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REVERSE WEIGHTED L_P–NORM INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper, we give the new type of reverse weighted L_p norm inequalities and their important applications to studying stability of some inverse problems. Especially, we will see their applications to inverse problems in non-homogeneous linear differential equations.

1. Introduction

We have the following famous reverse Hölder inequality

PROPOSITION 1. ([5]) *For two positive functions f and g satisfying*

$$0 < m \leqslant \frac{f}{g} \leqslant M < \infty \tag{1.1}$$

on the set X, and for $p,q > 1$, $p^{-1} + q^{-1} = 1$,

$$\left(\int_X f\,d\mu\right)^{\frac{1}{p}} \left(\int_X g d\mu\right)^{\frac{1}{q}} \leqslant A_{p,q}\left(\frac{m}{M}\right) \int_X f^{\frac{1}{p}} g^{\frac{1}{q}} d\mu, \tag{1.2}$$

if the right hand side integral converges. Here

$$A_{p,q}(t) = p^{-\frac{1}{p}} q^{-\frac{1}{q}} \frac{t^{-\frac{1}{pq}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}} \left(1-t^{\frac{1}{q}}\right)^{\frac{1}{q}}}.$$

By using Proposition 1, S. Saitoh, V. K. Tuan, M. Yamamoto ([7]) obtain the following reverse weighted L_p norm inequality

PROPOSITION 2. ([7]) *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(x) \leqslant M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(x) \leqslant M_2^{\frac{1}{p}} < \infty, \quad p > 1, \ x \in \mathbb{R}. \tag{1.3}$$

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Then for any positive continuous functions ρ_1 and ρ_2 we have the reverse L_p- weighted convolution inequality

$$\left\|((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_p \geqslant A_{p,q}^{-1}\left(\frac{m_1m_2}{M_1M_2}\right)\|F_1\|_{L_p(\mathbb{R},\rho_1)}\|F_2\|_{L_p(\mathbb{R},\rho_2)}, \quad (1.4)$$

if the left hand side is finite.

In formula (1.4) replacing ρ_2 by 1, and $F_2(x-\xi)$ by $G(x-\xi)$, and integrating with respect to x from c to d, the authors ([7]) arrive at the following inequality

$$\begin{aligned}&\int_c^d\left(\int_{-\infty}^{+\infty}F(\xi)\rho(\xi)G(x-\xi)d\xi\right)^p dx\\ &\quad\geqslant A_{p,q}^{-p}\left(\frac{m}{M}\right)\left(\int_{-\infty}^{+\infty}\rho(\xi)d\xi\right)^{p-1}\int_{-\infty}^{+\infty}F^p(\xi)\rho(\xi)d\xi\int_{c-\xi}^{d-\xi}G^p(x)dx\end{aligned} \quad (1.5)$$

if positive continuous functions ρ, F and G satisfy

$$0<m^{\frac{1}{p}}\leqslant F(\xi)G(x-\xi)\leqslant M^{\frac{1}{p}}<\infty,\quad x\in[c,d],\ \ \xi\in\mathbb{R}. \quad (1.6)$$

Inequality (1.5) is especially impotant when $G(x-\xi)$ is a Green's function.

Recently, N. D. V. Nhan and D. T. Duc([6]) introduced new type of convolution inequality in weighted $L_p(\mathbb{R}^2)$

PROPOSITION 3. ([6]) *For two non-vanishing functions $\rho_j(x,y)(j=1,2)$ belonging to $L_1(\mathbb{R}^2,dxdy)$ and for $p>1$ we have the L_p weighted convolution inequality*

$$\left\|((F_1\rho_1)*(F_2\rho_2))(\rho_1*\rho_2)^{\frac{1}{p}-1}\right\|_{L_p(\mathbb{R}^2)}\leqslant\|F_1\|_{L_p(\mathbb{R}^2,|\rho_1|)}\|F_2\|_{L_p(\mathbb{R}^2,|\rho_2|)}, \quad (1.7)$$

for $F_j(\xi,\tau)\in L_p(\mathbb{R}^2,|\rho_j(\xi,\tau)|d\xi d\tau)$ $(j=1,2)$. Equality holds for F_j if and only if F_j are represented in the form

$$F_j(\xi,\tau)=C_je^{\alpha\xi+\beta\tau};\quad C_j:\ \text{constant}, \quad (1.8)$$

where α,β are two constants such that $F_j(\xi,\tau)\in L_p(\mathbb{R}^2,|\rho_j(\xi,\tau)|d\xi d\tau)$ $(j=1,2)$.

Moreover, the authors ([6]) derived new type norm inequality which is important applications to non-homogeneous linear differential equations

PROPOSITION 4. ([6]) *For two non-vanishing functions $\rho_j(x,y)$ $(j=1,2)$ belonging to $L_1(\mathbb{R}^2,dxdy)$, we have the inequality*

$$\begin{aligned}&\int_{-\infty}^{+\infty}\frac{\left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau\right|^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\rho_1(\xi,\tau)||\rho_2(x-\xi,\tau)|d\xi d\tau\right)^{p-1}}dx\\ &\qquad\leqslant\left\|\|F_1\|^p_{L_p(\mathbb{R},|\rho_1|d\xi)}\right\|_{L_p(\mathbb{R},d\tau)}\left\|\|F_2\|^p_{L_p(\mathbb{R},|\rho_2|d\xi)}\right\|_{L_q(\mathbb{R},d\tau)},\end{aligned} \quad (1.9)$$

for $p>1,q>1,p^{-1}+q^{-1}=1$ and F_j $(j=1,2)$ are such that the right hand side of (1.9) *is finite.*

In this paper, by using the reverse Hölder inequality, we give the new type of reverse weighted L_p norm inequalities and their important applications to studying stability of some inverse problems.

