Journal of
Mathematical
Inequalities
Volume 2, Number 1 (2008), 75–95

MEANS AND HERMITE INTERPOLATION

ALAN HORWITZ

(*communicated by J. Matkowski*)

Abstract. Let $m_2 < m_1$ be two given nonnegative integers with $n = m_1 + m_2 + 1$. For suitably differentiable f, we let $P, Q \in \pi_n$ be the Hermite polynomial interpolants to f which satisfy $P^{(j)}(a) = f^{(j)}(a), j = 0, 1, ..., m_1$ and $P^{(j)}(b) = f^{(j)}(b), j = 0, 1, ..., m_2$, $Q^{(j)}(a) = f^{(j)}(a), j = 0, 1, ..., m_2$ and $Q^{(j)}(b) = f^{(j)}(b), j = 0, 1, ..., m_1$. Suppose that $f \in C^{n+2}(I)$ with $f^{(n+1)}(x) \neq 0$ for $x \in (a, b)$. If $m_1 - m_2$ is even, then there is a unique $x_0, a < x_0 < b$, such that $P(x_0) = Q(x_0)$. If $m_1 - m_2$ is odd, then there is a unique $x_0, a < x_0 < b$, such that $f(x_0) = \frac{1}{2}(P(x_0) + Q(x_0))$. x_0 defines a strict, symmetric mean, which we denote by $M_{f,m_1,m_2}(a, b)$. We prove various properties of these means. In particular, we show that $f(x) = x^{m_1+m_2+2}$ yields the arithmetic mean, $f(x) = x^{-1}$ yields the harmonic mean, and $f(x) = x^{(m_1+m_2+1)/2}$ yields the geometric mean.

1. Introduction

DEFINITION 1. A *mean* $m(a, b)$ in two variables is a continuous function on $\Re_2^+ = \{(a, b) : a, b > 0\}$ with $\min(a, b) \leqslant m(a, b) \leqslant \max(a, b)$. m is called

(1) *Strict* if $m(a, b) = \min(a, b)$ or $m(a, b) = \max(a, b)$ if and only if $a = b$ for all $(a, b) \in \Re_2^+$.

(2) *Symmetric* if $m(b, a) = m(a, b)$ for all $(a, b) \in \Re_2^+$.

(3) *Homogeneous* if $m(ka, kb) = km(a, b)$ for any $k > 0$ and for all $(a, b) \in \Re_2^+$.

Of course, in some cases a mean can be extended to all real numbers, such as with the arithmetic mean $m(a, b) = \frac{a+b}{2}$. In this paper we define means in two variables using intersections of Hermite polynomial interpolants to a given function, f. Throughout we assume, unless stated otherwise, that $m_2 < m_1$ are two given nonnegative integers with $n = m_1 + m_2 + 1$. If $f^{(k)}(a)$ and $f^{(k)}(b)$ each exist for $k = 0, 1, ..., m_1$, we let $P, Q \in \pi_n$ be the Hermite polynomial interpolants to f which satisfy

$$\begin{aligned} P^{(j)}(a) &= f^{(j)}(a),\ j = 0, 1, ..., m_1 \text{ and } P^{(j)}(b) = f^{(j)}(b),\ j = 0, 1, ..., m_2, \quad (1)\\ Q^{(j)}(a) &= f^{(j)}(a),\ j = 0, 1, ..., m_2 \text{ and } Q^{(j)}(b) = f^{(j)}(b),\ j = 0, 1, ..., m_1. \end{aligned}$$

Of course P and Q depend on $m_1, m_2,$ and f, but we supress that in our notation. Under suitable conditions on f (see Theorem 2 below), if $m_1 - m_2$ is even, then there is a unique $x_0, a < x_0 < b$, such that $P(x_0) = Q(x_0)$. If $m_1 - m_2$ is odd (see Theorem 4

Mathematics subject classification (2000): 26D10.

Key words and phrases: Mean, arithmetic mean, geometric mean, Hermite interpolation, Taylor polynomial mean.

© ELEMENT, Zagreb
Paper JMI-02-08

below), then there is a unique $x_0, a < x_0 < b,$ such that $f(x_0) = \frac{1}{2}(P(x_0) + Q(x_0))$. In either case, x_0 defines a strict, symmetric mean, which we denote by $M_{f,m_1,m_2}(a,b)$.

The means defined in this paper are similar to a class of means defined in [2] and [3], which were based on intersections of Taylor polynomials, each of order r. More precisely, for $f \in C^{r+1}(I), I = (a,b),$ let P_c denote the Taylor polynomial to f of order r at $x = c$, where r is odd. In [2] it was proved that if $f^{(r+1)}(x) \neq 0$ on $[a,b]$, then there is a unique $u, a < u < b,$ such that $P_a(u) = P_b(u)$. This defines a mean $m(a,b) \equiv u$. These means were extended to the case when r is even in [3] by defining $m(a,b)$ to be the unique solution in (a,b) of the equation $f(x) = \dfrac{1}{2}(P(x) + Q(x))$. However, many of the proofs in this paper are more complex than those in [2] and [3] because the means M_{f,m_1,m_2} depend on two nonnegative integers, m_1 and m_2, rather than just on the one nonnegative integer, r. In [2] the author also proved some minimal results for means involving intersections of Hermite interpolants to a given function, f. In particular we proved a version of Theorems 2, 6, and 7 below for the special case when $m_1 - m_2 = 2$. In this paper we prove much more along these lines.

2. Main Results

Our first result allows us to define a mean using intersections of Hermite interpolants when $m_1 - m_2$ is even.

THEOREM 2. *Suppose that $m_2 < m_1$ are two given nonnegative integers with $m_1 - m_2$ even. Let $n = m_1 + m_2 + 1$ and let $I = (a,b), 0 < a < b$ be a given open interval. Suppose that $f \in C^{n+2}(I)$ with $f^{(n+1)}(x) \neq 0$ for $x \in I$, and let P and Q satisfy the Hermite interpolation conditions given by (1). Then there is a unique $x_0, a < x_0 < b,$ such that $P(x_0) = Q(x_0)$.*

Proof. We may assume, without loss of generality, that $f^{(n+1)}(x) > 0$ on I. Let $E_P(x) = f(x) - P(x)$ and $E_Q(x) = f(x) - Q(x)$ denote the respective error functions for P and Q, and let $f[x_0, x_1, ..., x_n]$ denote the n-th order divided difference of f for distinct nodes $x_0, x_1, ..., x_n$. In general, divided differences at distinct points are defined inductively by $f[x_0, x_1, ..., x_j] = \frac{f[x_0,x_1,...,x_{j-1}] - f[x_1,...,x_j]}{x_0 - x_j}$ with $f[x_0] = f(x_0)$. For sufficiently differentiable f, one can extend the definition of divided difference in a continuous fashion when the nodes are not all distinct (see, for example, [4]). We let $f[x_0^{m_0}, x_1^{m_1}, ..., x_n^{m_n}]$ denote the divided difference where x_k appears m_k times. Using one well–known form of the error in Hermite interpolation, one has

$$E_P(x) = (x-a)^{m_1+1}(x-b)^{m_2+1} f[x, a^{m_1+1}, b^{m_2+1}] \tag{2}$$

and

$$E_Q(x) = (x-a)^{m_2+1}(x-b)^{m_1+1} f[x, a^{m_2+1}, b^{m_1+1}].$$

Let

$$h_1(x) = f[x, a^{m_1+1}, b^{m_2+1}], h_2(x) = f[x, a^{m_2+1}, b^{m_1+1}].$$

Now $P(x) = Q(x) \iff E_P(x) = E_Q(x) \iff$

$$(x-a)^{m_1-m_2} h_1(x) = (x-b)^{m_1-m_2} h_2(x). \tag{3}$$

By the Mean Value Theorem for divided differences (see [4]), $f[x,a^{m_1+1},b^{m_2+1}] = \frac{f^{(n+1)}(\zeta_1)}{(n+1)!}$ and $f[x,a^{m_2+1},b^{m_1+1}] = \frac{f^{(n+1)}(\zeta_2)}{(n+1)!}$, where $\zeta_1,\zeta_2 \in I$ if $x \in I$. Thus $f[x,a^{m_1+1},b^{m_2+1}] > 0$ and $f[x,a^{m_2+1},b^{m_1+1}] > 0$. Now $h_1'(x) = \frac{d}{dx}f[x,a^{m_1+1},b^{m_2+1}] = f[x,x,a^{m_1+1},b^{m_2+1}]$ (see [4]), which implies that

$$\begin{aligned}
&\tfrac{d}{dx}\left[(x-a)^{m_1-m_2}h_1(x)\right]\\
&\quad = (x-a)^{m_1-m_2}h_1'(x) + (m_1-m_2)(x-a)^{m-1}h_1(x)\\
&\quad = (x-a)^{m_1-m_2-1}\left[(x-a)f[x,x,a^{m_1+1},b^{m_2+1}] + (m_1-m_2)f[x,a^{m_1+1},b^{m_2+1}]\right].
\end{aligned}$$

Now $(x-a)^{m_1-m_2-1} \geqslant 0$ for $x \in I$. Simplifying the term in brackets using properties of divided differences yields

$$\begin{aligned}
(x-a)&f[x,x,b^{m_2+1},a^{m_1+1}] + (m_1-m_2)f[x,b^{m_2+1},a^{m_1+1}]\\
&= f[x,x,b^{m_2+1},a^{m_1}] - f[x,b^{m_2+1},a^{m_1+1}] + (m_1-m_2)f[x,b^{m_2+1},a^{m_1+1}]\\
&= f[x,x,b^{m_2+1},a^{m_1}] + (m_1-m_2-1)f[x,b^{m_2+1},a^{m_1+1}] > 0
\end{aligned}$$

again by the Mean Value Theorem for divided differences. Thus

$$\frac{d}{dx}\left[(x-a)^{m_1-m_2}h_1(x)\right] > 0 \Rightarrow (x-a)^{m_1-m_2}h_1(x)$$

is increasing on I. Similarly,

$$\begin{aligned}
&\frac{d}{dx}\left[(x-b)^{m_1-m_2}h_2(x)\right]\\
&\quad = (x-b)^{m_1-m_2}h_2'(x) + (m_1-m_2)(x-b)^{m_1-m_2-1}h_2(x)\\
&\quad = (x-b)^{m_1-m_2-1}\left[(x-b)f[x,x,a^{m_2+1},b^{m_1+1}] + (m_1-m_2)f[x,a^{m_2+1},b^{m_1+1}]\right]\\
&\quad = (x-b)^{m_1-m_2-1}\left(f[x,x,a^{m_2+1},b^{m_1}] + (m_1-m_2-1)f[x,a^{m_2+1},b^{m_1+1}]\right).
\end{aligned}$$

Since m_1-m_2-1 is odd, $\dfrac{d}{dx}\left[(x-b)^{m_1-m_2}h_2(x)\right] < 0$ on I, which implies that $(x-b)^{m_1-m_2}h_2(x)$ is decreasing on I. Thus $(x-a)^{m_1-m_2}h_1(x)$ is positive and increasing on I and vanishes at a, while $(x-b)^{m_1-m_2}h_2(x)$ is positive and decreasing on I and vanishes at b. Hence the equation in (3) has a unique solution $x_0 \in I$. Since $E_P(x_0) = E_Q(x_0)$, $P(x_0) = Q(x_0)$, which finishes the proof of Theorem 2. □

REMARK 3. (1) Theorem 1 was proven in [2] using a different approach and only for the case when $m_1 - m_2 = 2$.

