AN EXTENSION OF THE HILBERT–TYPE
INEQUALITY AND ITS REVERSE

BICHENG YANG

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Abstract. By introducing some parameters and the weight coefficient, one proves a new extension of the Hilbert-type inequality with a best constant factor. The reverse, some equivalent forms and a number of new particular cases are considered.

1. Introduction

If $a_n, b_n \geqslant 0$, $0 < \sum_{n=1}^{\infty} a_n^2 < \infty$ and $0 < \sum_{n=1}^{\infty} b_n^2 < \infty$, then one has two inequalities as follows (cf. [1]):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}} ; \qquad (1)$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m-n} < \pi^2 \left\{ \sum_{n=1}^{\infty} a_n^2 \sum_{n=1}^{\infty} b_n^2 \right\}^{\frac{1}{2}} , \qquad (2)$$

where the constant factors π and π^2 are the best possible. Inequality (1) is the well known Hilbert's inequality and (2) is named of Hilbert-type inequality. Both of them are important in Mathematical Analysis and its applications (see [2]). In 1925, Hardy and Riesz gave a best extension of (1) by introducing a (p, q)-parameter $(p > 1, \frac{1}{p} + \frac{1}{q} = 1)$ as (see [3]):

$$I_1 := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}} . \qquad (3)$$

Similarly, one still had a best extension of (2) as (see [1])

$$J_1 := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m-n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^2 \left\{ \sum_{n=1}^{\infty} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} b_n^q \right\}^{\frac{1}{q}} . \qquad (4)$$

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In 1997, by estimating the weight coefficient, Yang et al. [4] gave a strengthened version of (3) as

$$
I_1 < \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/p}} \right] a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \left[\frac{\pi}{\sin(\frac{\pi}{p})} - \frac{1-\gamma}{n^{1/q}} \right] b_n^q \right\}^{\frac{1}{q}}, \tag{5}
$$

where $1 - \gamma = 0.42278^+$ is the best value (γ is Euler constant). Recently, some best extensions of (1) and (3) have been proved by a number of mathematicians (cf. [5, 6, 7, 8]). In 2003, Yang et al. [9] analyzed some ways of using weight coefficient to do research for Hilbert-type inequalities. In 2005, by introducing some parameters $\lambda, \alpha > 0$, $0 < \phi_r \leqslant 1$ $(r = p, q)$, $\phi_p + \phi_q = \lambda \alpha$, Yang [10] gave an extension of (3) as

$$
\begin{aligned}
I(\lambda, \alpha) &:= \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{(m^\alpha + n^\alpha)^\lambda} \\
&< \frac{1}{\alpha} B\left(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha} \right) \left\{ \sum_{n=1}^{\infty} n^{p(1-\phi_q)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q \right\}^{\frac{1}{q}}, \quad (6)
\end{aligned}
$$

where the constant factor $k_{\lambda, \alpha} = \frac{1}{\alpha} B(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha})$ is the best possible, and $B(u, v)$ is the Beta function. Yang [11, 12] also considered the reverse of (3) and the integral analogue. The reverse of (6) was still obtained by [10] as: If $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $0 < \theta_p(n) = O(\frac{1}{n^{\phi_p}}) < 1$, then

$$
I(\lambda, \alpha) > k_{\lambda, \alpha} \left\{ \sum_{n=1}^{\infty} [1 - \theta_p(n)] n^{p(1-\phi_q)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\phi_p)-1} b_n^q \right\}^{\frac{1}{q}}, \tag{7}
$$

where the constant factor $k_{\lambda, \alpha} = \frac{1}{\alpha} B(\frac{\phi_p}{\alpha}, \frac{\phi_q}{\alpha})$ is still the best possible.

For $\lambda = \alpha = 1$, $\phi_p = \frac{1}{s}$, $\phi_q = \frac{1}{r}$ $(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$ in (6) and (7), one has

$$
I_1 < \frac{\pi}{\sin(\frac{\pi}{r})} \left\{ \sum_{n=1}^{\infty} n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}} \tag{8}
$$

and the reverse of (8) as: For $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < O(\frac{1}{n^{1/s}}) < 1$,

$$
I_1 > \frac{\pi}{\sin(\frac{\pi}{r})} \left\{ \sum_{n=1}^{\infty} \left[1 - O\left(\frac{1}{n^{1/s}} \right) \right] n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}}. \tag{9}
$$

In this paper, by introducing some parameters and the weight coefficient as [10], we prove a new extension of (2). The reverse, some equivalent forms and a number of new particular cases are considered.