2. New Reverse Weighted L_p Norm Inequalities

Our main results are the following

THEOREM 1. *Let F_1 and F_2 be positive functions satisfying*

$$0 < m_1^{\frac{1}{p}} \leqslant F_1(\xi,\tau) \leqslant M_1^{\frac{1}{p}} < \infty, \quad 0 < m_2^{\frac{1}{p}} \leqslant F_2(\xi,\tau) \leqslant M_2^{\frac{1}{p}} < \infty, \quad (\xi,\tau) \in \mathbb{R}^2. \tag{2.10}$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the reverse L_p $(p>1)-$ weighted convolution inequality

$$\left\| ((F_1\rho_1) * (F_2\rho_2))(\rho_1 * \rho_2)^{\frac{1}{p}-1} \right\|_{L_p(\mathbb{R}^2)} \geqslant A_{p,q}^{-2}\left(\frac{m_1m_2}{M_1M_2}\right) \|F_1\|_{L_p(\mathbb{R}^2,\rho_1)} \|F_2\|_{L_p(\mathbb{R}^2,\rho_2)}. \tag{2.11}$$

Inequality (2.11) and others should be understood in the sense that if the left hand side is finite, then so is the right hand side, and in this case the inequality holds.

Proof. Let

$$f(\xi) = F_1^p(\xi,\tau)F_2^p(x-\xi,y-\tau)\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)$$

and

$$g(\xi) = \rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau), \quad \forall (x,y) \in \mathbb{R}^2, \quad \forall (\xi,\tau) \in \mathbb{R}^2.$$

The condition (2.10) implies

$$m_1m_2 \leqslant \frac{f(\xi)}{g(\xi)} \leqslant M_1M_2, \quad \xi \in \mathbb{R}.$$

Hence, one can apply the reverse Hölder inequality (1.2) for f and g to get

$$\begin{aligned} &A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{-\infty}^{+\infty} F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,y-\tau)\rho_2(x-\xi,y-\tau)\,d\xi \\ &\geqslant \left\{\int_{-\infty}^{+\infty} F_1^p(\xi,\tau)\rho_1(\xi,\tau)F_2^p(x-\xi,y-\tau)\rho_2(x-\xi,y-\tau)\,d\xi\right\}^{\frac{1}{p}} \\ &\times \left\{\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)d\xi\right\}^{1-\frac{1}{p}}. \end{aligned} \tag{2.12}$$

Let

$$h(\tau) = \int_{-\infty}^{+\infty} F_1^p(\xi,\tau)\rho_1(\xi,\tau)F_2^p(x-\xi,y-\tau)\rho_2(x-\xi,y-\tau)\,d\xi$$

and

$$k(\tau) = \int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)d\xi, \quad \forall (x,y) \in \mathbb{R}^2.$$

The condition (2.10) implies

$$m_1m_2 \leqslant \frac{h(\tau)}{k(\tau)} \leqslant M_1M_2, \quad \tau \in \mathbb{R}.$$

Hence, one can apply the reverse Hölder inequality (1.2) for h and k to get

$$\begin{aligned}
&A_{p,q}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2^p(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi\right\}^{\frac{1}{p}}\\
&\qquad\times\left\{\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)d\xi\right\}^{1-\frac{1}{p}}d\tau\\
&\quad\geqslant\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2^p(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\right\}^{\frac{1}{p}}\\
&\qquad\times\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)d\xi d\tau\right\}^{1-\frac{1}{p}}.
\end{aligned}\tag{2.13}$$

Combining (2.12) and (2.13) yields

$$\begin{aligned}
&A_{p,q}^2\left(\frac{m_1m_2}{M_1M_2}\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\\
&\quad\geqslant\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2^p(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\right\}^{\frac{1}{p}}\\
&\qquad\times\left\{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,y-\tau)d\xi d\tau\right\}^{1-\frac{1}{p}}.
\end{aligned}\tag{2.14}$$

Hence,

$$\begin{aligned}
&A_{p,q}^{2p}\left(\frac{m_1m_2}{M_1M_2}\right)\frac{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\right)^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\right)^{p-1}}\\
&\quad\geqslant\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\,\rho_1(\xi,\tau)\,F_2^p(x-\xi,y-\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau.
\end{aligned}\tag{2.15}$$

Therefore,

$$\begin{aligned}
&\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,y-\tau)\rho_2(x-\xi,y-\tau)d\xi d\tau\right)^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\,\rho_2(x-\xi,y-\tau)\,d\xi d\tau\right)^{p-1}}dxdy\\
&\quad\geqslant A_{p,q}^{-2p}\left(\frac{m_1m_2}{M_1M_2}\right)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\,\rho_1(\xi,\tau)\,d\xi d\tau\\
&\qquad\times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_2^p(\xi,\tau)\,\rho_2(\xi,\tau)\,d\xi d\tau.
\end{aligned}\tag{2.16}$$

Raising both sides of the inequality (2.16) to power $\frac{1}{p}$ yields the inequality (2.11). □

REMARK 1. In formula (2.11) replacing ρ_2 by 1, and $F_2(x-\xi,y-\tau)$ by $G(x-\xi,y-\tau)$, and integrating with respect to x from a to b and respect to y from c to d, we arrive at the following inequality

$$\begin{aligned}\int_c^d\int_a^b\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F(\xi,\tau)\rho(\xi,\tau)G(x-\xi,y-\tau)d\xi d\tau\right)^p dxdy\\ \geqslant A_{p,q}^{-2p}\left(\frac{m}{M}\right)\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho(\xi,\tau)d\xi d\tau\right)^{p-1}\\ \times\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F^p(\xi,\tau)\rho(\xi,\tau)d\xi d\tau\int_{c-\tau}^{d-\tau}\int_{a-\xi}^{b-\xi}G^p(x,y)dxdy,\end{aligned} \tag{2.17}$$

if positive continuous functions ρ, F and G satisfy

$$0<m^{\frac{1}{p}}\leqslant F(\xi,\tau)G(x-\xi,y-\tau)<M^{\frac{1}{p}}<\infty,\quad (\xi,\tau)\in[a,b]\times[c,d],\quad (\xi,\tau)\in\mathbb{R}^2. \tag{2.18}$$

