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GENERALIZATIONS OF BERNOULLI'S INEQUALITY WITH APPLICATIONS

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Abstract. By using methods on the theory of majorization, some new generalizations of Bernoulli's inequality are established and some applications of the generalizations are given.

1. Introduction

Let $x > -1$ and n is a positive integer. Then

$$(1+x)^n \geqslant 1+nx. \tag{1.1}$$

(1.1) is known as the Bernoulli's inequality which play an important role in analysis and its applications. So, during the past few years, many researchers obtained various generalizations, extensions of inequality (1.1). For example, the following generalizations and variants of (1.1) were recorded in [1, pp. 127–128]:

THEOREM A. *Let $x > -1$. If $\alpha > 1$ or $\alpha < 0$, then*

$$(1+x)^\alpha \geqslant 1+\alpha x, \tag{1.2}$$

if $0 < \alpha < 1$, then

$$(1+x)^\alpha \leqslant 1+\alpha x. \tag{1.3}$$

In (1.2) and (1.3), equalities holding if and only if $x = 0$.

THEOREM B. *Let $a_i \geqslant 0$, $x_i > -1, i = 1,\dots,n$, and $\sum_{i=1}^n a_i \leqslant 1$. Then*

$$\prod_{i=1}^n (1+x_i)^{a_i} \leqslant 1+\sum_{i=1}^n a_i x_i, \tag{1.4}$$

if $a_i \geqslant 1$ or $a_i \leqslant 0$, and if $x_i > 0$, or $-1 < x_i < 0, i = 1,\dots,n$, then

$$\prod_{i=1}^n (1+x_i)^{a_i} \geqslant 1+\sum_{i=1}^n a_i x_i. \tag{1.5}$$

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For more information on the Bernoulli's inequality, please refer to [4, 6, 7, 8, 9, 10] and the references therein.

In this paper, some new generalizations of Bernoulli's inequality are established by the Schur-concatity of the elementary symmetric functions and the dual form of the elementary symmetric functions, and some applications of the generalizations are given. We obtain the following results.

THEOREM 1. *Let m,n is a positive integer, $k=1,\ldots,n$.*

(i) *If $m \geqslant n$ and $x>-1$, then*

$$C_m^k\left(1+\frac{n}{m}x\right)^k \geqslant \sum_{i=0}^{k} C_n^i C_{m-n}^{k-i}(1+x)^i, \tag{1.6}$$

and

$$k^{C_m^k}\left(1+\frac{n}{m}x\right)^{C_m^k} \geqslant \prod_{i=0}^{k}(ix+k)^{C_n^i C_{m-n}^{k-i}}. \tag{1.7}$$

(ii) *If $m<n$ and $x>-\frac{m}{n}$, then*

$$C_m^k(1+x)^k \geqslant \sum_{i=0}^{k} C_m^i C_{n-m}^{k-i}\left(1+\frac{m}{n}x\right)^i, \tag{1.8}$$

and

$$k^{C_m^k}(1+x)^{C_m^k} \geqslant \prod_{i=0}^{k}\left(\frac{m}{n}ix+k\right)^{C_m^i C_{n-m}^{k-i}}, \tag{1.9}$$

where $C_n^k=\frac{n!}{k!(n-k)!}$ is the number of combinations of n elements taken k at a time, defined $C_n^0=1$ and $C_n^k=0$ for $k>n$. In (1.6), (1.7), (1.8) and (1.9), equalities holding if and only if $x=0$.

REMARK 1. When $x=0$, (1.6), (1.7), (1.8) and (1.9) are deduce to Vandermonde identity.

