THE INTEGRAL OPERATOR ON THE CLASSES $S^\ast_\alpha(b)$ AND $C_\alpha(b)$

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(communicated by Th. Rassias)

Abstract. In this paper we present some properties for two general integral operators on the classes $S^\ast_\alpha(b)$ and $C_\alpha(b)$.

1. Introduction

Let $\mathcal{U} = \{z \in \mathbb{C}, |z| < 1\}$ be the open unit disc of the complex plane. Denote by $\mathcal{H}(U)$ and $\mathcal{A}$, the class of the holomorphic functions in $\mathcal{U}$ and the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $\mathcal{U}$, respectively.

In the paper [3], Frasin studied the classes $S^\ast_\alpha(b)$ and $C_\alpha(b)$.

A function $f(z) \in \mathcal{A}$ is said to be a starlike of complex order $b, (b \in \mathbb{C} - \{0\})$ and type $\alpha, (0 \leq \alpha < 1)$, that is $f \in S^\ast_\alpha(b)$, if and only if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > \alpha$$

for all $z \in \mathcal{U}$. (1)

A function $f(z) \in \mathcal{A}$ is said to be convex of complex order $b, (b \in \mathbb{C} - \{0\})$ and type $\alpha, (0 \leq \alpha < 1)$, that is $f \in C_\alpha(b)$, if and only if

$$\Re \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > \alpha$$

for all $z \in \mathcal{U}$. (2)


Key words and phrases: Integral operator, holomorphic functions, starlike functions, convex functions, complex order.

Supported by the GAR 20/2007.
Recently, the first author and N. Breaz in [1] introduced and studied the integral operator
\[ F_n(z) = \int_0^z \left( \frac{f_1(t)}{t} \right)^{\alpha_1} \cdots \left( \frac{f_n(t)}{t} \right)^{\alpha_n} \, dt \]  
and the integral operator
\[ F_{\alpha_1, \ldots, \alpha_n}(z) = \int_0^z (f'_1(t))^{\alpha_1} \cdots (f'_n(t))^{\alpha_n} \, dt \]  
for \( \alpha_i > 0 \), was introduced by the first author et al. in [2].

In the present paper, we consider two integral operators in above and study their properties on the classes \( S_\alpha(b) \) and \( C_\alpha(b) \).

2. Main results

**Theorem 1.** Let \( \alpha_i, i \in \{1, \ldots, n\} \) the real numbers with the properties \( \alpha_i > 0 \) for \( i \in \{1, \ldots, n\} \), \( \alpha \) the real number, \( 0 \leq \alpha < 1 \) and
\[ 0 \leq (\alpha - 1) \sum_{i=1}^n \alpha_i + 1 < 1. \]  
If \( f_i \in S_\alpha^*(b) \) for \( i = \{1, \ldots, n\} \) and \( b \in \mathbb{C} - \{0\} \), then \( F_n \in C_\gamma(b) \), where
\[ \gamma = (\alpha - 1) \sum_{i=1}^n \alpha_i + 1. \]

**Proof.** We calculate the derivatives of the first and second order for \( F_n \). From (3), we obtain
\[ F'_n(z) = \left( \frac{f_1(z)}{z} \right)^{\alpha_1} \cdots \left( \frac{f_n(z)}{z} \right)^{\alpha_n} \]  
and
\[ F''_n(z) = \sum_{i=1}^n \alpha_i \left( \frac{zf'_i(z) - f_i(z)}{zf_i(z)} \right) F'_n(z). \]  
From the above equalities, we have
\[ \frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left( \frac{zf'_1(z) - f_1(z)}{zf_1(z)} \right) + \cdots + \alpha_n \left( \frac{zf'_n(z) - f_n(z)}{zf_n(z)} \right) \]  
and
\[ \frac{F''_n(z)}{F'_n(z)} = \alpha_1 \left( \frac{f'_1(z)}{f_1(z)} - \frac{1}{z} \right) + \cdots + \alpha_n \left( \frac{f'_n(z)}{f_n(z)} - \frac{1}{z} \right). \]  
By multiplying the relation (6) with \( z \), we obtain
\[ \frac{zF''_n(z)}{F'_n(z)} = \sum_{i=1}^n \alpha_i \left( \frac{zf'_i(z)}{f_i(z)} - 1 \right). \]
Then by multiplying the relation (7) with \( \frac{1}{b} \), we obtain

\[
\frac{1}{b} z F''(z) = \frac{1}{b} \sum_{i=1}^{n} \alpha_i \left( \frac{f''_i(z)}{f_i(z)} - 1 \right) = \sum_{i=1}^{n} \alpha_i \left[ 1 + \frac{1}{b} \left( \frac{f''_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^{n} \alpha_i \tag{8}
\]