Moreover, by the reverse Hölder's inequality and Fubini's theorem and by changing the variables in integrals, we obtain the following inequalities

THEOREM 2. *Let F_1 and F_2 be positive functions satisfying*

$$0<m_1^{\frac{1}{p}}\leqslant F_1(\xi,\tau)\leqslant M_1^{\frac{1}{p}}<\infty,\quad 0<m_2^{\frac{1}{p}}\leqslant F_2(\xi,\tau)\leqslant M_2^{\frac{1}{p}}<\infty,\quad (\xi,\tau)\in\mathbb{R}^2 \tag{2.19}$$

and

$$0<m_3\leqslant\frac{\left\{\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\rho_1(\xi,\tau)d\xi\right\}^p}{\left\{\int_{-\infty}^{+\infty}F_2^p(\xi,\tau)\rho_2(\xi,\tau)d\xi\right\}^q}\leqslant M_3,\quad p>1,\quad p^{-1}+q^{-1}=1,\quad \tau\in\mathbb{R}. \tag{2.20}$$

Then for any positive continuous functions ρ_1 and ρ_2, we have the inequality

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau\right)^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau\right)^{p-1}}dx\\ \geqslant A_{p,q}^{-2p}\left(\frac{m_1m_2}{M_1M_2}\right)A_{p,q}^{-1}\left(\frac{m_3}{M_3}\right)\left\|\|F_1\|_{L_p(\mathbb{R},\rho_1d\xi)}^p\right\|_{L_p(\mathbb{R},d\tau)}\left\|\|F_2\|_{L_p(\mathbb{R},\rho_2d\xi)}^p\right\|_{L_q(\mathbb{R},d\tau)}.\end{aligned} \tag{2.21}$$

Proof. Similar to proof of Theorem 1, we have

$$\begin{aligned}A_{p,q}^{2p}\left(\frac{m_1m_2}{M_1M_2}\right)\frac{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau\right)^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau\right)^{p-1}}\\ \geqslant\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\rho_1(\xi,\tau)F_2^p(x-\xi,\tau)\rho_2(x-\xi,\tau)d\xi d\tau.\end{aligned} \tag{2.22}$$

Taking integration of both sides of (2.22) with respect to x from $-\infty$ to $+\infty$, we obtain

$$A_{p,q}^{2p}\left(\frac{m_1 m_2}{M_1 M_2}\right)\int_{-\infty}^{+\infty}\frac{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}F_1(\xi,\tau)\rho_1(\xi,\tau)F_2(x-\xi,\tau)\rho_2(x-\xi,\tau)\,d\xi d\tau\right)^p}{\left(\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\rho_1(\xi,\tau)\rho_2(x-\xi,\tau)\,d\xi d\tau\right)^{p-1}}dx$$

$$\geqslant \int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\rho_1(\xi,\tau)\,d\xi\right\}\left\{\int_{-\infty}^{+\infty}F_2^p(\xi,\tau)\rho_2(\xi,\tau)\,d\xi\right\}d\tau. \quad (2.23)$$

From the condition (2.20), we apply the reverse Hölder inequality (1.2) to get

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}F_1^p(\xi,\tau)\rho_1(\xi,\tau)\,d\xi\right\}\left\{\int_{-\infty}^{+\infty}F_2^p(\xi,\tau)\rho_2(\xi,\tau)\,d\xi\right\}d\tau\\ &\qquad\geqslant A_{p,q}^{-1}\left(\frac{m_3}{M_3}\right)\left\|\|F_1\|_{L_p(\mathbb{R},\rho_1 d\xi)}^p\right\|_{L_p(\mathbb{R},d\tau)}\left\|\|F_2\|_{L_p(\mathbb{R},\rho_2 d\xi)}^p\right\|_{L_q(\mathbb{R},d\tau)}.\end{aligned} \quad (2.24)$$

Combining (2.23) and (2.24) gives the inequality (2.21). The proof is complete. □

REMARK 2. In formula (2.21) replacing ρ_2 by 1, and $F_2(x-\xi,\tau)$ by $G(x-\xi,\tau)$, we arrive at the following inequality

$$\begin{aligned}&\int_{-\infty}^{+\infty}\left(\int_a^b d\tau\int_{-\infty}^{+\infty}F(\xi,\tau)\rho(\xi,\tau)G(x-\xi,\tau)d\xi\right)^p dx\\ &\geqslant A_{p,q}^{-2p}\left(\frac{m_1}{M_1}\right)A_{p,q}^{-1}\left(\frac{m_2}{M_2}\right)\left(\int_a^b d\tau\int_{-\infty}^{+\infty}\rho(\xi,\tau)d\xi\right)^{p-1}\\ &\quad\times\left[\int_a^b\left\{\int_{-\infty}^{+\infty}F^p(\xi,\tau)\rho(\xi,\tau)d\xi\right\}^p d\tau\right]^{\frac{1}{p}}\left[\int_a^b\left\{\int_{-\infty}^{+\infty}G^p(x,\tau)dx\right\}^q d\tau\right]^{\frac{1}{q}},\end{aligned} \quad (2.25)$$

if positive continuous functions ρ, F and G satisfy

$$0<m_1^{\frac{1}{p}}\leqslant F(\xi,\tau)G(x-\xi,\tau)<M_1^{\frac{1}{p}}<\infty,\quad x\in\mathbb{R},\quad \xi\in\mathbb{R},\quad \tau\in[a,b] \quad (2.26)$$

and

$$0<m_2\leqslant\frac{\left\{\int_{-\infty}^{+\infty}F^p(\xi,\tau)\rho(\xi,\tau)d\xi\right\}^p}{\left\{\int_{-\infty}^{+\infty}G^p(\xi,\tau)d\xi\right\}^q}\leqslant M_2,\quad p>1,\quad p^{-1}+q^{-1}=1,\quad \tau\in\mathbb{R}. \quad (2.27)$$