2. Some lemmas

LEMMA 1. *If* $0 < \phi, \ \psi \leqslant 1, \ \phi + \psi = \lambda$, *define the weight coefficient* $\omega_m(\phi, \psi)$ *and* $\omega_n(\psi, \phi)$ *(* $m, n \in \mathbf{N}$ *) by*

$$\omega_m(\phi, \psi) := \sum_{n=1}^{\infty} \frac{\ln\left(\frac{m}{n}\right)}{m^\lambda - n^\lambda} \left(\frac{m^\psi}{n^{1-\phi}}\right), \qquad \omega_n(\psi, \phi) := \sum_{m=1}^{\infty} \frac{\ln\left(\frac{n}{m}\right)}{n^\lambda - m^\lambda} \left(\frac{n^\phi}{m^{1-\psi}}\right).$$
(10)

Setting $\theta_m(\phi, \psi) = \left[\frac{\sin(\pi\phi/\lambda)}{\pi}\right]^2 \int_0^{m^{-\lambda}} \frac{\ln u}{u-1} u^{\frac{\phi}{\lambda}-1} du$, *then it follows* $0 < \theta_m(\phi, \psi) = O(\frac{\ln m}{m^\phi}) < 1 \ (m \to \infty)$, *and*

$$\left[\frac{\pi}{\sin(\pi\phi/\lambda)}\right]^2 [1 - \theta_m(\phi, \psi)] \ < \ \omega_m(\phi, \psi) < \left[\frac{\pi}{\sin(\pi\phi/\lambda)}\right]^2,$$

$$\omega_n(\psi, \phi) \ < \ \left[\frac{\pi}{\sin(\pi\phi/\lambda)}\right]^2.$$
(11)

Proof. Since the function $f(x) = \frac{\ln(m/x)}{m^\lambda - x^\lambda}$ is decreasing in $(0, \infty)$ (see [13], Lemma 2.2), and $1 - \phi \geqslant 0$, setting $u = (\frac{x}{m})^\lambda$, in view of the face that $\int_0^\infty \frac{\ln u}{u-1} u^{a-1} du = \left[\frac{\pi}{\sin(a\pi)}\right]^2 \ (0 < a < 1)$ (see [1]), we find

$$\omega_m(\phi, \psi) \ < \ \int_0^\infty \frac{\ln(\frac{m}{x})}{m^\lambda - x^\lambda} \frac{m^\psi}{x^{1-\phi}} dx = \frac{1}{\lambda^2} \int_0^\infty \frac{(\ln u) u^{\frac{\phi}{\lambda}-1}}{u-1} du = \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^2;$$

$$\omega_m(\phi, \psi) \ > \ \int_1^\infty \frac{\ln(m/x)}{m^\lambda - x^\lambda} (\frac{m^\psi}{x^{1-\phi}}) dx = \frac{1}{\lambda^2} \int_{m^{-\lambda}}^\infty \frac{\ln u}{u-1} u^{\frac{\phi}{\lambda}-1} du$$

$$= \ \left[\frac{\pi}{\sin(\pi\phi/\lambda)}\right]^2 [1 - \theta_m(\phi, \psi)] > 0.$$

By the same way, it follows $\omega_n(\psi, \phi) < \left[\frac{\pi}{\sin(\pi\phi/\lambda)}\right]^2$, and one obtains (11).

It is obvious that $0 < \theta_m(\phi, \psi) < 1$. Since we find

$$\lim_{x \to 0^+} \frac{\int_0^{x^\lambda} \frac{\ln u}{u-1} u^{\frac{\phi}{\lambda}-1} du}{-x^\phi \ln x} \ = \ \lim_{x \to 0^+} \frac{\lambda^2 \ln x}{(\phi \ln x + 1)(1 - x^\lambda)} = \frac{\lambda^2}{\phi};$$

$$\int_0^{m^{-\lambda}} \frac{\ln u}{u-1} u^{\frac{\phi}{\lambda}-1} du \ \sim \ \frac{\lambda^2 \ln m}{\phi m^\phi} \ (m \to \infty),$$

then it follows $\theta_m(\phi, \psi) = O(\frac{\ln m}{m^\phi})(m \to \infty)$. The lemma is proved. $\quad\square$