The relation (8) is equivalent to

\[
1 + \frac{1}{b} z F''(z) = \sum_{i=1}^{n} \alpha_i \left[ 1 + \frac{1}{b} \left( \frac{f''_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^{n} \alpha_i + 1 \tag{9}
\]

Lastly, we calculate the real part of both terms of (9) and obtain

\[
\text{Re} \left\{ 1 + \frac{1}{b} z F''(z) \right\} = \sum_{i=1}^{n} \alpha_i \text{Re} \left[ 1 + \frac{1}{b} \left( \frac{f''_i(z)}{f_i(z)} - 1 \right) \right] - \sum_{i=1}^{n} \alpha_i + 1 \tag{10}
\]

Since \( f_i \in \mathcal{S}_\alpha(b) \) for \( i = \{1, \ldots, n\} \), by applying in the relation (10) the inequality (1) we have

\[
\text{Re} \left\{ 1 + \frac{1}{b} z F''(z) \right\} > (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 \tag{11}
\]

Because \( 0 \leq (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1 \), we obtain \( F_n \in \mathcal{C}_\gamma(b) \), where \( \gamma = (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 \).

**COROLLARY 2.** Let \( \alpha_1 > 0 \) and \( \alpha \) the real number with the property \( 0 \leq \alpha < 1 \). If \( 0 \leq (\alpha - 1) \alpha_1 + 1 < 1 \) and the function \( f_1 \in \mathcal{S}_\alpha(b) \), then the integral operator \( F_1 \in \mathcal{C}_\rho(b) \), where \( \rho = (\alpha - 1) \alpha_1 + 1 \).

**THEOREM 3.** Let \( f_i \in \mathcal{C}_\alpha(b) \), \( 0 \leq \alpha < 1 \), \( \alpha_i > 0 \), \( b \in \mathbb{C} - \{0\} \) and

\[
0 \leq (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1
\]

for all \( i \in \{1, \ldots, n\} \). Then the integral operator \( F_{\alpha_1, \ldots, \alpha_n} \in \mathcal{C}_\eta(b) \), where

\[
\eta = (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1.
\]

**Proof.** If we make the similar operations to the proof of the Theorem 1, we have

\[
\frac{F''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} = \alpha_1 \frac{f''_1(z)}{f'_1(z)} + \ldots + \alpha_n \frac{f''_n(z)}{f'_n(z)}. \tag{12}
\]

Then by multiplying this relation with \( \frac{\alpha}{b} \), we obtain

\[
\frac{1}{b} z F''_{\alpha_1, \ldots, \alpha_n}(z) = \alpha_1 \frac{z f''_1(z)}{b f'_1(z)} + \ldots + \alpha_n \frac{z f''_n(z)}{b f'_n(z)}. \tag{13}
\]
From the relation (13), we obtain that
\[
\text{Re} \left( \frac{1}{b} \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} + 1 \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( 1 + \frac{1}{b} \frac{z f''_i(z)}{f'_i(z)} \right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{14}
\]

Since \( f_i \in \mathcal{C}_\alpha(b) \), we have
\[
\text{Re} \left( \frac{1}{b} \frac{zF''_{\alpha_1, \ldots, \alpha_n}(z)}{F'_{\alpha_1, \ldots, \alpha_n}(z)} + 1 \right) > (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1. \tag{15}
\]

Since \( 0 \leq (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 < 1 \), the relation (15) implies that the integral operator \( F_{\alpha_1, \ldots, \alpha_n} \in \mathcal{C}_{\eta}(b) \), where \( \eta = (\alpha - 1) \sum_{i=1}^{n} \alpha_i + 1 \). \( \square \)

Putting \( n = 1 \) in the Theorem 3, we have

**Corollary 4.** Let \( f_1 \in \mathcal{C}_\alpha(b) \), \( 0 \leq \alpha < 1 \). Then the integral operator \( F_{\alpha_1} \in \mathcal{C}_{\sigma}(b) \), where \( \sigma = (\alpha - 1)\alpha_1 + 1 \) and \( 0 \leq (\alpha - 1)\alpha_1 + 1 < 1 \).

**References**


(Received December 9, 2007)