Journal of
Mathematical
Inequalities
Volume 2, Number 1 (2008), 109–114

SOME MORE INEQUALITIES FOR ARITHMETIC MEAN, HARMONIC MEAN AND VARIANCE

RAJESH SHARMA

(*communicated by R. Bhatia*)

Abstract. We derive bounds on the variance of a random variable in terms of its arithmetic and harmonic means. Both discrete and continuous cases are considered, and an operator version is obtained. Some refinements of the Kantorovich inequality are obtained. Bounds for the largest and smallest eigenvalues of a positive definite matrix are also obtained.

1. Introduction

Let T be a self adjoint linear operator on a Hilbert space X. Suppose $m \leqslant T \leqslant M$, (i.e. for every unit vector u in X, $m \leqslant \langle Tu|u\rangle \leqslant M$). The well known Kantorovich inequality say

$$1 \leqslant \langle Tu|u\rangle \langle T^{-1}u|u\rangle \leqslant \frac{(M+m)^2}{4mM}, \quad m > 0. \tag{1.1}$$

See [1] for details. Improvements, generalizations and inequalities in a similar spirit have been obtained by several authors. In particular Bhatia and Davis [2] have proved a bound for the variance of T

$$\langle T^2u|u\rangle - \langle Tu|u\rangle^2 \leqslant (M - \langle Tu|u\rangle)(\langle Tu|u\rangle - m). \tag{1.2}$$

Note that the inequality (1.1) involves T and T^{-1} while (1.2) involves T and T^2. Our main result (Theorem-1 below) relates quantities involving T, T^{-1} and T^2 at the same time. This provides yet another refinement of the Kantorovich inequality (Theorem-2 below). Let $x_1, x_2, \ldots, x_n$ be positive numbers, $m \leqslant x_i \leqslant M, (i = 1, 2, \ldots, n)$. Let A, H and S be their arithmetic mean, harmonic mean and standard deviation, respectively. An interesting relation between A, H and S has been proved by Mercer [3]

$$A - H \geqslant \frac{S^2}{2M}. \tag{1.3}$$

We obtain a better lower bound for $A - H$ and also give a complementary upper bound (Corollary-1, 2.24, below). Finally as an application we obtain a lower bound for the largest eigenvalue and an upper bound for the smallest eigenvalue of a positive definite matrix A in terms of traces of A, A^2 and A^{-1}. Our bounds compare favourably with those obtain by Wolkowicz and Styan [4].

Mathematics subject classification (2000): 47A50, 15A42, 60E15.
Key words and phrases: Kantorovich inequality, largest eigenvalue, trace, variance.

Paper JMI-02-11

2. Main Results

THEOREM 1. *Let T be a self adjoint linear operator on a Hilbert space X such that $0 < m < T < M$. Then for every unit vector u in X*

$$\langle T^2u|u\rangle - \langle Tu|u\rangle^2 \leqslant \frac{M\left(M - \langle Tu|u\rangle\right)\left(\langle Tu|u\rangle\langle T^{-1}u|u\rangle - 1\right)}{M\langle T^{-1}u|u\rangle - 1} \tag{2.1}$$

and

$$\langle T^2u|u\rangle - \langle Tu|u\rangle^2 \geqslant \frac{m\left(\langle Tu|u\rangle - m\right)\left(\langle Tu|u\rangle\langle T^{-1}u|u\rangle - 1\right)}{1 - m\langle T^{-1}u|u\rangle}. \tag{2.2}$$

Proof. For $0 < m \leqslant x \leqslant M$

$$(x-\alpha)^2(x-M) \leqslant 0 \tag{2.3}$$

and

$$(x-\beta)^2(x-m) \geqslant 0, \tag{2.4}$$

where α and β are real numbers. From inequality (2.3), we get that

$$f(x) = (2\alpha+M)x - \alpha(\alpha+2M) + \alpha^2\frac{M}{x} - x^2 \geqslant 0. \tag{2.5}$$