LEMMA 2. *If* $p > 0 \ (p \neq 1), \ \frac{1}{p} + \frac{1}{q} = 1, \ 0 < \phi, \psi \leqslant 1, \ \phi + \psi = \lambda$, *then for* $0 < \varepsilon < \frac{p\psi}{2}$ *and*

$$\widetilde{J}_\lambda := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) m^{\phi-1-\frac{\varepsilon}{p}}}{m^\lambda - n^\lambda} n^{\psi-1-\frac{\varepsilon}{q}},$$
(12)

one has

$$\left\{\left[\frac{\pi}{\lambda\sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} - O(1)$$

$$< \widetilde{J}_\lambda < \left\{\left[\frac{\pi}{\lambda\sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}(\varepsilon\to 0^+). \qquad (13)$$

Proof. By the same idea of Lemma 1, setting $u = (x/n)^\lambda$, one has

$$\widetilde{J}_\lambda > \sum_{n=1}^{\infty} n^{\psi-1-\frac{\varepsilon}{q}}\int_1^{\infty}\frac{\ln(\frac{x}{n})x^{\phi-1-\frac{\varepsilon}{p}}}{x^\lambda - n^\lambda}dx$$

$$= \frac{1}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\int_{n^{-\lambda}}^{\infty}\frac{(\ln u)u^{\frac{\psi}{\lambda}-1-\frac{\varepsilon}{p\lambda}}}{u-1}du$$

$$= \frac{1}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left[\int_0^{\infty}\frac{(\ln u)u^{\frac{\phi}{\lambda}-1-\frac{\varepsilon}{p\lambda}}}{u-1}du - \int_0^{n^{-\lambda}}\frac{(\ln u)u^{\frac{\phi}{\lambda}-1-\frac{\varepsilon}{p\lambda}}}{u-1}du\right]$$

$$\geqslant \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{\left[\frac{\pi}{\lambda\sin(\frac{\pi\phi}{\lambda})}\right]^2 + o(1)\right\} + \frac{1}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{n^{-\lambda}}\ln u\sum_{k=0}^{\infty}u^{k+\frac{\phi}{2\lambda}-1}du$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{\left[\frac{\pi}{\lambda\sin(\frac{\pi\phi}{\lambda})}\right]^2 + o(1)\right\} + \frac{1}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=0}^{\infty}\frac{1}{k+\frac{\phi}{2\lambda}}\int_0^{n^{-\lambda}}\ln u\, du^{k+\frac{\phi}{2\lambda}}$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{\left[\frac{\pi}{\lambda\sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\}$$

$$-\frac{1}{\lambda^2}\sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{\psi}{2}}}\sum_{k=0}^{\infty}\frac{1}{k+\frac{\phi}{2\lambda}}\left[\ln n + \frac{1}{k+\frac{\phi}{2\lambda}}\right]\frac{1}{n^{\lambda k}}$$

$$\geqslant \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{\left[\frac{\pi}{\lambda\sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\}$$

$$-\frac{1}{\lambda^2}\left[\sum_{n=2}^{\infty}\frac{\ln n}{n^{1+\frac{\psi}{2}}}\sum_{k=0}^{\infty}\frac{1}{k+\frac{\phi}{2\lambda}}\cdot\frac{1}{2^{\lambda k}} + \sum_{n=1}^{\infty}\frac{1}{n^{1+\frac{\psi}{2}}}\sum_{k=0}^{\infty}\frac{1}{(k+\frac{\phi}{2\lambda})^2}\right]$$

$$= \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\left\{\left[\frac{\pi}{\lambda\sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\} - O(1)(\varepsilon\to 0^+).$$