(2) Heuristically speaking, we may consider the means defined in [2] as a special case of the means above, where $m_1 = r$ and $m_2 = -1$. The latter value means that no values of f or any of its derivatives are matched. However, the formulas we use do not actually work if $m_2 = -1$.

The proof of the following theorem is almost identical to the proof of Theorem 2 and we omit it.

THEOREM 4. *Suppose that* $m_2 < m_1$ *are two given nonnegative integers with* $m_1 - m_2$ *odd. Let* $n = m_1 + m_2 + 1$ *and let* $I = (a,b), 0 < a < b$ *be a given open* *interval. Suppose that* $f \in C^{n+2}(I)$ *with* $f^{(n+1)}(x) \neq 0$ *for* $x \in I$*, and let* P *and* Q *satisfy the Hermite interpolation conditions given by (1). Then there is a unique* $x_0, a < x_0 < b$*, such that* $f(x_0) = \frac{1}{2}(P(x_0) + Q(x_0))$.

The unique x_0 from Theorems 2 and 4 defines a strict, symmetric mean, which we denote by $x_0 = M_{f,m_1,m_2}(a,b)$. It is easy to unify the cases of $m_1 - m_2$ even or odd as follows: $M_{f,m_1,m_2}(a,b)$ is the unique solution, in (a,b), of the equation $E_P(x) = (-1)^{m_1-m_2} E_Q(x)$. Equivalently, $M_{f,m_1,m_2}(a,b)$ is the unique solution, in (a,b), of the equation

$$(x-a)^{m_1-m_2} f[x, a^{m_1+1}, b^{m_2+1}] = (b-x)^{m_1-m_2} f[x, a^{m_2+1}, b^{m_1+1}]. \tag{4}$$

As in [2] and [3], we shall see that some of the familiar means, such as the arithmetic, geometric, and harmonic means arise in certain special cases. For $f(x) = x^p$, we denote $M_{f,m_1,m_2}(a,b)$ by $M_{p,m_1,m_2}(a,b)$ for any real number p with $p \notin \{0,1,...,n\}$. If $p = k, k \in \{0,1,...,n\}$, one can define M_{p,m_1,m_2} using a limiting argument, or by defining M_{p,m_1,m_2} to be M_{f,m_1,m_2}, where $f(x) = x^k \log x$. This gives a continuous extension of M_{p,m_1,m_2} to all real numbers p.

REMARK 5. For any polynomial $R \in \pi_n$, $M_{f-R,m_1,m_2}(a,b) = M_{f,m_1,m_2}(a,b)$.

The following three theorems are the analogs of ([3], Theorems 1.3 and 1.4) and ([2], Theorem 1.8) for Hermite interpolation.

THEOREM 6. *If* $p = m_1 + m_2 + 2$*, then* $M_{p,m_1,m_2}(a,b) = A(a,b) = \frac{a+b}{2}$.

Proof. If $f(x) = x^{m_1+m_2+2}$, then by the Mean Value Theorem for divided differences, $f[x, a^{m_1+1}, b^{m_2+1}] = f[x, a^{m_2+1}, b^{m_1+1}] = (m_1 + m_2 + 2)!$. Thus the unique solution, x_0, in $I = (a,b)$ of the equation $E_P(x) = (-1)^{m_1-m_2} E_Q(x)$ is the unique solution of $(x-a)^{m_1-m_2} = (-1)^{m_1-m_2}(x-b)^{m_1-m_2}$, which implies that $x_0 = \frac{a+b}{2}$.
□

THEOREM 7. *If* $p = -1$*, then* $M_{p,m_1,m_2}(a,b) = H(a,b) = \frac{2ab}{a+b}$ *for any* m_1 *and* m_2.

Proof. If $f(x) = \frac{1}{x}$, then $f[x_0, x_1, ..., x_n] = \frac{(-1)^n}{x_0 x_1 \cdots x_n}$ (see [6], page 11, formula (4)]. It then follows easily that $f[x, a^{m_1+1}, b^{m_2+1}] = \frac{(-1)^{m_1+m_2}}{a^{m_1+1} b^{m_2+1} x}$ and $f[x, a^{m_2+1}, b^{m_1+1}] = \frac{(-1)^{m_1+m_2}}{a^{m_2+1} b^{m_1+1} x}$. Thus the unique solution, x_0, in $I = (a,b)$, of the equation $E_P(x) = (-1)^{m_1-m_2} E_Q(x)$ is the unique solution of

$$(x-a)^{m_1-m_2} \frac{(-1)^{m_1+m_2}}{a^{m_1+1} b^{m_2+1} x} = (-1)^{m_1-m_2}(x-b)^{m_1-m_2} \frac{(-1)^{m_1+m_2}}{a^{m_2+1} b^{m_1+1} x},$$

which is equivalent to $(x-a)^{m_1-m_2} b^{m_1-m_2} = (x-b)^{m_1-m_2} a^{m_1-m_2} \implies x = \frac{2ab}{a+b}$.
□

Theorems 6 and 7 show that the arithmetic and harmonic means arise as the x coordinates of the intersection point of Hermite interpolants. Our next result shows that the geometric mean arises as well, but the proof is considerably more difficult.

THEOREM 8. *If* $p = \frac{m_1+m_2+1}{2}$*, where* $m_1 + m_2$ *is even, then* $M_{p,m_1,m_2}(a,b) = G(a,b) = \sqrt{ab}$.

REMARK 9. Theorem 8 does not hold if $m_1 + m_2$ is odd. In that case, p is a positive integer strictly less than $m_1 + m_2 + 1$, which implies that $f^{(m_1+m_2+2)}(x) \equiv 0$.

Before proving Theorem 8, we need three lemmas.

LEMMA 10. *Let $m_1 \geqslant 0$ be any integer. Then*

$$\sum_{k=0}^{m_1}(-1)^k\binom{m_1/2}{k}(1-b)^k = b^{m_1}\sum_{k=0}^{m_1}\binom{m_1/2}{k}(1-b)^k b^{-k}, b \neq 0 \tag{5}$$

Proof. We use induction in m_1. First, it is trivial that (5) holds when $m_1 = 0$ or $m_1 = 1$. Now let $G_{m_1}(b) = \sum_{k=0}^{m_1}(-1)^k\binom{m_1/2}{k}(1-b)^k$ and $H_{m_1}(b) = b^{m_1}\sum_{k=0}^{m_1}\binom{m_1/2}{k}(1-b)^k b^{-k}$ denote the left and right hand sides of (5), respectively. Then

$$\begin{aligned}
G_{m_1+2}(b) &= \sum_{k=0}^{m_1+2}(-1)^k\binom{m_1/2+1}{k}(1-b)^k \\
&= (-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(1-b)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} \\
&\quad + \sum_{k=0}^{m_1}(-1)^k\binom{m_1/2}{k}(1-b)^k + \sum_{k=0}^{m_1-1}(-1)^{k+1}\binom{m_1/2}{k}(1-b)^{k+1} \\
&= (-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(1-b)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} \\
&\quad + G_{m_1}(b) - (1-b)G_{m_1}(b) - (-1)^{m_1+2}\binom{m_1/2}{m_1}(1-b)^{m_1+1} \\
&= bG_{m_1}(b) + (-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(1-b)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} \\
&\quad - (-1)^{m_1+1}\binom{m_1/2}{m_1}(1-b)^{m_1+1}.
\end{aligned}$$

It is easy to show that $b^{m_1}G_{m_1}\left(\frac{1}{b}\right) = H_{m_1}(b)$. Thus

$$\begin{aligned}
H_{m_1+2}(b) &= b^{m_1+2}G_{m_1+2}\left(\tfrac{1}{b}\right) \\
&= b^{m_1+2}(-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(1-1/b)^{m_1+1} + b^{m_1+2}(-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(1-1/b)^{m_1+2} \\
&\quad + b^{m_1+1}G_{m_1}\left(\tfrac{1}{b}\right) - b^{m_1+2}(-1)^{m_1+1}\binom{m_1/2}{m_1}(1-1/b)^{m_1+1} \\
&= bH_{m_1}(b) + b(-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(b-1)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(b-1)^{m_1+2} \\
&\quad - b(-1)^{m_1+1}\binom{m_1/2}{m_1}(b-1)^{m_1+1}.
\end{aligned}$$

Assuming that $G_{m_1}(b) = H_{m_1}(b)$, we have that

$$\begin{gathered}
G_{m_1+2}(b) = H_{m_1+2}(b) \iff \\
(-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(1-b)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} - (-1)^{m_1+1}\binom{m_1/2}{m_1}(1-b)^{m_1+1} \\
= b(-1)^{m_1+1}\binom{m_1/2+1}{m_1+1}(b-1)^{m_1+1} + (-1)^{m_1+2}\binom{m_1/2+1}{m_1+2}(b-1)^{m_1+2} \\
-b(-1)^{m_1+1}\binom{m_1/2}{m_1}(b-1)^{m_1+1}.
\end{gathered}$$