REMARK 3. In formula (2.23) replacing ρ_2 by 1, and $F_2(x-\xi,\tau)$ by $G(x-\xi,\tau)$, and integrating with respect to x from a to b we have

$$A_{p,q}^{2p}\left(\frac{m}{M}\right)\int_a^b\left(\int_c^d d\tau\int_{-\infty}^{+\infty}F(\xi,\tau)\rho(\xi,\tau)G(x-\xi,\tau)d\xi\right)^p dx$$
$$\geqslant\left(\int_c^d d\tau\int_{-\infty}^{+\infty}\rho(\xi,\tau)d\xi\right)^{p-1}\int_c^d d\tau\int_{-\infty}^{+\infty}F^p(\xi,\tau)\rho(\xi,\tau)d\xi\int_{a-\xi}^{b-\xi}G^p(x,\tau)dx, \tag{2.28}$$

if positive continuous functions ρ, F and G satisfy

$$0<m^{\frac{1}{p}}\leqslant F(\xi,\tau)G(x-\xi,\tau)<M^{\frac{1}{p}}<\infty,\quad x\in[a,b],\quad(\xi,\tau)\in\mathbb{R}^2. \tag{2.29}$$

Inequality (1.2) reverses the side if $0<p<1$. Hence, inequality (2.11) and inequality (2.21) reverse the side if $0<p<1$.

Inequality (2.17) is especially important when $G(x-\xi,y-\tau)$ is a Green's function. In the next section, we see some applications to stability of inverse problems in partial differential equations ([1], [2], [3], [4], [10]).

3. Applications

3.1. Wave Equation

For the function

$$\theta(x)=\begin{cases}1 & \text{for } x\geqslant 0\\ 0 & \text{for } x<0,\end{cases}$$

we consider the integral transform

$$\begin{aligned}u(x,t)&=\frac{1}{2c}\int_0^t d\tau\int_{x-c(t-\tau)}^{x+c(t-\tau)}F(\xi,\tau)\rho(\xi,\tau)d\xi\\&=\frac{1}{2c}\int_0^t d\tau\int_{-\infty}^{+\infty}\theta\left(c(t-\tau)-|x-\xi|\right)F(\xi,\tau)\rho(\xi,\tau)d\xi\end{aligned} \tag{3.30}$$

which gives the formal solution $u(x,t)$ of the wave equation

$$u_{tt}=c^2u_{xx}+F(x,t)\rho(x,t),\quad x\in\mathbb{R},\quad t>0,\ (c:\ \text{constant},\ >0), \tag{3.31}$$

satisfying the conditions

$$u(x,0)=u_t(x,0)=0,\quad \text{on } \mathbb{R}. \tag{3.32}$$

Take

$$G(\xi,\tau)=\theta\left(c(t-\tau)-|\xi|\right),\quad \xi\in\mathbb{R},\quad \tau\in[0,t].$$

The condition (2.26) reads

$$0<m_1^{\frac{1}{p}}\leqslant F(\xi,\tau)\theta\left(c(t-\tau)-|x-\xi|\right)\leqslant M_1^{\frac{1}{p}}\quad \xi\in\mathbb{R},\quad \tau\in(0,t). \tag{3.33}$$

It will be satisfied for $x \in [-a, a]$, if we have

$$0 < m_1^{\frac{1}{p}} \leqslant F(\xi, \tau) \leqslant M_1^{\frac{1}{p}}, \quad 0 < \tau < t, \;\; -a - c(t-\tau) < \xi < a + c(t-\tau). \tag{3.34}$$

We have

$$\int_{-\infty}^{+\infty} G^p(\xi, \tau) d\xi = 2c(t-\tau).$$

The condition (2.27) reads

$$0 < m_2 \leqslant \frac{\left\{\int_{-\infty}^{+\infty} F^p(\xi,\tau)\rho(\xi,\tau)d\xi\right\}^p}{\{2c(t-\tau)\}^q} \leqslant M_2 < \infty, \quad \tau \in (0,t). \tag{3.35}$$

It will be satisfied for $0 < \tau < t$, if we have

$$0 < m_2^{\frac{1}{q}} (2ct)^{\frac{1}{pq}} \leqslant \int_{-\infty}^{+\infty} F^p(\xi,\tau)\rho(\xi,\tau)d\xi \leqslant M_2^{\frac{1}{q}} (2ct)^{\frac{1}{pq}}. \tag{3.36}$$

Notice that

$$\left[\int_0^t \left\{\int_{-\infty}^{+\infty} G^p(x,\tau)dx\right\}^q d\tau\right]^{\frac{1}{q}} = \frac{2ct^{1+\frac{1}{q}}}{(1+q)^{\frac{1}{q}}}.$$

Thus the inequality (2.25) yields

$$\int_{-\infty}^{+\infty} u(x,t)^p dx \geqslant \left\{A_{p,q}\left(\frac{m_1}{M_1}\right)\right\}^{-2p} \left\{A_{p,q}\left(\frac{m_2}{M_2}\right)\right\}^{-1} \frac{t^{1+\frac{1}{q}}}{(2c)^{p-1}(1+q)^{\frac{1}{q}}}$$

$$\times \left(\int_0^t d\tau \int_{-\infty}^{+\infty} \rho(\xi,\tau)d\xi\right)^{p-1} \left[\int_0^t \left\{\int_{-\infty}^{+\infty} F^p(\xi,\tau)\rho(\xi,\tau)d\xi\right\}^p d\tau\right]^{\frac{1}{p}}. \tag{3.37}$$

Here we assume that ρ is a positive continuous function on $\mathbb{R} \times [0,t]$, and F satisfies (3.34) and (3.36).

