\to\infty$, we get

$$g(x_0)\leqslant n!\int_{\Sigma_n}g(a_0+Au)dm(u), \tag{6}$$

where A is the $n\times n$ matrix whose columns are the vectors

$$a_1-a_0,\ldots,a_r-a_0,\underbrace{x_0-a_0,\ldots,x_0-a_0}_{n-r\text{ times}}.$$

Consider the function $h:\Sigma_n\to\mathbf{R}$, defined by $h(u):=g(a_0+Au)$. Obviously, h is continuous on Σ_n. Since the point a_0+Au lies in $\bar{K}$ for every $u\in\Sigma_n$, it follows that

$$h(u)=g(a_0+Au)\leqslant g(x_0)\qquad\text{for every } u\in\Sigma_n.$$

On the other hand, $h(0_n)=g(a_0)=g(\bar{a}_0)=0<g(x_0)$. The continuity of h ensures now that

$$\begin{aligned} n!\int_{\Sigma_n}g(a_0+Au)dm(u) &= n!\int_{\Sigma_n}h(u)dm(u)\\ &< n!\int_{\Sigma_n}g(x_0)dm(u)\\ &= g(x_0),\end{aligned}$$

contradicting the inequality (6). This contradiction shows that f is convex.

$3^\circ\Rightarrow1^\circ$ Suppose that f is not convex. Then there exist two points $\bar{a}_0,\bar{b}_0\in C$ as well as a number $\alpha\in(0,1)$ such that

$$f(\bar{x})>\alpha f(\bar{a}_0)+(1-\alpha)f(\bar{b}_0),$$

where $\bar{x}:=\alpha\bar{a}_0+(1-\alpha)\bar{b}_0$.

Choose $a \in \mathbf{R}^n$ and $c \in \mathbf{R}$ such that the function $g : C \to \mathbf{R}$, defined by $g(x) := f(x) - \langle a, x \rangle - c$, satisfies $g(\bar{a}_0) = g(\bar{b}_0) = 0$ (this is possible because $\bar{a}_0 \neq \bar{b}_0$). It is immediately seen that g is continuous and satisfies an inequality similar to (2), namely

$$\frac{1}{m(K)} \int_K g(x)dm(x) \leqslant \frac{g(a_0) + g(a_1) + \cdots + g(a_n)}{n+1} \tag{7}$$

for every n-dimensional simplex $K = [a_0, a_1, \ldots, a_n] \subset C$. Likewise, one has $g(\bar{x}) > 0$. Set

$$\begin{aligned}
&S_1 := \{ \lambda \in (\alpha, 1] \mid g(\lambda \bar{a}_0 + (1-\lambda)\bar{b}_0) = 0 \}, \qquad \lambda_0 := \inf S_1,\\
&S_2 := \{ \mu \in [0, \alpha) \mid g(\mu \bar{a}_0 + (1-\mu)\bar{b}_0) = 0 \}, \qquad \mu_0 := \sup S_2,\\
&a_0 := \lambda_0 \bar{a}_0 + (1-\lambda_0)\bar{b}_0,\\
&b_0 := \mu_0 \bar{a}_0 + (1-\mu_0)\bar{b}_0.
\end{aligned}$$

The continuity of g ensures that

$$\begin{aligned}
&\bar{x} \in (a_0, b_0) \subset [\bar{a}_0, \bar{b}_0],\\
&g(a_0) = g(b_0) = 0,\\
&g(x) > 0 \qquad \text{for every } x \in (a_0, b_0).
\end{aligned}$$

Here (a_0, b_0) denotes the open line segment whose endpoints are a_0 and b_0.

Now we construct for each $j \in \{1, \ldots, n\}$ a sequence $(a_j^p)_{p \geqslant 1}$ of points in $\mathbf{R}^n$ with the following properties:

- $\lim_{p \to \infty} a_j^p = b_0$ for every $j \in \{1, \ldots, n\}$;
- $K_p := [a_0, a_1^p, \ldots, a_n^p]$ is an n-dimensional simplex contained in C.

According to (7), we have

$$\frac{1}{m(K_p)} \int_{K_p} g(x)dm(x) \leqslant \frac{g(a_1^p) + \cdots + g(a_n^p)}{n+1} \tag{8}$$

for every positive integer p.

Given any positive integer p, let A_p be the $n \times n$ matrix whose columns are the vectors $a_1^p - a_0, \ldots, a_n^p - a_0$. Then the transform $\varphi_p : \mathbf{R}^n \to \mathbf{R}^n$, defined by $\varphi_p(u) := a_0 + A_p u$, maps Σ_n onto K_p. Making the change of variables $x = \varphi_p(u) = a_0 + A_p u$ in the integral in (8), we deduce that

$$n! \int_{\Sigma_n} g(a_0 + A_p u)dm(u) \leqslant \frac{g(a_1^p) + \cdots + g(a_n^p)}{n+1}$$

for every positive integer p. Letting $p \to \infty$ and using the continuity of g, we obtain

$$n! \int_{\Sigma_n} g(a_0 + Au)dm(u) \leqslant 0, \tag{9}$$

where A is the $n \times n$ matrix whose columns are the vectors

$$\underbrace{b_0 - a_0, \ldots, b_0 - a_0}_{n \text{ times}}.$$

Consider the function $h:\Sigma_n \to \mathbf{R}$, defined by $h(u) := g(a_0 + Au)$. Clearly, h is continuous on Σ_n. Since

$$a_0 + Au = (1 - u_1 - \cdots - u_n)a_0 + (u_1 + \cdots + u_n)b_0 \in [a_0, b_0]$$

for every $u = (u_1, \ldots, u_n) \in \Sigma_n$, it follows that

$$h(u) = g(a_0 + Au) \geqslant 0 \qquad \text{for every } u \in \Sigma_n.$$

Taking into account that $g(x) > 0$ for $x \in (a_0, b_0)$, the function h does not vanish on Σ_n, hence

$$\int_{\Sigma_n} h(u)dm(u) = \int_{\Sigma_n} g(a_0 + Au)dm(u) > 0,$$

contradicting (9). This contradiction shows that f is convex. □

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