If m_1 is even, then the equality holds trivially since all terms involved are 0. So assume now that m_1 is odd. Then

$$
\begin{aligned}
& G_{m_1+2}(b) = H_{m_1+2}(b) \\
\iff\ & \tbinom{m_1/2+1}{m_1+1}(1-b)^{m_1+1} - \tbinom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} - \tbinom{m_1/2}{m_1}(1-b)^{m_1+1} \\
& \quad = b\tbinom{m_1/2+1}{m_1+1}(b-1)^{m_1+1} - \tbinom{m_1/2+1}{m_1+2}(b-1)^{m_1+2} - b\tbinom{m_1/2}{m_1}(b-1)^{m_1+1} \\
\iff\ & \tbinom{m_1/2+1}{m_1+1}(1-b)^{m_1+2} - 2\tbinom{m_1/2+1}{m_1+2}(1-b)^{m_1+2} - \tbinom{m_1/2}{m_1}(1-b)^{m_1+2} = 0 \\
\iff\ & \tbinom{m_1/2+1}{m_1+1} - 2\tbinom{m_1/2+1}{m_1+2} - \tbinom{m_1/2}{m_1} = 0 \\
\iff\ & \tbinom{m_1/2}{m_1+1} - 2\tbinom{m_1/2+1}{m_1+2} = 0 \\
\iff\ & \frac{\left(\frac{m_1}{2}\right)\left(\frac{m_1}{2}-1\right)\cdots\left(\frac{m_1}{2}-m_1\right)}{(m_1+1)!} - 2\frac{\left(\frac{m_1}{2}+1\right)\left(\frac{m_1}{2}-1\right)\cdots\left(\frac{m_1}{2}-m_1\right)}{(m_1+2)!} = 0 \\
\iff\ & \frac{m_1(m_1-2)\cdots(m_1-2m_1)}{2^{m_1+1}(m_1+1)!} - \frac{(m_1+2)(m_1-2)\cdots(m_1-2m_1)}{2^{m_1+1}(m_1+2)!} = 0 \\
\iff\ & (m_1+2)m_1(m_1-2)\cdots(-m_1) - (m_1+2)m_1(m_1-2)\cdots(-m_1) = 0.
\end{aligned}
$$

That completes the proof of Lemma 10. □

LEMMA 11. *Let m_1 and m_2 be any integers. Then if $y > 0$,*

$$\sum_{k=0}^{m_1} \binom{m_2+m_1-k}{m_2}\binom{(m_2+m_1)/2}{k}(-1)^k(1-y)^k = y^{m_1}\sum_{k=0}^{m_1} \binom{m_2+m_1-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k y^{-k}$$

Proof. It is not hard to show, using, for example, the methods in [8], that $\frac{1}{\binom{m_1+m_2}{m_2}}\sum_{k=0}^{m_1}\binom{m_2+m_1-k}{m_2}\binom{(m_2+m_1)/2}{k}x^k = {}_2F_1\left(-\frac{1}{2}m_1-\frac{1}{2}m_2, -m_1; -m_1-m_2; -x\right)$, where ${}_2F_1([a,b],[c],z)$ is the hypergeometric function $\sum_{k\geqslant 0}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$. Thus it suffices to prove that

$$
\begin{aligned}
& {}_2F_1\left(-\tfrac{1}{2}m_1-\tfrac{1}{2}m_2, -m_1; -m_1-m_2; 1-y\right) \\
& \qquad = y^{m_1}\,{}_2F_1\left(-\tfrac{1}{2}m_1-\tfrac{1}{2}m_2, -m_1; -m_1-m_2; \tfrac{y-1}{y}\right).
\end{aligned}
$$

The latter equality follows from the identity $(1-x)^b\,{}_2F_1(a,b;c;x) = {}_2F_1\left(c-a,b;c;\frac{x}{x-1}\right)$, $x\notin(1,\infty)$ with $a = -\frac{1}{2}m_1-\frac{1}{2}m_2, b=-m_1, c=-m_1-m_2$, and $x = 1-y$. That proves Lemma 11. □

We now use Lemmas 10 and 11 to prove the following identity.

LEMMA 12. *If $m_1 \geqslant 0$ and m_2 are any integers, if $p = \frac{m_1+m_2+1}{2}$, and if $b \neq -1$ and $b \neq 0$, then*

$$\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}(-1)^k\binom{p}{k}(1-b)^k(1+b)^{-l} = b^{m_1}\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}.$$

REMARK 13. The lemma actually holds if m_1 is a negative integer if one interprets both sides of the equality to be 0 in that case.

Proof. We denote the left and right hand sides in Lemma 12 by

$$\begin{aligned} L_{m_1,m_2}(b) &= \sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l}, \\ R_{m_1,m_2}(b) &= b^{m_1}\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{l}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}, \end{aligned}$$

respectively. Here we used the identity $\binom{m_2+l}{m_2}=\binom{m_2+l}{l}$. Thus, to prove Lemma 12, it suffices to prove

$$L_{m_1,m_2}(b)=R_{m_1,m_2}(b). \tag{6}$$

We shall first prove the recursion

$$\begin{aligned} &L_{m_1+1,m_2-1}(b) \\ &\quad = \tfrac{b}{b+1}L_{m_1,m_2}(b)+\sum_{k=0}^{m_1+1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-m_1-1+k} \end{aligned} \tag{7}$$

Throughout we use the identities $\binom{n+1}{k}=\binom{n}{k}+\binom{n}{k-1}$, which, in particular, implies that $\binom{m_2+l-1}{l}=\binom{m_2+l}{l}-\binom{m_2+l-1}{l-1}$. In addition we use the fact that $\binom{n}{-1}=0$ for any whole number, n. Now

$$\begin{aligned} &L_{m_1+1,m_2-1}(b) \\ &= \sum_{k=0}^{m_1+1}\sum_{l=0}^{m_1+1-k}\binom{m_2+l-1}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &= \binom{p}{m_1+1}(-1)^{m_1+1}(1-b)^{m_1+1}+\sum_{k=0}^{m_1}\sum_{l=0}^{m_1+1-k}\binom{m_2+l-1}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &= \binom{p}{m_1+1}(-1)^{m_1+1}(1-b)^{m_1+1}+\sum_{k=0}^{m_1}\sum_{l=0}^{m_1+1-k}\binom{m_2+l}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &\qquad -\sum_{k=0}^{m_1}\sum_{l=0}^{m_1+1-k}\binom{m_2+l-1}{l-1}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &= \binom{p}{m_1+1}(-1)^{m_1+1}(1-b)^{m_1+1}+\sum_{k=0}^{m_1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-m_1-1+k} \\ &\qquad +\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &\qquad -\sum_{k=0}^{m_1}\binom{m_2+m_1-k}{m_2}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-m_1-1+k} \\ &\qquad -\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l-1}{l-1}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \\ &= \sum_{k=0}^{m_1+1}\binom{m_2+m_1-k}{m_2-1}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-m_1-1+k} \\ &\qquad +\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{l}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-l} \end{aligned}$$

$$
\begin{aligned}
&- \sum_{k=0}^{m_1} \sum_{l=-1}^{m_1-k} \binom{m_2+l}{l} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-l-1} \\
&= \sum_{k=0}^{m_1+1} \binom{m_2+m_1-k}{m_2-1} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-m_1-1+k} \\
&\quad + \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{m_2+l}{l} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-l} \\
&\quad - (1+b)^{-1} \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{m_2+l}{l} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-l} \\
&= \sum_{k=0}^{m_1+1} \binom{m_2+m_1-k}{m_2-1} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-m_1-1+k} + L_{m_1,m_2}(b) - (1+b)^{-1} L_{m_1,m_2}(b) \\
&= \frac{b}{b+1} L_{m_1,m_2}(b) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (-1)^k (1-b)^k (1+b)^{-m_1-1+k}.
\end{aligned}
$$

That proves (7). It is easy to show that

$$
b^{m_1} L_{m_1,m_2}\left(\frac{1}{b}\right) = R_{m_1,m_2}(b). \tag{8}
$$

Thus by (8) and (7),

$$
\begin{aligned}
&R_{m_1+1,m_2-1}(b) = b^{m_1+1} L_{m_1+1,m_2-1}\left(\frac{1}{b}\right) \\
&= b^{m_1+1}\left(\frac{1}{b+1} L_{m_1,m_2}\left(\frac{1}{b}\right) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (-1)^k (1-1/b)^k (1+1/b)^{-m_1-1+k} \right) \\
&= b^{m_1+1}\left(\frac{1}{b+1} L_{m_1,m_2}\left(\frac{1}{b}\right) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (1-b)^k (1+b)^{-m_1-1+k} b^{m_1+1-2k} \right) \\
&= \left(\frac{b^{m_1+1}}{b+1} L_{m_1,m_2}\left(\frac{1}{b}\right) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (1-b)^k (1+b)^{-m_1-1+k} b^{2m_1+2-2k} \right) \\
&= \frac{b}{b+1} R_{m_1,m_2}(b) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (1-b)^k (1+b)^{-m_1-1+k} b^{2m_1+2-2k}.
\end{aligned}
$$

That yields the recursion

$$
\begin{aligned}
&R_{m_1+1,m_2-1}(b) \\
&\quad = \frac{b}{b+1} R_{m_1,m_2}(b) + \sum_{k=0}^{m_1+1} \binom{m_2+m_1+1-k}{m_2} \binom{p}{k} (1-b)^k (1+b)^{-m_1-1+k} b^{2m_1+2-2k}.
\end{aligned} \tag{9}
$$

In a similar fashion, one can also prove the recursions

$$
\begin{aligned}
L_{m_1-1,m_2+1}(b) &= \frac{b+1}{b} L_{m_1,m_2}(b) - \frac{b+1}{b} (-1)^{m_1} \binom{p}{m_1} (1-b)^{m_1} \\
&\quad - \frac{b+1}{b} \sum_{k=0}^{m_1-1} (-1)^k \binom{m_1+m_2-k+1}{m_2+1} \binom{p}{k} (1-b)^k (1+b)^{-(m_1-k)}
\end{aligned} \tag{10}
$$

and

$$
\begin{aligned}
R_{m_1-1,m_2+1}(b) &= \frac{b+1}{b} R_{m_1,m_2}(b) - \frac{b+1}{b} (-1)^{m_1} \binom{p}{m_1} (b-1)^{m_1} \\
&\quad - \frac{b+1}{b} \sum_{k=0}^{m_1-1} (-1)^k \binom{m_1+m_2-k+1}{m_2+1} \binom{p}{k} (b-1)^k (1+b)^{-m_1+k} b^{2m_1-2k}
\end{aligned} \tag{11}
$$

We now use induction to prove (6). We start the induction with $m_2 = -1$ and m_1 any fixed non–negative integer. In that case the only nonzero term on both sides of (6) occurs when $l = 0$, which yields $\sum_{k=0}^{m_1}(-1)^k\binom{m_1/2}{k}(1-b)^k = b^{m_1}\sum_{k=0}^{m_1}\binom{m_1/2}{k}(1-b)^k b^{-k}$, which is precisely Lemma 10. Proceeding with the induction, we assume now that $L_{m_1,m_2}(b) = R_{m_1,m_2}(b)$. Then, using (7) and (9), it follows that

$$L_{m_1+1,m_2-1}(b) = R_{m_1+1,m_2-1}(b) \iff$$

$$\sum_{k=0}^{m_1+1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(-1)^k(1-b)^k(1+b)^{-m_1-1+k}$$

$$= \sum_{k=0}^{m_1+1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(1-b)^k(1+b)^{-m_1-1+k}b^{2m_1+2-2k} \iff$$

$$\sum_{k=0}^{m_1+1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(-1)^k(1-b^2)^k = \left(b^2\right)^{m_1+1}\sum_{k=0}^{m_1+1}\binom{m_2+m_1+1-k}{m_2}\binom{p}{k}(1-b^2)^k\left(b^2\right)^{-k} \tag{12}$$

(12) now follows from Lemma 11, by replacing m_1+1 by m_1 and letting $y = b^2$. That proves, with the assumption $L_{m_1,m_2}(b) = R_{m_1,m_2}(b)$, that

$$L_{m_1+1,m_2-1}(b) = R_{m_1+1,m_2-1}(b). \tag{13}$$

Also, using (10) and (11), it follows that

$$L_{m_1-1,m_2+1}(b) = R_{m_1-1,m_2+1}(b) \iff$$

$$(-1)^{m_1}\binom{p}{m_1}(1-b)^{m_1} + \sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}(1-b)^k(1+b)^{-(m_1-k)}$$

$$= (-1)^{m_1}\binom{p}{m_1}(b-1)^{m_1} + \sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}(b-1)^k(1+b)^{-m_1+k}b^{2m_1-2k}.$$

To prove this equality we consider two cases.