Hence one obtains (13). The lemma is proved.   □

3.   Main results

THEOREM 1. *If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi, \psi \leqslant 1$, $\phi + \psi = \lambda$, $a_n, b_n \geqslant 0$, such that $0 < \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p < \infty$ and $0 < \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q < \infty$, then the following equivalent inequalities hold:*

$$
\begin{aligned}
J_\lambda & := \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n}) a_m b_n}{m^\lambda - n^\lambda} \\
& < \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^2 \left\{ \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q \right\}^{\frac{1}{q}} ;
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
L_\lambda & := \sum_{n=1}^{\infty} n^{p\psi-1} \left[\sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n}) a_m}{m^\lambda - n^\lambda} \right]^p \\
& < \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^{2p} \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p,
\end{aligned}
\tag{15}
$$

where the constant factor $\left[\frac{\pi}{\sin(\frac{\pi\phi}{\lambda})} \right]^2$ and $\left[\frac{\pi}{\sin(\frac{\pi\phi}{\lambda})} \right]^{2p}$ are the best possible. In particular,

(i)   *for $\lambda = 1$ and $\phi = \frac{1}{r}, \psi = \frac{1}{s}(r > 1, \frac{1}{r} + \frac{1}{s} = 1)$, one has*

$$
J_1 < \left[\frac{\pi}{\sin(\frac{\pi}{r})} \right]^2 \left\{ \sum_{n=1}^{\infty} n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}},
\tag{16}
$$

$$
\sum_{n=1}^{\infty} n^{\frac{p}{s}-1} \left[\sum_{m=1}^{\infty} \frac{\ln(m/n) a_m}{m - n} \right]^p < \left[\frac{\pi}{\sin(\frac{\pi}{r})} \right]^{2p} \sum_{n=1}^{\infty} n^{\frac{p}{s}-1} a_n^p;
\tag{17}
$$

(ii)   *for $0 < \phi = \psi = \frac{\lambda}{2} \leqslant 1$, one gets*

$$
J_\lambda < \left(\frac{\pi}{\lambda} \right)^2 \left\{ \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{\frac{1}{q}},
\tag{18}
$$

$$
\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{\ln(m/n) a_m}{m^\lambda - n^\lambda} \right]^p < \left(\frac{\pi}{\lambda} \right)^{2p} \sum_{n=1}^{\infty} n^{p(1-\frac{\lambda}{2})-1} a_n^p.
\tag{19}
$$

Proof. By Hölder's inequality with weight (see [14]), in view of (10), one obtains

$$
\begin{aligned}
J_\lambda &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})}{m^\lambda - n^\lambda}\left[\frac{m^{(1-\phi)/q}}{n^{(1-\psi)/p}}a_m\right]\left[\frac{n^{(1-\psi)/p}}{m^{(1-\phi)/q}}b_n\right] \\
&\leqslant \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\ln(\frac{m}{n})}{m^\lambda - n^\lambda}\frac{m^{(1-\phi)(p-1)}}{n^{1-\psi}}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})}{m^\lambda - n^\lambda}\frac{n^{(1-\psi)(q-1)}}{m^{1-\phi)}}b_n^q\right\}^{\frac{1}{q}} \\
&= \left\{\sum_{m=1}^{\infty}\omega_m(\phi,\psi)m^{p(1-\phi)-1}a_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}\omega_n(\psi,\phi)n^{q(1-\psi)-1}b_n^q\right\}^{\frac{1}{q}}.
\end{aligned}
$$

Hence by (11), inequality (14) follows.

For $0 < \varepsilon < \frac{p\psi}{2}$, setting $\widetilde{a}_n$, $\widetilde{b}_n$ as: $\widetilde{a}_n = n^{\phi-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{\psi-1-\frac{\varepsilon}{q}}(n \in \mathbf{N})$, and making the assumption that the positive constant $k_0 \leqslant \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^2$ is the best value of (14), one finds

$$
\begin{aligned}
\widetilde{J}_\lambda &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\ln(m/n)\widetilde{a}_m}{m^\lambda - n^\lambda}\widetilde{b}_n \\
&< k_0\left\{\sum_{m=1}^{\infty}m^{p(1-\phi)-1}\widetilde{a}_m^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty}n^{q(1-\psi)-1}\widetilde{b}_n^q\right\}^{\frac{1}{q}} = k_0\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}},
\end{aligned}
$$

and by (13), one has

$$
\left\{\left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\}\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} - O(1) < \widetilde{J}_\lambda < k_0\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}};
$$

$$
\left\{\left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2 + o(1)\right\} - \left(\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}}\right)^{-1}O(1) < k_0.
$$

Therefore $\left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2 \leqslant k_0(\varepsilon \to 0^+)$. It follows that the constant factor $k_0 = \left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2$ in (14) is the best possible.