3.2. Heat Equation

3.2.1. Example 1

We consider the integral transform

$$u(x,t) = \int_0^t d\tau \int_{-\infty}^{+\infty} \frac{F(\xi,\tau)\rho(\xi,\tau)}{2c\sqrt{\pi(t-\tau)}} \exp\left\{-\frac{(\xi-x)^2}{4c^2(t-\tau)}\right\} d\xi \tag{3.38}$$

which gives the solution $u(x,t)$ of the heat equation

$$u_t = c^2 u_{xx} + F(x,t)\rho(x,t), \quad x \in \mathbb{R}, \; t > 0 \tag{3.39}$$

satisfying the condition

$$u(x,0) = 0, \quad \text{on } \mathbb{R}. \tag{3.40}$$

Take

$$G(\xi,\tau)=\frac{1}{2c\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{\xi^2}{4c^2(t-\tau)}\right\}.$$

Let

$$x\in[-d,d],\quad \xi\in[f,f],\quad d+f\leqslant\sqrt{\frac{4c(t-\tau)}{p}\log\frac{M}{m}},\quad 0\leqslant\tau<t.$$

From

$$1\leqslant\exp\left\{\frac{(x-\xi)^2}{4c^2(t-\tau)}\right\}\leqslant\exp\left\{\frac{(d+f)^2}{4c^2(t-\tau)}\right\},$$

we have

$$0<m^{\frac{1}{p}}\leqslant\frac{F(\xi,\tau)}{2c\sqrt{\pi(t-\tau)}}\exp\left\{-\frac{(x-\xi)^2}{4c^2(t-\tau)}\right\}\leqslant M^{\frac{1}{p}},\tag{3.41}$$

if $\tau\in[0,t)$, $\xi\in[-f,f]$,

$$m^{\frac{1}{p}}2c\sqrt{\pi(t-\tau)}\exp\left\{\frac{(d+f)^2}{4c^2(t-\tau)}\right\}\leqslant F(\xi,\tau)\leqslant M^{\frac{1}{p}}2c\sqrt{\pi(t-\tau)}.\tag{3.42}$$

It is easy to see that

$$\int_{a-\xi}^{b-\xi}G^p(x,\tau)dx=\frac{1}{2^p\left(c\sqrt{\pi(t-\tau)}\right)^{p-1}}\left[\operatorname{erf}\left(\frac{\sqrt{p}(b-\xi)}{2c\sqrt{t-\tau}}\right)-\operatorname{erf}\left(\frac{\sqrt{p}(a-\xi)}{2c\sqrt{t-\tau}}\right)\right],$$

where

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\eta^2}d\eta$$

is the error function. Therefore, for $-d\leqslant a<b\leqslant d$, the inequality (2.28) yields

$$\int_a^b u(x,t)^p dx\geqslant\frac{1}{2}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p}\left(\int_0^t d\tau\int_{-f}^{f}\rho(\xi,\tau)d\xi\right)^{p-1}J,\tag{3.43}$$

where

$$J=\int_0^t d\tau\int_{-f}^{f}\frac{F^p(\xi,\tau)\rho(\xi,\tau)}{\left(2c\sqrt{\pi(t-\tau)}\right)^{p-1}}\left[\operatorname{erf}\left(\frac{\sqrt{p}(b-\xi)}{2c\sqrt{t-\tau}}\right)-\operatorname{erf}\left(\frac{\sqrt{p}(a-\xi)}{2c\sqrt{t-\tau}}\right)\right]d\xi.$$

Here we assume that ρ is a positive continuous function on $[-f,f]\times(0,t)$, and F satisfies (3.42).

3.2.2. Example 2

In the integral transform

$$u(x,y,t) = \frac{1}{(2c\sqrt{\pi t})^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left\{-\frac{(\xi-x)^2+(\tau-y)^2}{4c^2t}\right\}F(\xi,\tau)\rho(\xi,\tau)d\xi d\tau \tag{3.44}$$

which gives the solution $u(x,y,t)$ of the heat equation

$$u_t = c^2(u_{xx}+u_{yy}) \tag{3.45}$$

satisfying the condition

$$u(x,y,0) = F(x,y)\rho(x,y). \tag{3.46}$$

Take

$$G(\xi,\tau) = \exp\left\{-\frac{\xi^2+\tau^2}{4c^2t}\right\}.$$

Let

$$x\in[-a,a],\;\; y\in[-b,b],\;\; \xi\in[-d,d],\;\; \tau\in[-f,f],\;\; (a+d)^2+(b+f)^2 \geqslant \frac{4ct}{p}\log\frac{M}{m}.$$

From

$$1 \leqslant \exp\left\{\frac{(\xi-x)^2+(\tau-y)^2}{4c^2t}\right\} \leqslant \exp\left\{\frac{(a+d)^2+(b+f)^2}{4c^2t}\right\},$$

we have

$$0 < m^{\frac{1}{p}} \leqslant F(\xi,\tau)\exp\left\{-\frac{(\xi-x)^2+(\tau-y)^2}{4c^2t}\right\} \leqslant M^{\frac{1}{p}}, \tag{3.47}$$

if

$$m^{\frac{1}{p}}\exp\left\{\frac{(a+d)^2+(b+f)^2}{4c^2t}\right\} \leqslant F(\xi,\tau) \leqslant M^{\frac{1}{p}}, \quad (\xi,\tau)\in[-d,d]\times[-f,f]. \tag{3.48}$$