Case 1: m_1 is even

We must show that

$$\sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}(1-b^2)^k = b^{2m_1}\sum_{k=0}^{m_1-1}\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}\left(\frac{1-b^2}{b^2}\right)^k,$$

which is equivalent to

$$\sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k = y^{m_1}\sum_{k=0}^{m_1-1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k$$

upon replacing m_2 by m_2-1 and letting $y = b^2$. Hence we must prove that

$$\sum_{k=0}^{m_1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k - (-1)^{m_1}\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}$$

$$= y^{m_1}\sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k - y^{m_1}\binom{(m_2+m_1)/2}{m_1}\left(\frac{1-y}{y}\right)^{m_1} \iff$$

$$\sum_{k=0}^{m_1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k = y^{m_1}\sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k,$$

which follows from Lemma 11.

Case 2: m_1 is odd
We must show that

$$-\binom{p}{m_1}(1-b)^{m_1}+(1+b)^{-m_1}\sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}(1-b^2)^k$$
$$=\binom{p}{m_1}(1-b)^{m_1}+(1+b)^{-m_1}b^{2m_1}\sum_{k=0}^{m_1-1}\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}\left(\frac{1-b^2}{b^2}\right)^k \iff$$
$$-\binom{p}{m_1}(1-b^2)^{m_1}+\sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}(1-b^2)^k$$
$$=\binom{p}{m_1}(1-b^2)^{m_1}+b^{2m_1}\sum_{k=0}^{m_1-1}\binom{m_1+m_2-k+1}{m_2+1}\binom{p}{k}\left(\frac{1-b^2}{b^2}\right)^k.$$

Replace m_2 by m_2-1 and let $y=b^2$ to get

$$\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}+\sum_{k=0}^{m_1-1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k$$
$$=\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}+y^{m_1}\sum_{k=0}^{m_1-1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k \iff$$
$$-\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}+\sum_{k=0}^{m_1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k+\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}$$
$$=\binom{(m_2+m_1)/2}{m_1}(1-y)^{m_1}+y^{m_1}\sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k-y^{m_1}\binom{(m_2+m_1)/2}{m_1}\left(\frac{1-y}{y}\right)^{m_1}$$
$$\iff \sum_{k=0}^{m_1}(-1)^k\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}(1-y)^k=y^{m_1}\sum_{k=0}^{m_1}\binom{m_1+m_2-k}{m_2}\binom{(m_2+m_1)/2}{k}\left(\frac{1-y}{y}\right)^k,$$

which is again Lemma 11. That proves, with the assumption $L_{m_1,m_2}(b)=R_{m_1,m_2}(b)$, that

$$L_{m_1-1,m_2+1}(b)=R_{m_1-1,m_2+1}(b). \tag{14}$$

The case $m_2=-1$ and (14) shows that (6) holds when $m_1=m_2$ and m_1 is any non–negative integer, or when $m_1=m_2+1$ and m_1 is any non–negative integer. (13) now shows that (6) holds when $m_1\geqslant 0$ and m_2 are any integers. That finishes the proof of Lemma 12. □

We are now ready to prove Theorem 8.

Proof. We now use a formula due to Spitzbart (see [9], Theorem 2), which expresses divided differences of the form $f[x_0^{r_0+1},x_1^{r_1+1},\ldots,x_n^{r_n+1}]$ with confluent arguments as a linear combination of the values of f and its derivatives at $x_0,x_1,\ldots,x_n$. Using $f(x)=x^p, x_0=x, x_1=a, x_2=b, r_0=0, r_1=m_1,$ and $r_2=m_2$, one can write

$$f[x,a^{m_1+1},b^{m_2+1}]=A_1+B_1+C_1, \tag{15}$$

where

$$A_1 = \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{-m_2-1}{l} \binom{-1}{m_1-k-l} \binom{p}{k} (a-b)^{-m_2-1-l} (a-x)^{-m_1-1+k+l} a^{p-k},$$

$$B_1 = \sum_{k=0}^{m_2} \sum_{l=0}^{m_2-k} \binom{-m_1-1}{l} \binom{-1}{m_2-k-l} \binom{p}{k} (b-a)^{-m_1-1-l} (b-x)^{-m_2-1+k+l} b^{p-k},$$

and

$$C_1 = (x-a)^{-m_1-1}(x-b)^{-m_2-1} x^p. \tag{16}$$

Using the identities $\binom{-1}{m_j-k-l} = (-1)^{m_j}(-1)^{k+l}$ and $\binom{-m_j-1}{l} = (-1)^l \binom{m_j+l}{m_j}$, $j = 1,2$ to simplify the expressions for A_1 and B_1 yields

$$A_1 = a^p \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{m_2+l}{m_2} \binom{p}{k} (-1)^{m_1+k} (a-b)^{-m_2-1-l} (a-x)^{-m_1-1+k+l} a^{-k}, \tag{17}$$

$$B_1 = b^p \sum_{k=0}^{m_2} \sum_{l=0}^{m_2-k} \binom{m_1+l}{m_1} \binom{p}{k} (-1)^{m_2+k} (b-a)^{-m_1-1-l} (b-x)^{-m_2-1+k+l} b^{-k}$$

By switching m_1 and m_2 we obtain

$$f\,[x, a^{m_2+1}, b^{m_1+1}] = A_2 + B_2 + C_2, \tag{18}$$

where

$$A_2 = a^p \sum_{k=0}^{m_2} \sum_{l=0}^{m_2-k} \binom{m_1+l}{m_1} \binom{p}{k} (-1)^{m_2+k} (a-b)^{-m_1-1-l} (a-x)^{-m_2-1+k+l} a^{-k}, \tag{19}$$

$$B_2 = b^p \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{m_2+l}{m_2} \binom{p}{k} (-1)^{m_1+k} (b-a)^{-m_2-1-l} (b-x)^{-m_1-1+k+l} b^{-k},$$

$$C_2 = (x-a)^{-m_2-1}(x-b)^{-m_1-1} x^p.$$

Now letting $x = 1$ and $a = \frac{1}{b}$ in (15) and (18) yields

$$\begin{aligned} b^{m_2}(A_1 + B_1 + C_1) &= b^{m_2} f\,[1, (1/b)^{m_1+1}, b^{m_2+1}] \\ b^{m_1}(A_2 + B_2 + C_2) &= b^{m_1} f\,[1, (1/b)^{m_2+1}, b^{m_1+1}], \end{aligned} \tag{20}$$

After some simplification, we have

$$\begin{aligned}
b^{m_2} A_1 &= (-1)^{m_1} b^{m_1+2m_2+2-p} (1-b)^{-m_1-m_2-2} (1+b)^{-m_2-1} \\
&\quad \cdot \sum_{k=0}^{m_1} \sum_{l=0}^{m_1-k} \binom{m_2+l}{m_2} (-1)^k \binom{p}{k} (1-b)^k (1+b)^{-l}, \\
b^{m_2} B_1 &= (-1)^{m_1} b^{m_1+m_2+p+1} (1-b)^{-m_1-m_2-2} (1+b)^{-m_1-1} \\
&\quad \cdot \sum_{k=0}^{m_2} \sum_{l=0}^{m_2-k} \binom{m_1+l}{m_1} \binom{p}{k} (1-b)^k (1+b)^{-l} b^{l-k}, \\
b^{m_2} C_1 &= (-1)^{m_1+1} (1-b)^{-m_1-m_2-2} b^{m_1+m_2+1}, \\
b^{m_1} A_2 &= (-1)^{m_2} b^{m_2+2m_1+2-p} (1-b)^{-m_1-m_2-2} (1+b)^{-m_1-1} \\
&\quad \cdot \sum_{k=0}^{m_2} \sum_{l=0}^{m_2-k} \binom{m_1+l}{m_1} (-1)^k \binom{p}{k} (1-b)^k (1+b)^{-l},
\end{aligned}$$

$$b^{m_1}B_2 = (-1)^{m_2} b^{m_1+m_2+p+1}(1-b)^{-m_1-m_2-2}(1+b)^{-m_2-1} \cdot \sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k} \binom{m_2+l}{m_2}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k},$$

and $b^{m_1}C_2 = (-1)^{m_2+1}(1-b)^{-m_1-m_2-2}b^{m_1+m_2+1}$. We claim:

$$b^{m_2}A_1 = b^{m_1}B_2, \quad b^{m_1}A_2 = b^{m_2}B_1, \quad b^{m_2}C_1 = b^{m_1}C_2, \quad b \neq \pm 1. \tag{21}$$