$$C_m^k=\sum_{i=0}^{k} C_n^i C_{m-n}^{k-i}. \tag{1.10}$$

THEOREM 2. *If $a_i \geqslant 1$ or $a_i \leqslant 0$, and if $x_i>0$ or $0 \geqslant x_i \geqslant -1$. Then*

$$\begin{aligned} C_n^k\left(\frac{1}{n}\sum_{i=1}^{n}(1+x_i)^{a_i}\right)^k &\geqslant \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+x_{i_j}\right)^{a_{i_j}} \qquad (1.11)\\ &\geqslant \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+a_{i_j}x_{i_j}\right) \geqslant C_{n-1}^k + C_{n-1}^{k-1}\left(1+\sum_{i=1}^{n}a_ix_i\right). \end{aligned}$$

$$\left(\frac{k}{n}\sum_{i=1}^{n}(1+x_i)^{a_i}\right)^{C_n^k} \geqslant \prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\sum_{j=1}^{k}\left(1+x_{i_j}\right)^{a_{i_j}} \tag{1.12}$$

$$\geqslant \prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\sum_{j=1}^{k}\left(1+a_{i_j}x_{i_j}\right) \geqslant k^{C_{n-1}^k}\left(k+\sum_{i=1}^{n}a_ix_i\right)^{C_{n-1}^{k-1}}.$$

REMARK 2. When $k=n$, (1.11) is deduce to (1.5), and when $k=1$, (1.12) is deduce to (1.5) too.

2. Proof of Theorem

For our own convenience, we introduce the following notations. we assume that the set of n-dimensional row vector on real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x=(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i \geqslant 0, i=1,\ldots,n\},$$

$$\mathbb{R}^n_{++} = \{x=(x_1,\ldots,x_n)\in\mathbb{R}^n : x_i > 0, i=1,\ldots,n\}.$$

Let $\boldsymbol{x}=(x_1,\ldots,a_n)\in\mathbb{R}^n$. Its elementary symmetric functions are

$$E_k(\boldsymbol{x}) = E_k(x_1,\ldots,x_n) = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod\nolimits_{j=1}^{k}x_{i_j}, \quad k=1,\ldots,n.$$

In particular, $E_n(\boldsymbol{x})=\prod_{i=1}^n x_i$, $E_1(\boldsymbol{x})=\sum_{i=1}^n x_i$, and defined $E_0(\boldsymbol{x})=1$ and $E_k(\boldsymbol{x})=0$ for $k<0$ or $k>n$.

The dual form of the elementary symmetric functions are

$$E_k^*(\boldsymbol{x}) = E_k^*(x_1,\ldots,x_n) = \prod_{1\leqslant i_1<\ldots<i_k\leqslant n}\sum\nolimits_{j=1}^{k}x_{i_j}, \quad k=1,\ldots,n,$$

and defined $E_0^*(\boldsymbol{x})=1$, and $E_k^*(\boldsymbol{x})=0$ for $k<0$ or $k>n$.

We need the following definitions and lemmas.

DEFINITION 1. ([2, 3]) Let $\boldsymbol{x}=(x_1,\ldots,x_n)$ and $\boldsymbol{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

(i) $\boldsymbol{x}$ is said to be majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\boldsymbol{y}$) if $\sum_{i=1}^k x_{[i]} \leqslant \sum_{i=1}^k y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$, where $x_{[1]}\geqslant\cdots\geqslant x_{[n]}$ and $y_{[1]}\geqslant\cdots\geqslant y_{[n]}$ are rearrangements of $\boldsymbol{x}$ and $\boldsymbol{y}$ in a descending order, and $\boldsymbol{x}$ is said to strictly majorized by $\boldsymbol{y}$ (in symbols $\boldsymbol{x}\prec\prec\boldsymbol{y}$) if $\boldsymbol{x}$ is not permutation of $\boldsymbol{y}$.

(ii) $\boldsymbol{x}\geqslant\boldsymbol{y}$ means $x_i\geqslant y_i$ for all $i=1,2,\ldots,n$. let $\Omega\subset\mathbb{R}^n$, $\varphi\colon\ \Omega\to\mathbb{R}$ is said to be increasing if $\boldsymbol{x}\geqslant\boldsymbol{y}$ implies $\varphi(\boldsymbol{x})\geqslant\varphi(\boldsymbol{y})$. φ is said to be decreasing if and only if $-\varphi$ is increasing.

(iii) $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $\boldsymbol{x}$ and $\boldsymbol{y}\in\Omega$, where α and $\beta\in[0,1]$ with $\alpha+\beta=1$.