We have

$$\int_{i-\tau}^{j-\tau}dy\int_{g-\xi}^{h-\xi}G^p(x,y)dx = \frac{c^2\pi t}{p}\left[\operatorname{erf}\left(\frac{\sqrt{p}(h-\xi)}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{\sqrt{p}(g-\xi)}{2c\sqrt{t}}\right)\right] \times\left[\operatorname{erf}\left(\frac{\sqrt{p}(j-\tau)}{2c\sqrt{t}}\right)-\operatorname{erf}\left(\frac{\sqrt{p}(i-\tau)}{2c\sqrt{t}}\right)\right].$$

Therefore, for $-a\leqslant g<h\leqslant a,\;\; -b\leqslant i<j\leqslant b$, the inequality (2.17) yields

$$\int_i^j dy\int_g^h u(x,y,t)^p dx \geqslant \frac{1}{p4^p(c^2\pi t)^{p-1}}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} \times\left(\int_{-f}^{f}d\tau\int_{-d}^{d}\rho(\xi,\tau)d\xi\right)^{p-1}K, \tag{3.49}$$

where

$$K = \int_{-f}^{f} d\tau \int_{-d}^{d} F^p(\xi,\tau)\rho(\xi,\tau)\left[\operatorname{erf}\left(\frac{\sqrt{p}(h-\xi)}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\sqrt{p}(g-\xi)}{2c\sqrt{t}}\right)\right] \times \left[\operatorname{erf}\left(\frac{\sqrt{p}(j-\tau)}{2c\sqrt{t}}\right) - \operatorname{erf}\left(\frac{\sqrt{p}(i-\tau)}{2c\sqrt{t}}\right)\right] d\xi.$$

Here we assume that ρ is a positive continuous function on $[-d,d]\times[-f,f]$, and F satisfies (3.48).

3.3. Poisson Integrals

3.3.1. Example 1

We consider the Dirichlet problem for the Laplace Equation in a half-space of $\mathbb{R}^3$, i.e. the determination of the bounded solution of

$$\Delta_3 u(x,y,t) = 0, \quad t>0, \ \ (x,y)\in\mathbb{R}^2 \tag{3.50}$$

with the boundary condition

$$u(x,y,0) = F(x,y)\rho(x,y). \tag{3.51}$$

We have the solution of the Dirichlet problem (3.50), (3.51) in the form

$$u(x,y,t) = \frac{t}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{F(\xi,\tau)\rho(\xi,\tau)}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}} d\xi d\tau. \tag{3.52}$$

Take

$$G(x,y) = \frac{1}{[t^2+x^2+y^2]^{\frac{3}{2}}}.$$

Let

$$x\in[a,b], \quad \xi\in[c,d], \quad y\in[f,g], \quad \tau\in[h,k].$$

Denote

$$\alpha = \max\{|a-c|,|a-d|,|b-c|,|b-d|\}, \quad \beta = \max\{|f-h|,|f-g|,|g-h|,|g-k|\}.$$

We have

$$\frac{1}{[t^2+\alpha^2+(y-\tau)^2]^{\frac{3}{2}}} \leqslant \frac{1}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}} \leqslant \frac{1}{[t^2+(y-\tau)^2]^{\frac{3}{2}}}$$

and

$$\frac{1}{[t^2+\alpha^2+\beta^2]^{\frac{3}{2}}} \leqslant \frac{1}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}} \leqslant \frac{1}{t^3}.$$

Thus,

$$\int_{a-\xi}^{b-\xi} G^p(x,y-\tau)dx = \int_{a-\xi}^{b-\xi} \frac{dx}{[t^2+x^2+(\tau-y)^2]^{\frac{3p}{2}}} \geqslant \frac{(b-a)}{[t^2+\alpha^2+(y-\tau)^2]^{\frac{3p}{2}}}$$

and

$$\int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} G^p(x,y)dx = \int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} \frac{dx}{[t^2+x^2+y^2]^{\frac{3p}{2}}} \geqslant \frac{(b-a)(g-f)}{[t^2+\alpha^2+\beta^2]^{\frac{3p}{2}}}.$$

Hence, for a function F satisfying

$$\left[t^2+\alpha^2+(y-\tau)^2\right]^{\frac{3}{2}} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant \left[t^2+(y-\tau)^2\right]^{\frac{3}{2}} M^{\frac{1}{p}} \tag{3.53}$$

and ρ is a positive continuous function on $[c,d]\times\mathbb{R}$, we obtain

$$\begin{aligned}\int_a^b u^p(x,y,t)dx \geqslant & \frac{(b-a)t^p}{(2\pi)^p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p}\left(\int_{-\infty}^{+\infty} d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \\ & \times \int_{-\infty}^{+\infty} d\tau \int_c^d \frac{F^p(\xi,\tau)\rho(\xi,\tau)}{[t^2+\alpha^2+(y-\tau)^2]^{\frac{3p}{2}}} d\xi.\end{aligned} \tag{3.54}$$

Moreover, if the function F satisfies

$$\left[t^2+\alpha^2+\beta^2\right]^{\frac{3}{2}} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant t^3 M^{\frac{1}{p}} \tag{3.55}$$

and for a positive continuous function ρ on $[c,d]\times[h,k]$, we obtain

$$\begin{aligned}\int_f^g dy \int_a^b u^p(x,y,t)dx \geqslant & \frac{(b-a)(g-f)t^p}{(2\pi)^p\,[t^2+\alpha^2+\beta^2]^{\frac{3p}{2}}}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} \\ & \times \left(\int_h^k d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \int_h^k d\tau \int_c^d F^p(\xi,\tau)\rho(\xi,\tau)d\xi.\end{aligned} \tag{3.56}$$

Consider now the conjugate Poisson integral

$$v(x,y,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} F(\xi,\tau)\rho(\xi,\tau)\frac{x-\xi}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}}d\xi d\tau. \tag{3.57}$$