It is trivial that $b^{m_2}C_1 = b^{m_1}C_2$. Now

$$b^{m_2}A_1 = b^{m_1}B_2 \iff$$

$$(-1)^{m_1} b^{m_1+2m_2+2-p}(1-b)^{-m_1-m_2-2}(1+b)^{-m_2-1}\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}(-1)^k\binom{p}{k}(1-b)^k(1+b)^{-l}$$

$$= (-1)^{m_2} b^{m_1+m_2+p+1}(1-b)^{-m_1-m_2-2}(1+b)^{-m_2-1}\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}$$

$$\iff$$

$$\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}(-1)^k\binom{p}{k}(1-b)^k(1+b)^{-l} = b^{m_1}\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\binom{m_2+l}{m_2}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}, \tag{22}$$

and

$$b^{m_1}A_2 = b^{m_2}B_1 \iff$$

$$(-1)^{m_2} b^{m_2+2m_1+2-p}(1-b)^{-m_1-m_2-2}(1+b)^{-m_1-1}\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\binom{m_1+l}{m_1}(-1)^k\binom{p}{k}(1-b)^k(1+b)^{-l}$$

$$= (-1)^{m_1} b^{m_1+m_2+p+1}(1-b)^{-m_1-m_2-2}(1+b)^{-m_1-1}\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\binom{m_1+l}{m_1}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}$$

$$\iff$$

$$\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}(-1)^k\binom{m_1+l}{m_1}\binom{p}{k}(1-b)^k(1+b)^{-l} = b^{m_2}\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\binom{m_1+l}{m_1}\binom{p}{k}(1-b)^k(1+b)^{-l}b^{l-k}. \tag{23}$$

(22) is precisely Lemma 12, and the proof of (23) is very similar to the proof of Lemma 12. More simply, one can just interchange m_1 and m_2 in Lemma 12, since Lemma 12 actually holds for all integers m_1 and m_2 (see the remark following Lemma 12). That proves (21), which immediately gives

$$b^{m_2}(A_1+B_1+C_1) = b^{m_1}(A_2+B_2+C_2). \tag{24}$$

Now, if $f(x) = x^{(m_1+m_2+1)/2}$, then M_{p,m_1,m_2} is a homogeneous mean. Thus it suffices to prove that $M_{p,m_1,m_2}\left(\frac{1}{b}, b\right) = 1, b \neq 1, b \geqslant 0$, which is equivalent to

$$\left(1-\tfrac{1}{b}\right)^m f\left[1,(1/b)^{m_1+1},b^{m_2+1}\right] = (1-b)^m f\left[1,(1/b)^{m_2+1},b^{m_1+1}\right]$$

by (3) with $a = \frac{1}{b}$ and $x = 1$. A little simplification yields $b^{m_2} f[1, (1/b)^{m_1+1}, b^{m_2+1}] = b^{m_1} f[1, (1/b)^{m_2+1}, b^{m_1+1}]$, which follows directly from (24) using (20). $\square$

REMARK 14. There are various well known integral representations for divided differences which might be used to give a shorter proof of Theorem 8. This author, however, was not able to make such a proof work.

Before proving our next result, we need a theorem about Cauchy Mean Values, which have been discussed by many authors. In particular, we use results from the paper by Leach and Sholander [5]. Let I be an open interval of real numbers and consider two given functions $f, g \in C^n(I)$. Suppose that $g^{(n)}(x) \neq 0$ for $x \in I$ and that ϕ is monotone on I, where $\phi(x) = \frac{f^{(n)}(x)}{g^{(n)}(x)}$. Given $n+1$ numbers $\{x_0, x_1, ..., x_n\} \subseteq I$, there is a unique $c, \min\{x_0, x_1, ..., x_n\} \leqslant c \leqslant \max\{x_0, x_1, ..., x_n\}$, such that $\frac{f[x_0, x_1, ..., x_n]}{g[x_0, x_1, ..., x_n]} = \frac{f^{(n)}(c)}{g^{(n)}(c)}$. Of course, if the $x_0, x_1, ..., x_n$ are not distinct, we use the extended definition of the divided difference $f[x_0, x_1, ..., x_n]$ for confluent nodes. This defines a mean $c = M_{f,g}(x_0, x_1, ..., x_n)$. We state the following result of Leach and Sholander from ([5], Theorem 3) with the notation altered slightly for our purposes.

THEOREM 15. *If $\phi'(x)$ is never 0 on I, then $\frac{\partial}{\partial x_k} M_{f,g}(x_0, x_1, ..., x_n) > 0$ for $k = 0, 1, ..., n$.*

Now we prove the following lemma.

LEMMA 16. *Let $I = (a, b), 0 < a < b$ be a given open interval, let $m_2 < m_1$ be two given nonnegative integers, with $n = m_1 + m_2 + 1$, and suppose that $f, g \in C^{n+2}(I)$ with $f^{(n+1)}$ and $g^{(n+1)}$ nonzero on I. Assume also that $g^{(n+1)}(x)$ and $\phi'(x)$ are never 0 on I, where $\phi(x) = \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$. Let $\zeta_P, \zeta_Q \in I$ be the unique values satisfying $\frac{f[x, a^{m_1+1}, b^{m_2+1}]}{g[x, a^{m_1+1}, b^{m_2+1}]} = \frac{f^{(n+1)}(\zeta_P)}{g^{(n+1)}(\zeta_P)}$ and $\frac{f[x, a^{m_2+1}, b^{m_1+1}]}{g[x, a^{m_2+1}, b^{m_1+1}]} = \frac{f^{(n+1)}(\zeta_Q)}{g^{(n+1)}(\zeta_Q)}$. Then $\zeta_P < \zeta_Q$.*

Proof. $\frac{f[x, a^{m_1+1}, b^{m_2+1}]}{g[x, a^{m_1+1}, b^{m_2+1}]} = \frac{f[x_0, x_1, ..., x_n]}{g[x_0, x_1, ..., x_n]}$ where $x_0 = x, x_1 = \cdots = x_{m_1+1} = a$, and $x_{m_1+2} = \cdots = x_{m_1+m_2+2} = b$, while $\frac{f[x, a^{m_2+1}, b^{m_1+1}]}{g[x, a^{m_2+1}, b^{m_1+1}]} = \frac{f[x_0, x_1, ..., x_n]}{g[x_0, x_1, ..., x_n]}$ where $x_0 = x, x_1 = \cdots = x_{m_2+1} = a$, and $x_{m_2+2} = \cdots = x_{m_1+m_2+2} = b$. Then $\zeta_P = M_{f,g}\left(x, a^{m_1+1}, b^{m_2+1}\right)$ and $\zeta_Q = M_{f,g}\left(x, a^{m_2+1}, b^{m_1+1}\right)$, where $M_{f,g}$ denotes the mean defined above. Since $m_2 < m_1$ and $a < b$, by Theorem 15, $\zeta_P < \zeta_Q$. $\square$

Recall that the means discussed in this paper are denoted by $M_{f,m_1,m_2}(a, b)$, where $M_{f,m_1,m_2}(a, b)$ is the unique solution, in (a, b), of the equation $E_P(x) = (-1)^{m_1 - m_2} E_Q(x)$, $E_P(x)$ and $E_Q(x)$ given by (2). We now prove a result about when M_{f,m_1,m_2} and M_{g,m_1,m_2} are comparable. For any sufficiently smooth f, we let P_f and Q_f denote the Hermite interpolants satisfying (1). We also let $E_{P,f} = f - P_f$ and so on.

THEOREM 17. *Suppose that $\phi = \frac{f^{(n+1)}}{g^{(n+1)}}$ is strictly monotonic on $(0, \infty)$, where $f, g \in C^{n+1}(0, \infty)$. Then the means M_{f,m_1,m_2} and M_{g,m_1,m_2} are strictly comparable.*

That is, either $M_{f,m_1,m_2}(a,b) < M_{g,m_1,m_2}(a,b)$ *or* $M_{f,m_1,m_2}(a,b) > M_{g,m_1,m_2}(a,b)$ *for all* $(a,b) \in \Re_2^+$.

Proof. Suppose that $M_{f,m_1,m_2}(a,b) = M_{g,m_1,m_2}(a,b) = x_0$ for *some* $(a,b) \in O = \{(x,y) : 0 < x < y\}$. Note that $g(x_0) - P_g(x_0) \neq 0$ and $g(x_0) - Q_g(x_0) \neq 0$ since $g^{(n+1)}$ is nonzero on I. Then $E_{P,f}(x_0) = (-1)^{m_1-m_2} E_{Q,f}(x_0)$ and $E_{P,g}(x_0) = (-1)^{m_1-m_2} E_{Q,g}(x_0)$, which implies that $\frac{E_{P,f}(x_0)}{E_{P,g}(x_0)} = \frac{E_{Q,f}(x_0)}{E_{Q,g}(x_0)}$. By (2), we then have $\frac{f[x_0,a^{m_1+1},b^{m_2+1}]}{g[x_0,a^{m_1+1},b^{m_2+1}]} = \frac{f[x_0,a^{m_2+1},b^{m_1+1}]}{g[x_0,a^{m_2+1},b^{m_1+1}]}$. Let

$$\zeta_P = \phi^{-1}\left(\frac{f[x_0,a^{m_1+1},b^{m_2+1}]}{g[x_0,a^{m_1+1},b^{m_2+1}]}\right) \quad \text{and} \quad \zeta_Q = \phi^{-1}\left(\frac{f[x_0,a^{m_2+1},b^{m_1+1}]}{g[x_0,a^{m_2+1},b^{m_1+1}]}\right).$$

By Lemma 16, $\zeta_P < \zeta_Q$, which contradicts the fact that $\frac{f[x_0,a^{m_1+1},b^{m_2+1}]}{g[x_0,a^{m_1+1},b^{m_2+1}]} = \frac{f[x_0,a^{m_2+1},b^{m_1+1}]}{g[x_0,a^{m_2+1},b^{m_1+1}]}$. Thus $M_{f,m_1,m_2}(a,b)$ and $M_{g,m_1,m_2}(a,b)$ are never equal on O. Since M_{f,m_1,m_2} and M_{g,m_1,m_2} are each continuous on O and O is connected, that proves that either $M_{f,m_1,m_2}(a,b) < M_{g,m_1,m_2}(a,b)$ or $M_{f,m_1,m_2}(a,b) > M_{g,m_1,m_2}(a,b)$ for all $(a,b) \in O$ by the intermediate value theorem. Since the means M_{f,m_1,m_2} are symmetric, that proves Theorem 17. □

THEOREM 18. *Let* $m_2 < m_1$ *be two given nonnegative integers, with* $n = m_1+m_2+1$, *and suppose that* $f,g \in C^{n+2}(0,\infty)$. *Then* $M_{f,m_1,m_2}(a,b) = M_{g,m_1,m_2}(a,b)$ *for all* $(a,b) \in \Re_2^+$ *if and only if* $g(x) = cf(x) + p(x)$ *for some constant* c *and some polynomial* $p \in \pi_n$.

Proof. ($\Longleftarrow$ Suppose that $g(x) = cf(x) + p(x)$ for some constant c and some polynomial $p \in \pi_n$. Then it is trivial that $P_f = P_g$ and $Q_f = Q_g$, which implies that $M_{f,m_1,m_2}(a,b) = M_{g,m_1,m_2}(a,b)$ for all $(a,b) \in \Re_2^+$.