(iv) let $\Omega \subset \mathbb{R}^n$, $\varphi\colon\ \Omega \to \mathbb{R}$ is said to be a Schur-convex function on Ω if $\boldsymbol{x} \prec \boldsymbol{y}$ on Ω implies $\varphi(\boldsymbol{x}) \leqslant \varphi(\boldsymbol{y})$. φ is said to be a Schur-concave function on Ω if and only if $-\varphi$ is Schur-convex function. φ is said to be a strictly Schur-convex function on Ω if $\boldsymbol{x} \prec\prec \boldsymbol{y}$ on Ω implies $\phi(\boldsymbol{x}) < \phi(\boldsymbol{y})$, φ is said to be a strictly Schur-concave function on Ω if and only $-\varphi$ is strictly Schur-convex on Ω.

LEMMA 1. [2, p. 5] *Let* $\boldsymbol{x} \in \mathbb{R}^n$ *and* $\bar{\boldsymbol{x}} = \frac{1}{n}\sum_{i=1}^n x_i$. *Then* $(\bar{\boldsymbol{x}}, \ldots, \bar{\boldsymbol{x}}) \prec \boldsymbol{x}$.

Proof of Theorem 1. From Lemma 1, we have

$$\boldsymbol{p} := \Bigg(\underbrace{1+\frac{n}{m}x, \ldots, 1+\frac{n}{m}x}_{m}\Bigg) \prec \Bigg(\underbrace{1+x, \ldots, 1+x}_{n}, \underbrace{1, \ldots, 1}_{m-n}\Bigg) := \boldsymbol{q}$$

and $\boldsymbol{p} \prec\prec \boldsymbol{q}$ for $x \neq 0$. If $m \geqslant n$, from $x > -1$, we have $x+1 > 0$ and $1 + \frac{n}{m}x > 1 - \frac{n}{m} > 0$, i.e. $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^n_{++}$. Since $E_k(\boldsymbol{x})$ be increasing and Schur-concave on $\mathbb{R}^n_+$ and it be increasing and strictly Schur-concave on $\mathbb{R}^n_{++}$ for $k > 1$ (see Proposition 6.7 in [2]), we have $E_k(\boldsymbol{p}) \geqslant E_k(\boldsymbol{q})$, i.e. (1.6) holds, and equality holding if and only if $x = 0$.

Since $E^*_k(\boldsymbol{x})$ be increasing and Schur-concave on $\mathbb{R}^n_+$ and it be increasing and strictly Schur-concave on $\mathbb{R}^n_{++}$ for $k > 1$ (see [3, p.86], [5]), we have $E^*_k(\boldsymbol{p}) \geqslant E^*_k(\boldsymbol{q})$, i.e. (1.7) holds, and equality holding if and only if $x = 0$.

If $m < n$, from Lemma 1, we have

$$\boldsymbol{p'} := \Bigg(\underbrace{1+x, \ldots, 1+x}_{n}\Bigg) \prec \Bigg(\underbrace{1+\frac{n}{m}x, \ldots, 1+\frac{n}{m}x}_{m}, \underbrace{1, \ldots, 1}_{n-m}\Bigg) := \boldsymbol{q'}$$

and $\boldsymbol{p'} \prec\prec \boldsymbol{q'}$ for $x \neq 0$. From $x > -\frac{m}{n}$, we have $1 + \frac{n}{m}x > 1 - \frac{n}{m} \cdot \frac{n}{m} > 0$, i.e. $\boldsymbol{p'}, \boldsymbol{q'} \in \mathbb{R}^n_{++}$. Thus we have $E_k(\boldsymbol{p'}) \geqslant E_k(\boldsymbol{q'})$ and $E^*_k(\boldsymbol{p'}) \geqslant E^*_k(\boldsymbol{q'})$, i.e. (1.8)and (1.9) hold, and equality holding if and only if $x = 0$.