For a large enough integer k, which makes $\sum_{m=1}^{k}m^{p(1-\phi)-1}a_m^p > 0$, if one sets $b_n(k) = n^{p\psi-1}\left[\sum_{m=1}^{k}\frac{\ln(m/n)a_m}{m^\lambda - n^\lambda}\right]^{p-1} > 0$, for $n \leqslant k$ (assuming that for $n > k$, $b_n(k) = a_n = 0$), and uses (14) to obtain

$$
\begin{aligned}
0 &< \sum_{n=1}^{k}n^{q(1-\psi)-1}b_n^q(k) = \sum_{n=1}^{k}n^{p\psi-1}\left[\sum_{m=1}^{k}\frac{\ln(m/n)a_m}{m^\lambda - n^\lambda}\right]^p = \sum_{n=1}^{k}\sum_{m=1}^{k}\frac{\ln(m/n)}{m^\lambda - n^\lambda}a_m b_n(k) \\
&< \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^2\left\{\sum_{n=1}^{k}n^{p(1-\phi)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=1}^{k}n^{q(1-\psi)-1}b_n^q(k)\right\}^{\frac{1}{q}};
\end{aligned}
\tag{20}
$$

$$0 \; < \; \sum_{n=1}^{k} n^{q(1-\psi)-1} b_n^q(k) = \sum_{n=1}^{k} n^{p\psi-1} \left[\sum_{m=1}^{k} \frac{\ln(m/n)a_m}{m^\lambda - n^\lambda} \right]^p$$

$$< \; \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^{2p} \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p < \infty, \qquad (21)$$

then $0 < \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q(\infty) < \infty$, and for $k \to \infty$, both (20) and (21) still preserve their strict sign-inequalities by (14). Thus (15) follows.

Assuming that (15) is valid, by Hölder's inequality (see [14]), one has

$$J_\lambda = \sum_{n=1}^{\infty} \left[n^{\psi - \frac{1}{p}} \sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})a_m}{m^\lambda - n^\lambda} \right] \left[n^{\frac{1}{p} - \psi} b_n^q \right] \leqslant L_\lambda^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q \right\}^{\frac{1}{q}}. \qquad (22)$$

Then by (15), inequality (14) holds. Hence inequalities (14) and (15) are equivalent. One conforms that the constant factor in (15) is the best possible. Otherwise, one can cause up with a contradiction by (22) that the constant factor in (14) is not the best possible. Hence the theorem is proved. $\qquad \square$

THEOREM 2. *If* $0 < p < 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < \phi, \psi \leqslant 1$, $\phi + \psi = \lambda$, $a_n \geqslant 0$, $b_n > 0$, *such that* $0 < \sum_{n=1}^{\infty} n^{p(1-\phi)-1} a_n^p < \infty$ *and* $0 < \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q < \infty$, *then the following equivalent inequalities hold:*

$$J_\lambda \; > \; \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^2 \left\{ \sum_{n=1}^{\infty} [1 - \theta_n(\phi, \psi)] n^{p(1-\phi)-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q \right\}^{\frac{1}{q}}; \quad (23)$$

$$L_\lambda \; > \; \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^{2p} \sum_{n=1}^{\infty} [1 - \theta_n(\phi, \psi)] n^{p(1-\phi)-1} a_n^p; \qquad (24)$$

$$M_\lambda \; := \; \sum_{m=1}^{\infty} \frac{m^{q\phi-1}}{[1 - \theta_m(\phi, \psi)]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{\ln(\frac{m}{n})b_n}{m^\lambda - n^\lambda} \right]^q$$

$$< \; \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^{2q} \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q, \qquad (25)$$

where the constant factor $\left[\frac{\pi}{\sin(\frac{\pi\phi}{\lambda})} \right]^2$ *and* $\left[\frac{\pi}{\sin(\frac{\pi\phi}{\lambda})} \right]^{2r}$ *($r = p, q$) are the best possible and* $0 < \theta_m(\phi, \psi) = O(\frac{\ln m}{m^\phi}) < 1 (m \to \infty)$.