($\Longrightarrow$ Suppose that $M_{f,m_1,m_2}(a,b) = M_{g,m_1,m_2}(a,b)$ for all $(a,b) \in \Re_2^+$, and assume that $\phi(x) = \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$ is not a constant function on $(0,\infty)$. Then ϕ is strictly monotone on some open interval I since ϕ' is continuous. Arguing exactly as in the proof of Theorem 17, with I replacing $(0,\infty)$, we conclude that either $M_{f,m_1,m_2}(a,b) < M_{g,m_1,m_2}(a,b)$ or $M_{f,m_1,m_2}(a,b) > M_{g,m_1,m_2}(a,b)$ for all $a,b \in I$, which is a contradiction. Thus $\frac{f^{(n+1)}(x)}{g^{(n+1)}(x)}$ must be a constant function on $(0,\infty)$, which then implies that $g(x) = cf(x) + p(x)$ for some constant c and some polynomial $p \in \pi_n$. □

The proof of the following theorem is very similar to the proofs of ([2], Lemma 1.2) and ([2], Theorem 1.4 and its Corollary), and we omit them.

THEOREM 19. *Suppose that* $f \in C^{n+2}(0,\infty)$ *and that* M_{f,m_1,m_2} *is a homogeneous mean. Then* $f^{(n+1)}(x) = cx^p$ *for some real numbers* c *and* p.

Theorem 19 implies that the means M_{p,m_1,m_2} are the only homogeneous means among the general class of means M_{f,m_1,m_2}.

THEOREM 20. $M_{p,m_1,m_2}(a,b)$ *is increasing in* p *for each fixed* $m_1, m_2, a,$ *and* b.

Proof. Let $f(x) = x^{p_1}, g(x) = x^{p_2}$, where $p_1 < p_2$. Then $\phi(x) = \frac{f^{(n+1)}(x)}{g^{(n+1)}(x)} = x^{p_1-p_2}$ is strictly monotonic on $(0,\infty)$. Let $0 < a < b$ be fixed and let $O = \{(p_1,p_2) \in \Re_2 : p_1 < p_2\}$. By Theorem 17, $M_{p_1,m_1,m_2}(a,b) \neq M_{p_2,m_1,m_2}(a,b)$ for all $(p_1,p_2) \in O$. Since O is connected and $M_{p,m_1,m_2}(a,b)$ is a continuous function of p, either $M_{p_1,m_1,m_2}(a,b) < M_{p_2,m_1,m_2}(a,b)$ or $M_{p_1,m_1,m_2}(a,b) > M_{p_2,m_1,m_2}(a,b)$ for all $(p_1,p_2) \in O$ by the intermediate value theorem. By Theorems 6 and 7, we must have $M_{p_1,m_1,m_2}(a,b) < M_{p_2,m_1,m_2}(a,b)$ for all $(p_1,p_2) \in O$ since it is well known that $H(a,b) \leqslant A(a,b)$. Since $a < b$ was arbitrary and M_{p_1,m_1,m_2} is symmetric, that proves Theorem 20. □

The following theorem discusses the asymptotic behavior of M_{p,m_1,m_2} as p approaches ∞ or $-\infty$.

THEOREM 21. $\lim_{p\to\infty} M_{p,m_1,m_2}(a,b) = \max\{a,b\}$ *and* $\lim_{p\to-\infty} M_{p,m_1,m_2}(a,b) = \min\{a,b\}$.

Proof. Since $M_{p,m_1,m_2}(a,b)$ is symmetric, we may assume that $a < b$. We prove that $\lim_{p\to\infty} M_{p,m_1,m_2}(a,b) = b$, the proof of the other case being similar. By (4), (15), and (18), $M_{p,m_1,m_2}(a,b)$ is the unique solution, in (a,b), of the equation $(x-a)^{m_1-m_2}(A_1+B_1+C_1) = (b-x)^{m_1-m_2}(A_2+B_2+C_2)$, where $f(x) = x^p$ and $A_j, B_j, C_j, j = 1,2$ are given by (17), (19), and (16). For $a \leqslant x \leqslant b$, it follows easily that $\frac{A_1}{\binom{p}{m_1}b^p}, \frac{B_1}{\binom{p}{m_1}b^p}, \frac{C_1}{\binom{p}{m_1}b^p}, \frac{A_2}{\binom{p}{m_1}b^p}$, and $\frac{C_2}{\binom{p}{m_1}b^p}$ each approach 0 as $p \to \infty$. In the double summation for B_2, take $k = m_1$, which implies that $l = 0$ and thus $\frac{B_2}{\binom{p}{m_1}b^p} \to (b-a)^{-m_2-1}(b-x)^{-1}b^{-m_1}$ as $p\to\infty$. Thus $(x-a)^{m_1-m_2}(A_1+B_1+C_1) - (b-x)^{m_1-m_2}(A_2+B_2+C_2) \to -(x-b)^{m_1-m_2-1}(b-a)^{-m_2-1}b^{-m_1}$ as $p\to\infty$, which easily implies that $M_{p,m_1,m_2}(a,b)$ must be approaching b if $m_1 - m_2 > 1$. We now consider the case $m_1 = 1$, $m_2 = 0$ separately. Then $M_{p,m_1,m_2}(a,b)$ is the unique solution, in (a,b), of the equation $(x-a)f[x,a,a,b] + (x-b)f[x,a,b,b] = 0, f(x) = x^p$. Using $f[x,a,a,b] = \frac{\frac{f(x)-f(a)-(x-a)f'(a)}{(x-a)^2} - \frac{f(b)-f(a)-(b-a)f'(a)}{(b-a)^2}}{x-b}$ and $f[x,a,b,b] = \frac{\frac{f(x)-f(b)-(x-b)f'(b)}{(x-b)^2} - \frac{f(a)-f(b)-(a-b)f'(b)}{(b-a)^2}}{x-a}$, some simplification yields the equation $L_p(x) = 0$, where $L_p(x) = 2(x^p-a^p)(b-a) - 2(b^p-a^p)(x-a) - p\left(b^{p-1}-a^{p-1}\right)(x-b)(x-a)$. For $a \leqslant x \leqslant b$, $\frac{L_p(x)}{p(b^p-a^p)} \to \frac{1}{b}(x-b)(x-a)$ as $p\to\infty$. Since M_{p,m_1,m_2} is increasing in p by Theorem 20, $M_{p,m_1,m_2}(a,b)$ must be approaching b as $p\to\infty$.
□

3. Special Cases

We now investigate the special case when $m_1 - m_2 = 2$, where $m_1 + m_2$ is even. In this case, the mean M_{f,m_1,m_2} can be obtained by solving a linear equation. In particular, if $f(x) = x^p$ where p is an integer, then M_{p,m_1,m_2} is a rational mean. Since $P^{(j)}(a) = Q^{(j)}(a)$ and $P^{(j)}(b) = Q^{(j)}(b)$, $j = 0,1,...,m_2$, $P-Q$ has zeros of multiplicity m_2+1

at $x=a$ and at $x=b$. Thus $P(x)-Q(x)=(x-a)^{m_2+1}(x-b)^{m_2+1}R(x)$, where R is a polynomial of degree m_1-m_2-1. Using the formulas in [1] for Hermite interpolation, one can directly compute the polynomials P and Q which satisfy (1).

$$\begin{aligned} P(x) &= \left(\tfrac{x-b}{a-b}\right)^{m_2+1}\sum_{j=0}^{m_1}\sum_{k=0}^{m_1-j}\tfrac{(x-a)^j}{j!}\tbinom{m_2+k}{k}\left(\tfrac{x-a}{b-a}\right)^k f^{(j)}(a) \\ &\quad+\left(\tfrac{x-a}{b-a}\right)^{m_1+1}\sum_{j=0}^{m_2}\sum_{k=0}^{m_2-j}\tfrac{(x-b)^j}{j!}\tbinom{m_1+k}{k}\left(\tfrac{x-b}{a-b}\right)^k f^{(j)}(b) \end{aligned} \tag{25}$$

and

$$\begin{aligned} Q(x) &= \left(\tfrac{x-b}{a-b}\right)^{m_1+1}\sum_{j=0}^{m_2}\sum_{k=0}^{m_2-j}\tfrac{(x-a)^j}{j!}\tbinom{m_1+k}{k}\left(\tfrac{x-a}{b-a}\right)^k f^{(j)}(a) \\ &\quad+\left(\tfrac{x-a}{b-a}\right)^{m_2+1}\sum_{j=0}^{m_1}\sum_{k=0}^{m_1-j}\tfrac{(x-b)^j}{j!}\tbinom{m_2+k}{k}\left(\tfrac{x-b}{a-b}\right)^k f^{(j)}(b) \end{aligned} \tag{26}$$

Since $m_1-m_2=2$, R is a linear polynomial, which implies that $P(x)-Q(x)=(x-a)^{m_2+1}(x-b)^{m_2+1}(cx+d)$. We now determine c and d. First,

$$\begin{aligned} d &= \tfrac{P(0)-Q(0)}{a^{m_2+1}b^{m_2+1}} = \tfrac{E_Q(0)-E_P(0)}{a^{m_2+1}b^{m_2+1}} = \tfrac{a^{m_2+1}b^{m_1+1}f[0,a^{m_2+1},b^{m_1+1}]-a^{m_1+1}b^{m_2+1}f[0,a^{m_1+1},b^{m_2+1}]}{a^{m_2+1}b^{m_2+1}} \\ &= b^{m_1-m_2}f[0,a^{m_2+1},b^{m_1+1}]-a^{m_1-m_2}f[0,a^{m_1+1},b^{m_2+1}] \implies \end{aligned}$$

$$d=b^2f[0,a^{m_2+1},b^{m_2+3}]-a^2f[0,a^{m_2+3},b^{m_2+1}] \tag{27}$$

Again, using the formula discussed earlier due to Spitzbart (see [9], Theorem 2),

$$\begin{aligned} f[0,a^{m_1+1},b^{m_2+1}] &= \sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\tfrac{1}{k!}(-1)^{m_1+k}\tbinom{m_2+l}{m_2}(a-b)^{-m_2-1-l}a^{-m_1-1+k+l}f^{(k)}(a) \\ &\quad+\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_1+l}{m_1}(b-a)^{-m_1-1-l}b^{-m_2-1+k+l}f^{(k)}(b) \\ &\quad+a^{-m_1-1}b^{-m_2-1}f(0), \end{aligned}$$

and

$$\begin{aligned} f[0,a^{m_2+1},b^{m_1+1}] &= \sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_1+l}{m_1}(a-b)^{-m_1-1-l}a^{-m_2-1+k+l}f^{(k)}(a) \\ &\quad+\sum_{k=0}^{m_1}\sum_{l=0}^{m_1-k}\tfrac{1}{k!}(-1)^{m_1+k}\tbinom{m_2+l}{m_2}(b-a)^{-m_2-1-l}b^{-m_1-1+k+l}f^{(k)}(b) \\ &\quad+a^{-m_2-1}b^{-m_1-1}f(0). \end{aligned}$$

