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ASYMPTOTIC BEHAVIOR OF INTERMEDIATE POINTS IN CERTAIN MEAN VALUE THEOREMS

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(*communicated by A. Čižmešija*)

Abstract. The paper deals with asymptotic behavior of intermediate points in certain mean value theorems: the Cauchy–Taylor mean value theorem, a generalization due to I. Pawlikowska of Flett's mean value theorem, and a Cauchy version of Pawlikowska's mean value theorem.

1. Introduction

Let I be an open interval in $\mathbb{R}$, let a be an arbitrary point in I, let $n \in \mathbb{N}$, and let $f : I \to \mathbb{R}$ be a function whose derivative $f^{(n)}$ exists on I. Then for any other point x in I one can expand $f(x)$ about the point a up to nth power by the Lagrange–Taylor formula to obtain

$$f(x) = T_{n-1}(f;a)(x) + \frac{f^{(n)}(\xi)}{n!}(x-a)^n, \tag{1}$$

where

$$T_{n-1}(f;a)(x) := f(a) + f'(a)(x-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}$$

denotes the Taylor polynomial of degree $n-1$ associated to f at a. In (1) the intermediate point (or points) ξ lies (lie) strictly between a and x. In the special case when $n = 1$, formula (1) becomes the classical (Lagrange) mean value theorem

$$f(x) - f(a) = f'(\xi)(x-a). \tag{2}$$

In the last three decades there was some interest in the asymptotic behavior of the intermediate point $\xi = \xi(x)$ in (1), (2) and other mean value theorems, when $x \to a$. Thus, A. G. Azpeitia [4] proved that given $p \in \mathbb{N}$ one has

$$\xi = \xi(x) = a + \binom{n+p}{n}^{-1/p}(x-a) + o(|x-a|) \qquad (x \to a) \tag{3}$$

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if $f^{(n+p)}$ exists on I and is continuous at a with $f^{(n+j)}(a) = 0 \ (1 \leqslant j < p)$ and $f^{(n+p)}(a) \neq 0$. This result was generalized by U. Abel [1]. He derived for ξ a complete asymptotic expansion of the form

$$\xi = \xi(x) = a + \sum_{k=1}^{\infty} \frac{c_k}{k!} (x-a)^k \qquad (x \to a),$$

provided that f possesses derivatives of sufficiently high order at a.

A well-known generalization of (2) is the Cauchy mean value theorem: consider two functions $f, g : I \to \mathbb{R}$ such that the derivatives f' and g' exist both on I. If g' does not vanish in I, then for every $x \neq a$ in I one has

$$g'(\xi)[f(x) - f(a)] = f'(\xi)[g(x) - g(a)], \tag{4}$$

with intermediate point (or points) ξ strictly between a and x. In a recent paper, D. I. Duca and O. Pop [6] proved that given $p \in \mathbb{N}$, the point ξ in (4) satisfies

$$\xi = \xi(x) = a + \frac{1}{\sqrt[p]{p+1}} (x-a) + o(|x-a|) \qquad (x \to a)$$

whenever the derivatives $f^{(p+1)}$ and $g^{(p+1)}$ exist on I and are both continuous at a, $f^{(j)}(a)g'(a) = f'(a)g^{(j)}(a) \ \ (2 \leqslant j \leqslant p)$, and $f^{(p+1)}(a)g'(a) \neq f'(a)g^{(p+1)}(a)$.

In the present paper we are concerned with the asymptotic behavior of the intermediate points in other mean value theorems: the Cauchy–Taylor mean value theorem (which is a common generalization of (1), (2) and (4)), a generalization due to I. Pawlikowska [8] of Flett's mean value theorem (see T. M. Flett [7] or P. K. Sahoo and T. Riedel [10]), and a Cauchy version of Pawlikowska's mean value theorem.

2. Asymptotic behavior of the intermediate point in the Cauchy–Taylor mean value theorem

Let I be an open interval in $\mathbb{R}$, let a be an arbitrary point in I, and let $n \in \mathbb{N}$. Further, let $f, g : I \to \mathbb{R}$ be functions whose derivatives $f^{(n)}$ and $g^{(n)}$ exist both on I. According to the Cauchy–Taylor mean value theorem, if $g^{(n)}$ does not vanish in I, then for every $x \neq a$ in I one has

$$g^{(n)}(\xi)\,[f(x) - T_{n-1}(f;a)(x)] = f^{(n)}(\xi)\,[g(x) - T_{n-1}(g;a)(x)], \tag{5}$$

with intermediate point (or points) ξ strictly between a and x.

The following theorem, pointing out the asymptotic behavior of ξ in (5), is a common generalization of the aforementioned results by A. G. Azpeitia on one hand and by D. I. Duca and O. Pop on the other hand.