In particular,
(i) for $\lambda = 1$, $\phi = \frac{1}{r}$, $\psi = \frac{1}{s}$ ($r > 1$, $\frac{1}{r} + \frac{1}{s} = 1$), one gets $0 < \theta_m(r) := \theta_m(\frac{1}{r}, \frac{1}{s}) = O(\frac{\ln m}{m^{1/r}}) < 1$, and

$$J_1 > \left[\frac{\pi}{\sin(\frac{\pi}{r})} \right]^2 \left\{ \sum_{n=1}^{\infty} [1 - \theta_n(r)] n^{\frac{p}{s}-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q \right\}^{\frac{1}{q}}, \qquad (26)$$

$$\sum_{n=1}^{\infty} n^{\frac{p}{s}-1} \left[\sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})a_m}{m-n} \right]^p > \left[\frac{\pi}{\sin(\frac{\pi}{r})} \right]^{2p} \sum_{n=1}^{\infty} [1 - \theta_n(r)] n^{\frac{p}{s}-1} a_n^p, \qquad (27)$$

$$\sum_{m=1}^{\infty} \frac{m^{\frac{q}{r}-1}}{[1 - \theta_m(r)]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{\ln(\frac{m}{n})b_n}{m-n} \right]^q < \left[\frac{\pi}{\sin(\frac{\pi}{r})} \right]^{2q} \sum_{n=1}^{\infty} n^{\frac{q}{r}-1} b_n^q; \qquad (28)$$

(ii)   for $0 < \phi = \psi = \frac{\lambda}{2} \leqslant 1$, one obtains $0 < \widetilde{\theta}_m(\lambda) := \theta_m(\frac{\lambda}{2}, \frac{\lambda}{2}) = O(\frac{\ln m}{m^{\lambda/2}}) < 1$, and

$$J_\lambda > \left(\frac{\pi}{\lambda} \right)^2 \left\{ \sum_{n=1}^{\infty} [1 - \widetilde{\theta}_n(\lambda)] n^{p(1-\frac{\lambda}{2})-1} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q \right\}^{\frac{1}{q}}, \qquad (29)$$

$$\sum_{n=1}^{\infty} n^{\frac{p\lambda}{2}-1} \left[\sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})a_m}{m^\lambda - n^\lambda} \right]^p > \left(\frac{\pi}{\lambda} \right)^{2p} \sum_{n=1}^{\infty} [1 - \widetilde{\theta}_n(\lambda)] n^{p(1-\frac{\lambda}{2})-1} a_n^p, \qquad (30)$$

$$\sum_{m=1}^{\infty} \frac{m^{\frac{q\lambda}{2}-1}}{[1 - \widetilde{\theta}_m(\lambda)]^{q-1}} \left[\sum_{n=1}^{\infty} \frac{\ln(\frac{m}{n})b_n}{m^\lambda - n^\lambda} \right]^q < \left(\frac{\pi}{\lambda} \right)^{2q} \sum_{n=1}^{\infty} n^{q(1-\frac{\lambda}{2})-1} b_n^q. \qquad (31)$$

Proof. In view of Lemma 1, it follows that $0 < \theta_m(\phi, \psi) = O(\frac{\ln m}{m^\phi}) < 1 (m \to \infty)$. Applying the reverse Hölder's inequality with weight (see [14]) and (10), one finds

$$J_\lambda \geqslant \left\{ \sum_{m=1}^{\infty} \omega_m(\phi, \psi) m^{p(1-\phi)-1} a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \omega_n(\psi, \phi) n^{q(1-\psi)-1} b_n^q \right\}^{\frac{1}{q}}.$$

Hence by (11), in view of $0 < p < 1$ and $q < 0$, inequality (23) follows.

For $0 < \varepsilon < \frac{p\psi}{2}$, setting $\widetilde{a}_n$, $\widetilde{b}_n$ as: $\widetilde{a}_n = n^{\phi-1-\frac{\varepsilon}{p}}$, $\widetilde{b}_n = n^{\psi-1-\frac{\varepsilon}{q}}$ $(n \in \mathbf{N})$, and making the assumption that the positive constant $k_0 \geqslant \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^2$ is the best value of (23), then by (13), one finds

$$\left\{ \left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)} \right]^2 + o(1) \right\} \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}}$$

$$> \widetilde{J}_\lambda > k_0 \left\{ \sum_{m=1}^{\infty} [1 - \theta_m(\phi, \psi)] m^{p(1-\phi)-1} \widetilde{a}_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q(1-\psi)-1} \widetilde{b}_n^q \right\}^{\frac{1}{q}}$$

$$\geqslant k_0 \sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \left\{ 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right)^{-1} \left[\sum_{m=1}^{\infty} \frac{1}{m} O\left(\frac{\ln m}{m^\phi} \right) \right] \right\}^{\frac{1}{p}}.$$