Letting $m_1=m_2+2$ gives

$$\begin{aligned} f[0,a^{m_2+1},b^{m_2+3}] &= \sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_2+l+2}{m_2+2}(a-b)^{-m_2-3-l}a^{-m_2-1+k+l}f^{(k)}(a) \\ &\quad+\sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k}\tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_2+l}{m_2}(b-a)^{-m_2-1-l}b^{-m_2-3+k+l}f^{(k)}(b) \\ &\quad+a^{-m_2-1}b^{-m_2-3}f(0), \end{aligned}$$

and

$$f[0,a^{m_2+3},b^{m_2+1}] = \sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k} \tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_2+l}{m_2}(a-b)^{-m_2-1-l}a^{-m_2-3+k+l}f^{(k)}(a)$$
$$+\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k} \tfrac{1}{k!}(-1)^{m_2+k}\tbinom{m_2+2+l}{m_2+2}(b-a)^{-m_2-3-l}b^{-m_2-1+k+l}f^{(k)}(b)$$
$$+a^{-m_2-3}b^{-m_2-1}f(0).$$

Hence, by (27),

$$\begin{aligned}
d &= b^2 f[0,a^{m_2+1},b^{m_2+3}] - a^2 f[0,a^{m_2+3},b^{m_2+1}] \\
&= \sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+l+2}{m_2+2} b^2(a-b)^{-m_2-3-l}a^{-m_2-1+k+l}f^{(k)}(a) \\
&\quad+\sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+l}{m_2}(b-a)^{-m_2-1-l}b^{-m_2-1+k+l}f^{(k)}(b) \\
&\quad+a^{-m_2-1}b^{-m_2-1}f(0) - \sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+l}{m_2}(a-b)^{-m_2-1-l}a^{-m_2-1+k+l}f^{(k)}(a) \\
&\quad-\sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+2+l}{m_2+2}a^2(b-a)^{-m_2-3-l}b^{-m_2-1+k+l}f^{(k)}(b) - a^{-m_2-1}b^{-m_2-1}f(0) \\
&= \sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+l+2}{m_2+2}(a-b)^{-m_2-3-l} \\
&\qquad\cdot\left(b^2a^{-m_2-1+k+l}f^{(k)}(a) + (-1)^{m_2+l}a^2b^{-m_2-1+k+l}f^{(k)}(b)\right) \\
&\quad+\sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k} \tfrac{(-1)^{m_2+k}}{k!}\tbinom{m_2+l}{m_2}(b-a)^{-m_2-1-l} \\
&\qquad\cdot\left(b^{-m_2-1+k+l}f^{(k)}(b) + (-1)^{m_2+l}a^{-m_2-1+k+l}f^{(k)}(a)\right).
\end{aligned}$$

Now we find c. It is not hard to show, using (1), that the coefficient, c_{P,m_1,m_2}, of the highest power in P, which is $x^{m_1+m_2+1}$, is given by

$$\sum_{j=0}^{m_1}\frac{\binom{m_2+m_1-j}{m_2}f^{(j)}(a)}{j!(a-b)^{m_2+1}(b-a)^{m_1-j}} + \sum_{j=0}^{m_2}\frac{\binom{m_2+m_1-j}{m_1}f^{(j)}(b)}{j!(b-a)^{m_1+1}(a-b)^{m_2-j}}$$

or

$$c_{P,m_1,m_2} = \frac{(-1)^{m_2}}{(b-a)^{m_1+m_2+1}}\left(\sum_{j=0}^{m_2}\frac{(-1)^j\binom{m_2+m_1-j}{m_1}(b-a)^jf^{(j)}(b)}{j!} - \sum_{j=0}^{m_1}\frac{\binom{m_2+m_1-j}{m_2}(b-a)^jf^{(j)}(a)}{j!}\right) \tag{28}$$

Similarly, the coefficient, c_{Q,m_1,m_2}, of the highest power in Q, which is $x^{m_1+m_2+1}$, is given by

$$\sum_{j=0}^{m_2}\frac{\binom{m_2+m_1-j}{m_1}f^{(j)}(a)}{j!(a-b)^{m_1+1}(b-a)^{m_2-j}} + \sum_{j=0}^{m_1}\frac{\binom{m_2+m_1-j}{m_2}f^{(j)}(b)}{j!(b-a)^{m_2+1}(a-b)^{m_1-j}}$$

or

$$c_{Q,m_1,m_2} = \frac{(-1)^{m_1}}{(b-a)^{m_1+m_2+1}}\left(\sum_{j=0}^{m_1}\frac{(-1)^j\binom{m_2+m_1-j}{m_2}(b-a)^jf^{(j)}(b)}{j!} - \sum_{j=0}^{m_2}\frac{\binom{m_2+m_1-j}{m_1}(b-a)^jf^{(j)}(a)}{j!}\right) \tag{29}$$

Hence

$$
\begin{aligned}
c \;=\;& \tfrac{(-1)^{m_2+1}}{(b-a)^{m_1+m_2+1}}\left(\sum_{j=0}^{m_1}\tfrac{\binom{m_2+m_1-j}{m_2}(b-a)^j f^{(j)}(a)}{j!}-\sum_{j=0}^{m_2}\tfrac{(-1)^j\binom{m_2+m_1-j}{m_1}(b-a)^j f^{(j)}(b)}{j!}\right)\\
&-\tfrac{(-1)^{m_1+1}}{(b-a)^{m_1+m_2+1}}\left(\sum_{j=0}^{m_2}\tfrac{\binom{m_2+m_1-j}{m_1}(b-a)^j f^{(j)}(a)}{j!}-\sum_{j=0}^{m_1}\tfrac{(-1)^j\binom{m_2+m_1-j}{m_2}(b-a)^j f^{(j)}(b)}{j!}\right)\implies\\
c \;=\;& \tfrac{1}{(b-a)^{m_1+m_2+1}}\sum_{j=0}^{m_1}\tfrac{\binom{m_2+m_1-j}{m_2}(b-a)^j((-1)^{m_2+1}f^{(j)}(a)-(-1)^{m_1+j}f^{(j)}(b))}{j!} \qquad (30)\\
&+\tfrac{1}{(b-a)^{m_1+m_2+1}}\sum_{j=0}^{m_2}\tfrac{\binom{m_2+m_1-j}{m_1}(b-a)^j((-1)^{m_1}f^{(j)}(a)+(-1)^{m_2+j}f^{(j)}(b))}{j!}.
\end{aligned}
$$

Using $m_1=m_2+2$ gives

$$M_{f,m_1,m_2}(a,b)=-\tfrac{d}{c},$$

where

$$
\begin{aligned}
d \;=\;& \sum_{k=0}^{m_2}\sum_{l=0}^{m_2-k}\tfrac{1}{k!}(-1)^{m_2+k}\binom{m_2+l+2}{m_2+2}(a-b)^{-m_2-3-l}\\
&\cdot\left(b^2a^{-m_2-1+k+l}f^{(k)}(a)+(-1)^{m_2+l}a^2b^{-m_2-1+k+l}f^{(k)}(b)\right)\\
&+\sum_{k=0}^{m_2+2}\sum_{l=0}^{m_2+2-k}\tfrac{1}{k!}(-1)^{m_2+k}\binom{m_2+l}{m_2}(b-a)^{-m_2-1-l} \qquad (31)\\
&\cdot\left(b^{-m_2-1+k+l}f^{(k)}(b)+(-1)^{m_2+l}a^{-m_2-1+k+l}f^{(k)}(a)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
c \;=\;& \tfrac{(-1)^{m_2}}{(b-a)^{2m_2+3}}\sum_{j=0}^{m_2+2}\tfrac{\binom{2m_2+2-j}{m_2}(b-a)^j((-1)^{j+1}f^{(j)}(b)-f^{(j)}(a))}{j!} \qquad (32)\\
&+\tfrac{(-1)^{m_2}}{(b-a)^{2m_2+3}}\sum_{j=0}^{m_2}\tfrac{\binom{2m_2+2-j}{m_2+2}(b-a)^j((-1)^j f^{(j)}(b)+f^{(j)}(a))}{j!}
\end{aligned}
$$

We now examine three special cases. For $m_1=4$ and $m_2=2$, using (31) and (32), we have

$$
\begin{aligned}
24\,(b-a)^6\,d = {}& 840(f(b)-f(a))+120(3a-4b)f'(b)+120(4a-3b)f'(a)\\
&+60\,(b-a)\left((2b-a)f''(b)-(b-2a)f''(a)\right)\\
&-4\,(b-a)^2\left((b-4a)f'''(a)\right.\\
&\left.+(4b-a)f'''(b)\right)+(b-a)^3\left(af''''(a)+bf''''(b)\right))
\end{aligned}
$$

and

$$
\begin{aligned}
-24\,(b-a)^6\,c = {}& -120(f'(b)-f'(a))+60(b-a)(f''(b)+f''(a))\\
&-12\,(b-a)^2\left(f'''(b)-f'''(a)\right)+(b-a)^3\left(f''''(b)+f''''(a)\right).
\end{aligned}
$$

Thus

$$M_{f,4,2}(a,b) = \frac{840(f(b)-f(a))+120(3a-4b)f'(b)+120(4a-3b)f'(a)+60(b-a)((2b-a)f''(b)-(b-2a)f''(a))}{-120(f'(b)-f'(a))+60(b-a)(f''(b)+f''(a))-12(b-a)^2(f'''(b)-f'''(a))+(b-a)^3(f''''(b)+f''''(a))} + \frac{-4(b-a)^2((b-4a)f'''(a)+(4b-a)f'''(b))+(b-a)^3(af''''(a)+bf''''(b))}{-120(f'(b)-f'(a))+60(b-a)(f''(b)+f''(a))-12(b-a)^2(f'''(b)-f'''(a))+(b-a)^3(f''''(b)+f''''(a))}$$