Therefore

$$\langle f(T)u|u\rangle \geqslant 0. \tag{2.6}$$

Inequality (2.6) gives

$$\langle T^2u|u\rangle \leqslant (2\alpha+M)\langle Tu|u\rangle - \alpha(\alpha+2M) + \alpha^2M\langle T^{-1}u|u\rangle. \tag{2.7}$$

Inequality (2.7) is valid for any real number α. It therefore must also hold good for that value of α for which the right hand side expression in (2.7) is minimum. On using derivatives we find that the function

$$g(\alpha) = (2\alpha+M)\langle Tu|u\rangle - \alpha(\alpha+2M) + \alpha^2M\langle T^{-1}u|u\rangle \tag{2.8}$$

has minimum at

$$\alpha = \frac{M - \langle Tu|u\rangle}{M\langle T^{-1}u|u\rangle - 1}. \tag{2.9}$$

Substituting the value of α from (2.9) in (2.7); inequality (2.1) follows immediately. In a similar way we can deduce inequality (2.2) from inequality (2.4). □

THEOREM 2. *Let T be a self adjoint linear operator on a Hilbert space X such that $0 < m \leqslant T \leqslant M$. Then for every unit vector u in X*

$$1 \leqslant \frac{(M-s)^2}{M(M-2s)} \leqslant \langle Tu|u\rangle\langle T^{-1}u|u\rangle \leqslant \frac{(m+s)^2}{m(m+2s)} \leqslant \frac{(M+m)^2}{4mM}, \tag{2.10}$$

where

$$s = \left\{\langle T^2u|u\rangle - \langle Tu|u\rangle^2\right\}^{\frac{1}{2}}. \tag{2.11}$$

Inequalities in (2.10) *give refinements of Kantorovich inequality.*

Proof. We find from inequality (2.2) that

$$\langle Tu|u\rangle \langle T^{-1}u|u\rangle \leqslant \frac{1}{m} \frac{m\langle Tu|u\rangle^2 - m^2\langle Tu|u\rangle + s^2\langle Tu|u\rangle}{\langle Tu|u\rangle^2 - m\langle Tu|u\rangle + s^2}. \tag{2.12}$$

On using derivatives we see that the right hand side expression in (2.12) has maximum when $\langle Tu|u\rangle = m+s$, therefore

$$\langle Tu|u\rangle \langle T^{-1}u|u\rangle \leqslant \frac{(m+s)^2}{m(m+2s)}. \tag{2.13}$$

Also

$$\frac{(m+s)^2}{m(m+2s)} \leqslant \frac{(M+m)^2}{4mM} \tag{2.14}$$

if and only if

$$s \leqslant \frac{M-m}{2}.$$

This is true. Also, from (2.1)

$$\langle Tu|u\rangle \langle T^{-1}u|u\rangle \geqslant \frac{1}{M} \frac{M^2\langle Tu|u\rangle - M\langle Tu|u\rangle^2 - s^2\langle Tu|u\rangle}{M\langle Tu|u\rangle - \langle Tu|u\rangle^2 - s^2}. \tag{2.15}$$

The right hand side expression in (2.15) has minimum when $\langle Tu|u\rangle = M-s$, therefore

$$\langle Tu|u\rangle \langle T^{-1}u|u\rangle \geqslant \frac{(M-s)^2}{M(M-2s)}. \tag{2.16}$$

The extreme left hand side inequality in (2.10) is true as $s^2 \geqslant 0$. □

THEOREM 3. *Let T be a self adjoint linear operator on a Hilbert space X such that $0 < m \leqslant T \leqslant M$. Then for every unit vector u in X*

$$\langle Tu|u\rangle - \langle T^{-1}u|u\rangle^{-1} \geqslant \frac{(M-m)s^2}{M(M-m)-s^2} \tag{2.17}$$

and

$$\langle Tu|u\rangle - \langle T^{-1}u|u\rangle^{-1} \leqslant \frac{(M-m)s^2}{m(M-m)+s^2}. \tag{2.18}$$