Hence one has

$$\left\{ \left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)} \right]^2 + o(1) \right\} > k_0 \left\{ 1 - \left(\sum_{n=1}^{\infty} \frac{1}{n^{1+\varepsilon}} \right)^{-1} \left[\sum_{m=1}^{\infty} \frac{1}{m} O\left(\frac{\ln m}{m^\phi} \right) \right] \right\}^{\frac{1}{p}}.$$

Therefore $\left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2 \geqslant k_0(\varepsilon \rightarrow 0^+)$. It follows that the constant factor $k_0 = \left[\frac{\pi}{\lambda \sin(\pi\phi/\lambda)}\right]^2$ in (23) is the best possible.

One conforms that $L_\lambda > 0$. If $L_\lambda = \infty$, then (24) is valid; if $0 < L_\lambda < \infty$, setting $b_n = n^{p\psi - 1}\left[\sum_{m=1}^{\infty} \frac{\ln(m/n)a_m}{m^\lambda - n^\lambda}\right]^{p-1}$, then by (23), it follows

$$\infty > \sum_{n=1}^{\infty} n^{q(1-\psi)-1}b_n^q = L_\lambda = J_\lambda$$

$$> \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^2 \left\{\sum_{n=1}^{\infty}[1-\theta_n(\phi, \psi)]n^{p(1-\phi)-1}a_n^p\right\}^{\frac{1}{p}} \left\{\sum_{n=1}^{\infty} n^{q(1-\psi)-1}b_n^q\right\}^{\frac{1}{q}}; \quad (32)$$

$$\sum_{n=1}^{\infty} n^{q(1-\psi)-1}b_n^q = L_\lambda > \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^{2p} \sum_{n=1}^{\infty}[1 - \theta_n(\phi, \psi)]n^{p(1-\phi)-1}a_n^p. \quad (33)$$

Hence one has (24). Assuming that (24) is valid, by the reverse Hölder's inequality (see [14]), one has

$$J_\lambda = \sum_{n=1}^{\infty}\left[n^{\psi - \frac{1}{p}} \sum_{m=1}^{\infty} \frac{\ln(\frac{m}{n})a_m}{m^\lambda - n^\lambda}\right]\left[n^{\frac{1}{p} - \psi}b_n\right] \geqslant L_\lambda^{\frac{1}{p}}\left\{\sum_{n=1}^{\infty} n^{q(1-\psi)-1}b_n^q\right\}^{\frac{1}{q}}. \quad (34)$$

Then by (24), inequality (23) holds. Hence (23) and (24) are equivalent. If the constant in (24) is not the best possible, one can causes up with a contradiction by (34) that the constant factor in (23) is not the best possible.

For a large enough integer k, which makes $\sum_{n=1}^{k} n^{q(1-\psi)-1}b_n^q > 0$, if one sets $a_m(k) = \frac{m^{q\phi-1}}{[1-\theta_m(\phi,\psi)]^{q-1}}\left[\sum_{n=1}^{k} \frac{\ln(m/n)b_n}{m^\lambda - n^\lambda}\right]^{q-1} > 0$, for $n \leqslant k$ (assuming that for $n > k$, $a_n(k) = b_n = 0$), and uses (23) to obtain

$$\sum_{m=1}^{k}[1 - \theta_m(\phi, \psi)]m^{p(1-\phi)-1}a_m^p(k)$$

$$= \sum_{m=1}^{k} \frac{m^{q\phi-1}}{[1 - \theta_m(\phi, \psi)]^{q-1}}\left[\sum_{n=1}^{k} \frac{\ln(\frac{m}{n})b_n}{m^\lambda - n^\lambda}\right]^q = \sum_{n=1}^{k}\sum_{m=1}^{k} \frac{\ln(\frac{m}{n})}{m^\lambda - n^\lambda}a_m(k)b_n$$