For $m_1 = 3$ and $m_2 = 1$, again using (31) and (32), we have

$$\begin{aligned} 6(b-a)^{-4} d = &-60(f(b)-f(a)) + 12(3b-2a)f'(b) - 12(3a-2b)f'(a) \\ &+ 3(b-a)((a-3b)f''(b) + (b-3a)f''(a)) \\ &+ (b-a)^2 (bf'''(b) - af'''(a)) \end{aligned}$$

and

$$-6(b-a)^4 c = -12(f'(b)-f'(a))+6(b-a)(f''(b)+f''(a))-(b-a)^2(f'''(b)-f'''(a)).$$

Thus

$$M_{f,3,1}(a,b) = \frac{-60(f(b)-f(a))+12(3b-2a)f'(b)-12(3a-2b)f'(a)+3(b-a)((a-3b)f''(b)+(b-3a)f''(a))}{12(f'(b)-f'(a))-6(b-a)(f''(b)+f''(a))+(b-a)^2(f'''(b)-f'''(a))} + \frac{(b-a)^2(bf'''(b)-af'''(a))}{12(f'(b)-f'(a))-6(b-a)(f''(b)+f''(a))+(b-a)^2(f'''(b)-f'''(a))}$$

For $m_1 = 2$ and $m_2 = 0$ we have

$$d = \frac{(b^2-ab)f''(b)+(ab-a^2)f''(a)+(2a-4b)f'(b)+(4a-2b)f'(a)+6(f(b)-f(a))}{2(b-a)^2}$$

and

$$c = -\frac{(b-a)(f''(b)+f''(a))-2(f'(b)-f'(a))}{2(b-a)^2}.$$

Thus

$$M_{f,2,0}(a,b) = \frac{(b-a)(bf''(b)+af''(a))+2(a-2b)f'(b)+2(2a-b)f'(a)+6(f(b)-f(a))}{(b-a)(f''(b)+f''(a))-2(f'(b)-f'(a))}$$

If $f(x) = x^p$, then some simplification yields

$$M_{f,2,0}(a,b) = \frac{1}{p}\frac{b^{p-2}(p(p-5)b^2+p(3-p)ab+6b^2)-a^{p-2}(p(p-5)a^2+p(3-p)ab+6a^2)}{b^{p-2}((p-1)(b-a)-2b)+a^{p-2}((p-1)(b-a)+2a)},\ p \notin \{0,1,2,3\}.$$

The omitted cases for p can be obtained as limiting values, or one can just let $f(x) = x^p \log x$ for $p \in \{0,1,2,3\}$. That yields

$$M_{\log x,2,0}(a,b) = 3ab\frac{b^2-a^2-2ab(\ln b - \ln a)}{(b-a)^3} = 3ab\frac{b^2-a^2-2ab\ln\left(\frac{b}{a}\right)}{(b-a)^3},$$

$$M_{x\log x,2,0}(a,b) = 2ab\frac{(a+b)\ln\left(\frac{b}{a}\right)-2(b-a)}{b^2-a^2-2ab\ln\left(\frac{b}{a}\right)},$$

$$M_{x^2\log x,2,0}(a,b) = \frac{1}{2}\frac{b^2-a^2-2ab\ln\left(\frac{b}{a}\right)}{(a+b)\ln\left(\frac{b}{a}\right)-2(b-a)},$$

and

$$M_{x^3 \log x,2,0}(a,b) = \frac{1}{3}\frac{(b-a)^3}{b^2-a^2-2ab\ln\left(\frac{b}{a}\right)}.$$

Finally, we consider the case $m_1 = 1$ and $m_2 = 0$, so that m_1+m_2 is odd. As noted earlier, $M_{p,m_1,m_2}(a,b)$ is the unique solution, in (a,b), of the equation $2(x^p - a^p)(b-a) - 2(b^p - a^p)(x-a) - p\left(b^{p-1} - a^{p-1}\right)(x-b)(x-a) = 0$. For $p = 4$, after dividing thru by $2(x-a)(b-a)(b-x)$, we have $2(x-a)(b-a)(b-x)\left(b^2 - xb + ab - xa + a^2 - x^2\right) = 0$. This can be solved exactly to obtain $M_{4,1,0}(a,b) = \frac{1}{2}\sqrt{5b^2 + 6ab + 5a^2} - \frac{a+b}{2}$. For $p = 5$, after dividing thru by $(x-a)(b-a)(b-x)$, we have $2x^3 + 2bx^2 + 2ax^2 + 2b^2x + 2xab + 2xa^2 - 3a^3 - 3b^2a - 3ba^2 - 3b^3 = 0$. The root in (a,b) is given by $M_{5,1,0}(a,b) = \frac{1}{6}\sqrt[3]{s(a,b) + 6\sqrt{t(a,b)}} - \frac{2}{3}\frac{2b^2+ab+2a^2}{\sqrt[3]{s(a,b)+6\sqrt{t(a,b)}}} - \frac{a+b}{3}$, where $s(a,b) = 10(a+b)\left(19a^2 + 2ab + 19b^2\right)$ and $t(a,b) = 1017b^6 + 2238b^5a + 3495b^4a^2 + 4500b^3a^3 + 3495b^2a^4 + 2238a^5b + 1017a^6$.

4. Comparisons with Taylor polynomial means

As noted earlier, the means defined in this paper are similar to a class of means defined in [2], which were based on intersections of Taylor polynomials. For $f \in C^{r+1}(I), I = (a,b)$, let P_c denote the Taylor polynomial to f of order r at $x = c$, where r is an odd positive integer. In [2] it was proved that if $f^{(r+1)}(x) \neq 0$ on $[a,b]$, then there is a unique $u, a < u < b$, such that $P_a(u) = P_b(u)$. This defines a mean $m(a,b) \equiv u$, which we denote by $M_f^r(a,b)$. The arithmetic, geometric, and harmonic means arise for both classes of means. We now show that there are means defined in this paper which do not occur as intersections of Taylor polynomials. In particular, consider the mean $M_{\log x,2,0}(a,b) = 3ab\frac{b^2-a^2-2ab(\ln b-\ln a)}{(b-a)^3}$ discussed earlier. Then $h(b) = M_{\log x,2,0}(1,b) = 3b\frac{b^2-1-2b\ln b}{(b-1)^3}$, $\lim\limits_{b\to 1} h'(b) = \frac{1}{2}$, $\lim\limits_{b\to 1} h''(b) = -\frac{2}{5}$, $\lim\limits_{b\to 1} h'''(b) = \frac{3}{5}$, and $\lim\limits_{b\to 1} h''''(b) = -\frac{48}{35}$. Since $M_{\log x,2,0}$ is a homogeneous mean, if $M_{\log x,2,0} = M_f^r$ for some f, then we may assume that $f(x) = x^p$ for some real number p by ([2], Theorem 1.4). Let $k(b) = M_p^r(1,b) = M_f^r(1,b)$, where $f(x) = x^p$. From ([2], Theorem 4.1), $k''(1) = \frac{p-r-1}{2(r+2)}$, $k'''(1) = \frac{-3(p-r-1)}{4(r+2)}$, and $k''''(1) = \frac{p-r-1}{8(r+2)^3(r+4)}(12r^3 + 8(p+13)r^2 - 4(p^2 - 12p - 73)r - 16(2p^2 - p - 15)$. Setting $\frac{p-r-1}{2(r+2)} = -\frac{2}{5}$ and $\frac{-3(p-r-1)}{4(r+2)} = \frac{3}{5}$ implies that $r = 5p + 3$. Substituting into $k''''(1)$ gives $-\frac{12}{125}\frac{70p+99}{5p+7}$. Setting $-\frac{12}{125}\frac{70p+99}{5p+7} = -\frac{48}{35}$ implies that $p = -\frac{7}{10}$. Then $r = 5\left(-\frac{7}{10}\right) + 3 = -\frac{1}{2}$, which is not a positive integer. Thus $M_{\log x,2,0}$ *cannot* occur as one of the means M_f^r.

5. Open Questions and Future Research

In [3] it was shown that $\lim\limits_{r\to\infty} M_p^r(a,b) = H(a,b) = \frac{2ab}{a+b}$, where M_f^r are the Taylor polynomial means defined above. There is strong evidence that a similar result holds for the means defined in this paper. That is, CONJECTURE 22. $\lim_{n\to\infty} M_{p,m_1,m_2}(a,b) = H(a,b)$, *where* $n = m_1 + m_2 + 1$.

More generally, analyze the asymptotic behavior of M_{f,m_1,m_2} as $n \to \infty$. As in [3], it should follow that the arithmetic mean arises as $\lim_{n\to\infty} M_{f,m_1,m_2}$. It is then natural to ask:

QUESTION. Are the arithmetic and harmonic means the only means which arise as $\lim_{n\to\infty} M_{f,m_1,m_2}$?

We showed in Theorem 7 that $M_{-1,m_1,m_2}(a,b) = H(a,b) = \frac{2ab}{a+b}$ for any m_1 and m_2. Thus for $f(x) = \frac{1}{x}$, M_{f,m_1,m_2} is independent of m_1 and m_2.

CONJECTURE 23. *Show that the only function, f, for which M_{f,m_1,m_2} is independent of m_1 and m_2 is $f(x) = \frac{C}{x}$.*

REFERENCES

[1] RAVI P. AGARWAL & PATRICIA J. Y. WONG, *Error inequalities in polynomial interpolation and their applications*, Kluwer, 1993.

[2] ALAN HORWITZ, *Means and Taylor Polynomials*, Journal of Mathematical Analysis and Applications 149(1990), 220–235.

[3] ALAN HORWITZ, *Means and Averages of Taylor Polynomials*, Journal of Mathematical Analysis and Applications 176(1993), 404–412.

[4] E. ISAACSON AND H. B. KELLER, *Analysis of Numerical Methods*, Wiley, New York, 1966.

[5] E. B. LEACH AND M. C. SHOLANDER, *Multi–variable Extended Mean Values*, Journal of Mathematical Analysis and Applications 104(1984), 390–407.

[6] A. M. OSTROWSKI, *Solution of Equations in Euclidean and Banach Spaces* , 3rd ed., Academic Press, New York and London, 1973.

[7] JOSIP PECARIC & PATRICIA J. Y. WONG, *Polynomial interpolation and generalizations of mean value theorem*, Nonlinear Funct. Anal. Appl. 6 (2001), no. 3, 329–340.

[8] MARKO PETKOVŠEK, HERBERT WILF, DORON ZEILBERGER, A. K. PETERS, *A=B*, Massachusetts, 1996.

[9] A. SPITZBART, *A Generalization of Hermite's Interpolation Formula*, American Mathematical Monthly, Vol. 67, No. 1. (Jan., 1960), pp. 42–46.

(Received December 14, 2007)

Alan Horwitz
Penn State University
25 Yearsley Mill Road
Media, PA 19063
USA
e-mail: alh4@psu.edu