Proof. From inequalities (2.1) and (2.2) we respectively find that

$$\langle Tu|u\rangle - \langle T^{-1}u|u\rangle^{-1} \geqslant \langle Tu|u\rangle - \frac{M\left(M\langle Tu|u\rangle - \langle Tu|u\rangle^2 - s^2\right)}{M^2 - M\langle Tu|u\rangle - s^2} \tag{2.19}$$

and

$$\langle Tu|u\rangle - \langle T^{-1}u|u\rangle^{-1} \leqslant \langle Tu|u\rangle - \frac{m\left(\langle Tu|u\rangle^2 - m\langle Tu|u\rangle + s^2\right)}{m\langle Tu|u\rangle - m^2 + s^2}. \tag{2.20}$$

The right hand expression in (2.19) is an increasing function of $\langle Tu|u\rangle$ for $0 < m \leqslant T \leqslant M$ and assumes its minimum when $\langle Tu|u\rangle = m$. Substituting $\langle Tu|u\rangle = m$ in (2.19), we get inequality (2.17). Inequality (2.18) follows from (2.20) on using similar arguments. □

COROLLARY 1. *Let A, H and S respectively denote the arithmetic mean, harmonic mean and standard deviation of a random variable which is discrete or continuous and takes values in the interval $[m, M]$, $m > 0$. Then*

$$S^2 \leqslant \frac{M(A-H)(M-A)}{M-H}, \tag{2.21}$$

$$S^2 \geqslant \frac{m(A-m)(A-H)}{H-m}, \tag{2.22}$$

$$\frac{(M-S)^2}{M(M-2S)} \leqslant \frac{A}{H} \leqslant \frac{(m+S)^2}{m(m+2S)} \tag{2.23}$$

and

$$\frac{(M-m)S^2}{M(M-m)-S^2} \leqslant A-H \leqslant \frac{(M-m)S^2}{m(M-m)+S^2}. \tag{2.24}$$

The left hand side inequality in (2.24) *affects an improvement in inequality* (1.3) *and the right hand side inequality gives a complementary upper bound.*

Proof. The inequalities in this corollary can be proved in the similar ways as the corresponding inequalities for operators are proved in Theorem 1-Theorem 3. Alternatively, we can deduce these inequalities from the corresponding operator inequalities. For instance, consider the Euclidean Hilbert space R^n with inner product $\langle u|v \rangle = \sum_{i=1}^{n} x_i y_i$, where $u = (x_1, x_2, \ldots, x_n)$ and $v = (y_1, y_2, \ldots, y_n)$. Let $x_i = \sqrt{p_i} (i = 1, 2, \ldots, n)$, where p_i are positive real numbers such that $\sum_{i=1}^{n} p_i = 1$. Then u is a unit vector. Let T be a linear operator on R^n defined by $Tu = \left(\sqrt{p_1}x_1, \sqrt{p_2}x_2, \ldots, \sqrt{p_n}x_n\right)$. For $0 < m \leqslant T \leqslant M$, we have $\langle Tu|u \rangle = A$, $\langle T^{-1}u|u \rangle^{-1} = H$ and $\langle T^2u|u \rangle - \langle Tu|u \rangle^2 = S^2$. On substituting these values in inequalities in Theorem 1-Theorem 3; inequalities given in this corollary follow immediately. □

COROLLARY 2. *If a random variable is discrete or continuous and takes values in the interval $[m, M]$, $m > 0$, then*

$$M \geqslant \frac{A^2 - AH + S^2 + \sqrt{(A^2 - AH + S^2)^2 - 4HS^2(A-H)}}{2(A-H)} \tag{2.25}$$

and

$$m \leqslant \frac{A^2 - AH + S^2 - \sqrt{(A^2 - AH + S^2)^2 - 4HS^2(A-H)}}{2(A-H)}. \tag{2.26}$$