$$> \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})}\right]^2 \left\{\sum_{m=1}^{k}[1 - \theta_m(\phi, \psi)]m^{p(1-\phi)-1}a_m^p(k)\right\}^{\frac{1}{p}}$$

$$\times \left\{\sum_{n=1}^{k} n^{q(1-\psi)-1}b_n^q\right\}^{\frac{1}{q}} > 0; \quad (35)$$

$$0 < \sum_{m=1}^{k} [1 - \theta_m(\phi, \psi)] m^{p(1-\phi)-1} a_m^p(k)$$

$$= \sum_{m=1}^{k} \frac{m^{q\phi-1}}{[1 - \theta_m(\phi, \psi)]^{q-1}} \left[\sum_{n=1}^{k} \frac{\ln(\frac{m}{n})}{m^\lambda - n^\lambda} b_n \right]^q$$

$$< \left[\frac{\pi}{\lambda \sin(\frac{\pi\phi}{\lambda})} \right]^{2q} \sum_{n=1}^{\infty} n^{q(1-\psi)-1} b_n^q < \infty, \qquad (36)$$

then $0 < \sum_{m=1}^{\infty} m^{p(1-\phi)-1} a_m^p(\infty) < \infty$, for $k \to \infty$, both (35) and (36) still preserve their strict sign-inequalities by (23). Thus (25) follows.

Assuming that (25) is valid, by the reverse Hölder's inequality (see [14]), one has

$$J_\lambda = \sum_{m=1}^{\infty} \left\{ [1 - \theta_m(\phi, \psi]^{\frac{1}{p}} m^{\frac{1}{q} - \phi} a_m \right\} \left\{ \frac{m^{\phi - \frac{1}{q}}}{[1 - \theta_m(\phi, \psi]^{\frac{1}{p}}} \sum_{n=1}^{\infty} \frac{\ln(\frac{m}{n}) b_n}{m^\lambda - n^\lambda} \right\}$$

$$\geqslant \left\{ \sum_{m=1}^{\infty} [1 - \theta_m(\phi, \psi)] m^{p(1-\phi)-1} a_m^p \right\}^{\frac{1}{p}} M_\lambda^{\frac{1}{q}}. \qquad (37)$$

Then by (25), inequality (23) follows. Applying the same idea as above, one may prove that (23) and (25) are equivalent and the constant factor in (15) is the best possible by using (37). Hence inequalities (23), (24) and (25) are equivalent. This proves the theorem.   □

REMARKS. (a) For $\phi = \frac{\lambda}{q}$, $\psi = \frac{\lambda}{p}$ in (14), one obtains $0 < \lambda \leqslant 2$ and

$$J_\lambda < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^2 \left\{ \sum_{n=1}^{\infty} n^{(p-1)(1-\lambda)} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{(q-1)(1-\lambda)} b_n^q \right\}^{\frac{1}{q}}, \qquad (38)$$

which is a best extension of (4).

(b) For $\phi = \frac{\lambda}{p}$, $\psi = \frac{\lambda}{q}$ $(0 < \lambda \leqslant 2)$ in (14), one implies inequality (3.1) in [15]. In particular, for $\lambda = 1$, the dual form of (4) is reduced as follows:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m - n} < \left[\frac{\pi}{\sin(\frac{\pi}{p})} \right]^2 \left\{ \sum_{n=1}^{\infty} n^{p-2} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} n^{q-2} b_n^q \right\}^{\frac{1}{q}}, \qquad (39)$$

which is still a best extension of (2).

(c) For $\lambda = 2$ $(0 < \widetilde{\theta}_n(2) < 1)$ in (18) and (19), one obtains a new Hilbert-type inequality and its reverse as follows:

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m^2 - n^2} < \frac{\pi^2}{4} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}, \qquad (40)$$

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\ln(m/n) a_m b_n}{m^2 - n^2} > \frac{\pi^2}{4} \left\{ \sum_{n=1}^{\infty} [1 - \widetilde{\theta}_n(2)] \frac{1}{n} a_n^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} \frac{1}{n} b_n^q \right\}^{\frac{1}{q}}. \qquad (41)$$

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Bicheng Yang
Department of Mathematics
Guangdong Education Institute
Guangzhou
Guangdong 510303
P. R. China
e-mail: bcyang@pub.guangzhou.gd.cn