The proof of Theorem 1 is completed. □

Proof of Theorem 2. Set $y = \frac{1}{n}\sum_{i=1}^n (1+x_i)^{a_i}$, from (1.2), we have $(1+x_i)^{a_i} \geqslant 1 + a_i x_i$, $i = 1, \ldots, n$, and by lemma 1 , it follows that

$$\begin{aligned}\underbrace{(y, \ldots, y)}_{n} &\prec ((1+x_1)^{a_1}, \ldots, (1+x_n)^{a_n})\\ &\geqslant (1+a_1x_1, \ldots, 1+a_nx_n) \prec \Bigg(1 + \sum_{i=1}^n a_ix_i, \underbrace{1, \ldots, 1}_{n-1}\Bigg).\end{aligned}$$

And then, since $E_k(\boldsymbol{x})$ and $E_k^*(\boldsymbol{x})$ are increasing and Schur-concave on $\mathbb{R}_+^n$ and are increasing and strictly Schur-concave on $\mathbb{R}_{++}^n$ for $k > 1$, we have

$$E_k(\underbrace{y,\ldots,y}_{n}) \geqslant E_k\left((1+x_1)^{a_1},\ldots,(1+x_n)^{a_n}\right)$$

$$\geqslant E_k\left(1+a_1x_1,\ldots,1+a_nx_n\right) \geqslant E_k\left(1+\sum_{i=1}^{n}a_ix_i,\underbrace{1,\ldots,1}_{n-1}\right),$$

i.e. (1.11) is holds. And

$$E_k^*(\underbrace{y,\ldots,y}_{n}) \geqslant E_k\left((1+x_1)^{a_1},\ldots,(1+x_n)^{a_n}\right)$$

$$\geqslant E_k^*\left(1+a_1x_1,\ldots,1+a_nx_n\right) \geqslant E_k^*\left(1+\sum_{i=1}^{n}a_ix_i,\underbrace{1,\ldots,1}_{n-1}\right),$$

i.e. (1.12) is holds.

The proof of Theorem 2 is completed. □

3. Applications

THEOREM 3. *Let $a_i \geqslant 1$ and $x_i \geqslant 1$, $i = 1,\cdots,n, n \in \mathbb{N}, n \geqslant 2$. Then for $k = 1,\cdots,n$, we have*

$$\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+x_{i_j}\right)^{a_{i_j}} \geqslant \frac{2^{A_k}C_{n-1}^{k-1}}{1+A_k}\left(\frac{n}{k}(1+A_k)-A_n+\sum_{i=1}^{n}a_ix_i\right), \tag{3.1}$$

where $A_k = \min_{1\leqslant i_1<\ldots<i_k\leqslant n}\sum_{i=j}^{k}a_{i_j}$.

Proof. Firstly, since $x_i \geqslant 1$ implies $\frac{1+x}{2} \geqslant 0$, from Theorem 2 we have

$$\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+\frac{x_{i_j}-1}{2}\right)^{a_{i_j}} \geqslant C_{n-1}^{k}+C_{n-1}^{k-1}\left(1+\sum_{i=1}^{n}\frac{a_i(x_i-1)}{2}\right) \tag{3.2}$$

$$= C_{n-1}^{k-1}\left(\frac{n}{k}+\sum_{i=1}^{n}\frac{a_i(x_i-1)}{2}\right) \geqslant C_{n-1}^{k-1}\left(\frac{n}{k}+\sum_{i=1}^{n}\frac{a_i(x_i-1)}{1+A_k}\right)$$

$$= \frac{C_{n-1}^{k-1}}{1+A_k}\left(\frac{n}{k}(1+A_k)-A_n+\sum_{i=1}^{n}a_ix_i\right).$$

On the other hand, we have

$$\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+x_{i_j}\right)^{a_{i_j}} = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n} 2^{\sum_{j=1}^{k} a_{i_j}}\prod_{j=1}^{k}\left(1+\frac{x_{i_j}-1}{2}\right)^{a_{i_j}} \geqslant 2^{A_k}\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+\frac{x_{i_j}-1}{2}\right)^{a_{i_j}}. \tag{3.3}$$

□

Combining (3.2) with (3.3), we get (3.1). The proof of Theorem 3 is completed.