Proof. From inequality (2.21) we find that

$$(A-H)M^2 - (A^2 - AH + S^2)M + HS^2 \geqslant 0. \tag{2.27}$$

Inequality (2.25) now follows from inequality (2.27). Similarly we can deduce inequality (2.26) from inequality (2.22).
□

COROLLARY 3. *Let C be a complex $n \times n$ matrix with eigenvalues $0 < \lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_n$. Then*

$$\lambda_n \geqslant \frac{\operatorname{tr} C^2 \operatorname{tr} C^{-1} - n \operatorname{tr} C + \sqrt{(\operatorname{tr} C^2 \operatorname{tr} C^{-1} - n \operatorname{tr} C)^2 - 4(\operatorname{tr} C \operatorname{tr} C^{-1} - n^2)(n \operatorname{tr} C^2 - (\operatorname{tr} C)^2)}}{2\,(\operatorname{tr} C \operatorname{tr} C^{-1} - n^2)} \tag{2.28}$$

and

$$\lambda_1 \leqslant \frac{\operatorname{tr} C^2 \operatorname{tr} C^{-1} - n \operatorname{tr} C - \sqrt{(\operatorname{tr} C^2 \operatorname{tr} C^{-1} - n \operatorname{tr} C)^2 - 4(\operatorname{tr} C \operatorname{tr} C^{-1} - n^2)(n \operatorname{tr} C^2 - (\operatorname{tr} C)^2)}}{2\,(\operatorname{tr} C \operatorname{tr} C^{-1} - n^2)}, \tag{2.29}$$

where $\operatorname{tr} C^k =$ trace of C^k $(k = -1, 1$ *and* $2)$.

Proof. Corollary 3 can be deduced from Corollary 2. We note that $A = \dfrac{\operatorname{tr} C}{n}$, $H = \dfrac{n}{\operatorname{tr} C^{-1}}$ and $S^2 = \dfrac{\operatorname{tr} C^2}{n} - \left(\dfrac{\operatorname{tr} C}{n}\right)^2$. □

EXAMPLE 1. Let

$$C = \begin{bmatrix} 4 & 0 & 2 & 3 \\ 0 & 5 & 0 & 1 \\ 2 & 0 & 6 & 0 \\ 3 & 1 & 0 & 7 \end{bmatrix}.$$

We have, $\operatorname{tr} C = 22$, $\operatorname{tr} C^2 = 154$ and $\operatorname{tr} C^{-1} = \frac{481}{410}$. From inequalities (2.28) and (2.29) we respectively have $\lambda_4 \geqslant 7.6987$ and $\lambda_1 \leqslant 1.7478$, whereas Wolkowicz and Styan [4] have shown that $\lambda_4 \geqslant 7.158$ and $\lambda_1 \leqslant 3.842$.

EXAMPLE 2. Let

$$C = \begin{bmatrix} 4 & 1 & 1 & 2 & 2 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 6 & 1 & 1 \\ 2 & 1 & 1 & 7 & 1 \\ 2 & 1 & 1 & 1 & 8 \end{bmatrix}.$$

We have, $\operatorname{tr} C = 30$, $\operatorname{tr} C^2 = 222$ and $\operatorname{tr} C^{-1} = \frac{4589}{4377}$. From inequalities (2.28) and (2.29) we respectively have $\lambda_5 \geqslant 9.3393$ and $\lambda_1 \leqslant 3.4845$, whereas Wolkowicz and Styan [4] have shown that $\lambda_5 \geqslant 7.449$ and $\lambda_1 \leqslant 4.551$.

Acknowledgements. The author wants to express his thanks to Prof. Rajendra Bhatia who encouraged, paved the way and guided me before and during the investigation of present work. My thanks are also due to the Indian Statistical Institute, New Delhi for providing me all the support and facilities during my visits to the institution.

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(Received March 6, 2008)

Rajesh Sharma
Department of Mathematics
Himachal Pradesh University
Shimla -5
India – 171 005
e-mail: rajesh_hpu_math@yahoo.co.in