Take

$$G(x,y) = \frac{x}{[t^2+x^2+y^2]^{\frac{3}{2}}}.$$

Let

$$x\in[a,b],\quad \xi\in[c,d],\quad y\in[f,g],\quad \tau\in[h,k],\quad (d<a).$$

Denote

$$\beta = \max\{|f-h|,|f-g|,|g-h|,|g-k|\}.$$

We have

$$\frac{a-d}{[t^2+(b-c)^2+(y-\tau)^2]^{\frac{3}{2}}} \leqslant \frac{x-\xi}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}} \leqslant \frac{b-c}{[t^2+(a-d)^2+(y-\tau)^2]^{\frac{3}{2}}}$$

and

$$\frac{a-d}{[t^2+(b-c)^2+\beta^2]^{\frac{3}{2}}} \leqslant \frac{x-\xi}{[t^2+(\xi-x)^2+(\tau-y)^2]^{\frac{3}{2}}} \leqslant \frac{b-c}{[t^2+(a-d)^2]^{\frac{3}{2}}}.$$

Thus,

$$\int_{a-\xi}^{b-\xi} G^p(x,y-\tau)dx = \int_{a-\xi}^{b-\xi} \frac{(x-\xi)^p dx}{[t^2+x^2+(\tau-y)^2]^{\frac{3p}{2}}} \geqslant \frac{(b-a)(a-d)^p}{[t^2+(b-c)^2+(y-\tau)^2]^{\frac{3p}{2}}}$$

and

$$\int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} G^p(x,y)dx = \int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} \frac{(x-\xi)^p dx}{[t^2+x^2+y^2]^{\frac{3p}{2}}} \geqslant \frac{(b-a)(g-f)(a-d)^p}{[t^2+(b-c)^2+\beta^2]^{\frac{3p}{2}}}.$$

Hence, for a function F satisfying

$$\frac{\left[t^2+(b-c)^2+(y-\tau)^2\right]^{\frac{3}{2}}}{a-d} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant \frac{\left[t^2+(a-d)^2+(y-\tau)^2\right]^{\frac{3}{2}}}{b-c} M^{\frac{1}{p}}$$

and for a positive continuous function ρ on $[c,d]\times\mathbb{R}$, we obtain

$$\int_a^b v^p(x,y,t)dx \geqslant \frac{(b-a)(a-d)^p}{(2\pi)^p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} L \tag{3.58}$$

where

$$L = \left(\int_{-\infty}^{+\infty} d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \int_{-\infty}^{+\infty} d\tau \int_c^d \frac{F^p(\xi,\tau)\rho(\xi,\tau)}{[t^2+\alpha^2+(y-\tau)^2]^{\frac{3p}{2}}} d\xi.$$

Moreover, if the function F satisfies

$$\frac{\left[t^2+(b-c)^2+\beta^2\right]^{\frac{3}{2}}}{a-d} \leqslant F(\xi,\tau) \leqslant \frac{\left[t^2+(a-d)^2\right]^{\frac{3}{2}}}{b-c} M^{\frac{1}{p}},$$

and for a positive continuous function ρ on $[c,d]\times[h,k]$, we obtain

$$\begin{aligned} \int_f^g dy \int_a^b v^p(x,y,t)dx &\geqslant \frac{(b-a)(g-f)(a-d)^p}{(2\pi)^p\,[t^2+(b-c)^2+\beta^2]^{\frac{3p}{2}}}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} \\ &\times \left(\int_h^k d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \int_h^k d\tau \int_c^d F^p(\xi,\tau)\rho(\xi,\tau)d\xi. \end{aligned} \tag{3.59}$$

3.3.2. Example 2

The solution of the Poisson equation

$$\Delta_2 u(x,y) = F(x,y)\rho(x,y), \qquad (x,y) \in \mathbb{R}^2 \tag{3.60}$$

is given by

$$u(x,y) = \frac{1}{4\pi}\int_{-\infty}^{+\infty} d\tau \int_{-\infty}^{+\infty} F(\xi,\tau)\rho(\xi,\tau)\log\left[(x-\xi)^2 + (y-\tau)^2\right] d\xi. \tag{3.61}$$

Take

$$G(x,y) = \log\left[x^2 + y^2\right].$$

Let

$$x \in [a,b], \quad \xi \in [c,d], \quad y \in [f,g], \quad \tau \in [h,k], \quad (c > b+1 \text{ or } a < d+1).$$

Denote

$$\alpha = \max\{|b-c|, |a-d|\}, \quad \beta = \max\{|f-h|, |f-g|, |g-h|, |g-k|\},$$
$$\gamma = \min\{|b-c|, |a-d|\}.$$

We obtain

$$0 < \log\left[\gamma^2 + (y-\tau)^2\right] \leqslant \log\left[(x-\xi)^2 + (y-\tau)^2\right] \leqslant \log\left[\alpha^2 + (y-\tau)^2\right]$$

and

$$0 < 2\log\gamma \leqslant \log\left[(x-\xi)^2 + (y-\tau)^2\right] \leqslant \log\left[\alpha^2 + \beta^2\right].$$

Thus,

$$\int_{a-\xi}^{b-\xi} G^p(x, y-\tau)dx = \int_{a-\xi}^{b-\xi} \log^p\left[x^2 + (y-\tau)^2\right] dx \geqslant (b-a)\log^p\left[\gamma^2 + (y-\tau)^2\right]$$

and

$$\int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} G^p(x,y)dx = \int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} \log^p\left[x^2 + (y-\tau)^2\right] dx$$
$$\geqslant 2^p(b-a)(g-f)\log^p\gamma.$$