REMARK 3. When $k=n$, (3.1) is deduce to (7.5) in [4, p. 69]:

$$\prod_{i=1}^{n}(1+x_i)^{a_i}\geqslant\frac{2^{A_n}}{1+A_n}\left(1+\sum_{i=1}^{n}a_ix_i\right). \tag{3.4}$$

THEOREM 4. *Let* $a_i\geqslant 1$ *or* $a_i\leqslant 0$ *and* $0>x_i>-1$ *or* $x_i>0$, $i=1,\cdots,n,n\in\mathbb{N},n\geqslant 2$. *Then for* $k=1,\cdots,n$, *we have*

$$\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+x_{i_j}\right)^{-a_{i_j}} \geqslant \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1-a_{i_j}x_{i_j}(1+x_{i_j})^{-1}\right) \geqslant C_{n-1}^{k}+C_{n-1}^{k-1}\left(1-a_{i_j}x_{i_j}(1+x_{i_j})^{-1}\right) \tag{3.5}$$

Proof. Since $0>x_i>-1$ or $x_i>0$ implies $-x_i\left(1+x_i\right)^{-1}$ or $0>-x_i\left(1+x_i\right)^{-1}>-1$, from Theorem 2 we have

$$\sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+x_{i_j}\right)^{-a_{i_j}} = \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1-x_{i_j}\left(1+x_{i_j}\right)^{-1}\right)^{a_{i_j}} \geqslant \sum_{1\leqslant i_1<\ldots<i_k\leqslant n}\prod_{j=1}^{k}\left(1+a_{i_j}x_{i_j}(1+x_{i_j})^{-1}\right) \geqslant C_{n-1}^{k}+C_{n-1}^{k-1}\left(1+a_{i_j}x_{i_j}(1+x_{i_j})^{-1}\right).$$

□

The proof of Theorem 4 is completed.

REMARK 4. When $k=n$, from (3.5) we have

$$\prod_{i=1}^{n}(1+x_i)^{-a_i}\geqslant 1-\sum_{i=1}^{n}a_ix_i\left(1+x_{i_j}\right)^{-1},$$

i.e.

$$\prod_{i=1}^{n}(1+x_i)^{a_i}\leqslant\left(1-\sum_{i=1}^{n}a_ix_i\left(1+x_{i_j}\right)^{-1}\right)^{-1}. \tag{3.6}$$

(3.6) is (7.3) in [4, p. 69].

REFERENCES

[1] JI-CHANG KUANG, *Applied Inequalities* (Chang yong bu deng shi) 3nd ed, Shandong Press of science and technology, Jinan, China, 2004. (Chinese).

[2] BO-YING WANG, *Foundations of Majorization Inequalities* (Kong zhi bu deng shi ji chu), Beijing Normal Univ. Press, Beijing, China, 1990. (Chinese)

[3] A. M. MARSHALL AND I. OLKIN, *Inequalities:theory of majorization and its application*. New York: Academies Press, 1979.

[4] D. S. MITRINOVIC, J. E. PECARIC AND A. M. FLNK, *Classical and New Inequalitis in Analysis*, Kluwer Academic Publishers, 1993, 65–82

[5] HUAN-NAN SHI, *Schur-Concavity of Dual Form for Elementary symmetric function with applications*, Journal of NortheastNormal University (Natural Sciences Edition), 2001, 33

[6] XIANG-DONG WANG, HUA-MING SU AND FANG-UAN WANG, *Theory and method of inequalities* (Bu deng shi.li lun.fang fa), Henan education Press China, 1994. (Chinese)

[7] ALFRED WITKOWSKI, *A new of the monotonicity of power menas*, J. Ineq. Pure. Appl. Math., 5 (1) (2004), Article 6, *URL:* http//jipam.vu.edu.au./

[8] JIA-JIN WEN AND ZHAO LUO, *Optimal Generalization of Bernoulliis Inequality and Its Applications*, Journal of Chengdu University (Natural science) 20 (4) (2001), 1–8

[9] JIAN-YA GE, *Some extensions on Bernoulli's inequality*, Journal of Zhejiang University (Natural science) 26 (2) (2003), 119–122.

[10] CHANG-XING XUE, *Separations and extensions on Bernoulli's inequality*, Journal of Gansu Education College(Natural Scienc Edition), 1999, 13 (3): 5–7

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