Hence, for a function F satisfying

$$\frac{1}{\log\left[\gamma^2 + (y-\tau)^2\right]} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant \frac{1}{\log\left[\alpha^2 + (y-\tau)^2\right]} M^{\frac{1}{p}} \tag{3.62}$$

and for a positive continuous function ρ on $[c,d]\times\mathbb{R}$, we obtain

$$\begin{aligned}\int_a^b u^p(x,y,t)dx \geqslant & \frac{(b-a)}{(4\pi)^p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p}\left(\int_{-\infty}^{+\infty} d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \\ & \times \int_{-\infty}^{+\infty} d\tau \int_c^d F^p(\xi,\tau)\rho(\xi,\tau)\log^p\left[\gamma^2 + (y-\tau)^2\right] d\xi.\end{aligned} \tag{3.63}$$

Moreover, if the function F satisfies

$$\frac{1}{2\log\gamma}m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant \frac{1}{\log\left[\alpha^2+\beta^2\right]}M^{\frac{1}{p}} \tag{3.64}$$

and for a positive continuous function ρ on $[c,d]\times[h,k]$, we obtain

$$\begin{aligned}\int_f^g dy\int_a^b u^p(x,y,t)dx \geqslant \frac{(b-a)(g-f)\log^p\gamma}{(2\pi)^p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p}\\ \times\left(\int_h^k d\tau\int_c^d \rho(\xi,\tau)d\xi\right)^{p-1}\int_h^k d\tau\int_c^d F^p(\xi,\tau)\rho(\xi,\tau)d\xi.\end{aligned} \tag{3.65}$$

3.4. Biharmonic equation

The solution of the biharmonic equation

$$\Delta_3^2 u(x,y,t)=0, \quad t>0, \quad (x,y)\in\mathbb{R}^2, \quad \Delta_3^2=\Delta_3(\Delta_3) \tag{3.66}$$

with the boundary conditions

$$u(x,y,0)=F(x,y)\rho(x,y), \quad u_t(x,y,0)=0 \tag{3.67}$$

is given by

$$u(x,y,t)=\frac{3t^3}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{F(\xi,\tau)\rho(\xi,\tau)}{\left[t^2+(\xi-x)^2+(\tau-y)^2\right]^{\frac{5}{2}}}d\xi d\tau. \tag{3.68}$$

Take

$$G(x,y)=\frac{1}{\left[t^2+x^2+y^2\right]^{\frac{5}{2}}}.$$

Let

$$x\in[a,b], \quad \xi\in[c,d], \quad y\in[f,g], \quad \tau\in[h,k].$$

Denote

$$\alpha=\max\{|a-c|,|a-d|,|b-c|,|b-d|\}, \quad \beta=\max\{|f-h|,|f-g|,|g-h|,|g-k|\}.$$

We have

$$\frac{1}{\left[t^2+\alpha^2+(y-\tau)^2\right]^{\frac{5}{2}}} \leqslant \frac{1}{\left[t^2+(\xi-x)^2+(\tau-y)^2\right]^{\frac{5}{2}}} \leqslant \frac{1}{\left[t^2+(y-\tau)^2\right]^{\frac{5}{2}}}$$

and

$$\frac{1}{\left[t^2+\alpha^2+\beta^2\right]^{\frac{5}{2}}} \leqslant \frac{1}{\left[t^2+(\xi-x)^2+(\tau-y)^2\right]^{\frac{5}{2}}} \leqslant \frac{1}{t^5}.$$

Thus,

$$\int_{a-\xi}^{b-\xi} G^p(x,y-\tau)dx=\int_{a-\xi}^{b-\xi}\frac{dx}{\left[t^2+x^2+(\tau-y)^2\right]^{\frac{5p}{2}}} \geqslant \frac{(b-a)}{\left[t^2+\alpha^2+(y-\tau)^2\right]^{\frac{5p}{2}}}$$

and

$$\int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} G^p(x,y)dx = \int_{f-\tau}^{g-\tau} dy \int_{a-\xi}^{b-\xi} \frac{dx}{[t^2+x^2+y^2]^{\frac{5p}{2}}} \geqslant \frac{(b-a)(g-f)}{[t^2+\alpha^2+\beta^2]^{\frac{5p}{2}}}.$$

Hence, for a function F satisfying

$$\left[t^2+\alpha^2+(y-\tau)^2\right]^{\frac{5}{2}} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant \left[t^2+(y-\tau)^2\right]^{\frac{5}{2}} M^{\frac{1}{p}} \tag{3.69}$$

and for a positive continuous function ρ on $[c,d]\times\mathbb{R}$, we obtain

$$\begin{aligned}\int_a^b u^p(x,y,t)dx \geqslant & \frac{(b-a)3^p t^{3p}}{(2\pi)^p}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p}\left(\int_{-\infty}^{+\infty} d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \\ & \times \int_{-\infty}^{+\infty} d\tau \int_c^d \frac{F^p(\xi,\tau)\rho(\xi,\tau)}{[t^2+\alpha^2+(y-\tau)^2]^{\frac{5p}{2}}} d\xi.\end{aligned} \tag{3.70}$$

Moreover, if the function F satisfies

$$\left[t^2+\alpha^2+\beta^2\right]^{\frac{5}{2}} m^{\frac{1}{p}} \leqslant F(\xi,\tau) \leqslant t^5 M^{\frac{1}{p}} \tag{3.71}$$

and for a positive continuous function ρ on $[c,d]\times[h,k]$, we obtain

$$\int_f^g dy \int_a^b u^p(x,y,t)dx \geqslant \frac{(b-a)(g-f)3^p t^{3p}}{(2\pi)^p\,[t^2+\alpha^2+\beta^2]^{\frac{5p}{2}}}\left\{A_{p,q}\left(\frac{m}{M}\right)\right\}^{-2p} H \tag{3.72}$$

where

$$H = \left(\int_h^k d\tau \int_c^d \rho(\xi,\tau)d\xi\right)^{p-1} \int_h^k d\tau \int_c^d F^p(\xi,\tau)\rho(\xi,\tau)d\xi.$$

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