THEOREM 1. *Under the above assumptions let p and q be positive integers and suppose that f and g fulfil the following conditions:*

(i) *the derivatives $f^{(n+p+q)}$ and $g^{(n+p+q)}$ exist on I and they are both continuous at a,*

(ii) $f^{(n+j)}(a)g^{(n)}(a) = f^{(n)}(a)g^{(n+j)}(a)$ *for* $1 \leqslant j < p$,

(iii) $f^{(n+p)}(a)g^{(n)}(a) \neq f^{(n)}(a)g^{(n+p)}(a)$.
Then the point ξ *in* (5) *satisfies* (3).

Proof. Note first that, since $g^{(n)}(a) \neq 0$, by (ii) it follows immediately that

$$f^{(n+k)}(a)g^{(n+j)}(a) = f^{(n+j)}(a)g^{(n+k)}(a) \qquad \text{for all } 0 \leqslant j,k < p. \tag{6}$$

On the other hand, under the given assumptions, we may apply the Lagrange–Taylor formula to expand $f(x)-T_{n-1}(f;a)(x)$ and $g(x)-T_{n-1}(g;a)(x)$ up to the $(n+p+q)$th power and then, once more, to expand $f^{(n)}(\xi)$ and $g^{(n)}(\xi)$ in (5) up to the $(p+q)$th power. We get

$$f(x) - T_{n-1}(f;a)(x) = \sum_{k=0}^{p+q-1} \frac{f^{(n+k)}(a)}{(n+k)!}(x-a)^{n+k} + \frac{f^{(n+p+q)}(\xi_1)}{(n+p+q)!}(x-a)^{n+p+q},$$
$$g(x) - T_{n-1}(g;a)(x) = \sum_{k=0}^{p+q-1} \frac{g^{(n+k)}(a)}{(n+k)!}(x-a)^{n+k} + \frac{g^{(n+p+q)}(\xi_2)}{(n+p+q)!}(x-a)^{n+p+q},$$

with ξ_1 and ξ_2 strictly between a and x, and

$$f^{(n)}(\xi) = \sum_{j=0}^{p+q-1} \frac{f^{(n+j)}(a)}{j!}(\xi-a)^j + \frac{f^{(n+p+q)}(\xi_3)}{(p+q)!}(\xi-a)^{p+q},$$
$$g^{(n)}(\xi) = \sum_{j=0}^{p+q-1} \frac{g^{(n+j)}(a)}{j!}(\xi-a)^j + \frac{g^{(n+p+q)}(\xi_4)}{(p+q)!}(\xi-a)^{p+q},$$

with ξ_3 and ξ_4 strictly between a and ξ, respectively. Replacing all these into (5) we obtain

$$\begin{aligned} 0 \; = \; & \sum_{j=0}^{p+q-1}\sum_{k=0}^{p+q-1} \frac{f^{(n+k)}(a)g^{(n+j)}(a) - f^{(n+j)}(a)g^{(n+k)}(a)}{j!\,(n+k)!}(\xi-a)^j(x-a)^{n+k} \\ & + \frac{(x-a)^{n+p+q}}{(n+p+q)!}\sum_{j=0}^{p+q-1} \frac{f^{(n+p+q)}(\xi_1)g^{(n+j)}(a) - f^{(n+j)}(a)g^{(n+p+q)}(\xi_2)}{j!}(\xi-a)^j \\ & + \frac{(x-a)^n(\xi-a)^{p+q}}{(p+q)!}\sum_{k=0}^{p+q-1} \frac{f^{(n+k)}(a)g^{(n+p+q)}(\xi_4) - f^{(n+p+q)}(\xi_3)g^{(n+k)}(a)}{(n+k)!} \\ & \qquad \times (x-a)^k \\ & + \frac{f^{(n+p+q)}(\xi_1)g^{(n+p+q)}(\xi_4) - f^{(n+p+q)}(\xi_3)g^{(n+p+q)}(\xi_2)}{(p+q)!\,(n+p+q)!} \\ & \qquad \times (x-a)^{n+p+q}(\xi-a)^{p+q}. \end{aligned}$$

Taking into account that $|\xi - a| < |x-a|$, from (i), (6) and the above equality we deduce that

$$\begin{aligned} 0 \;=\; & \frac{f^{(n)}(a)g^{(n+p)}(a)-f^{(n+p)}(a)g^{(n)}(a)}{p!\,n!}\,(\xi-a)^p(x-a)^n \\ & +\frac{f^{(n+p)}(a)g^{(n)}(a)-f^{(n)}(a)g^{(n+p)}(a)}{(n+p)!}\,(x-a)^{n+p} \\ & +o\left(|x-a|^{n+p}\right) \qquad (x\to a). \end{aligned}$$

Multiplying both sides by $\dfrac{p!\,n!}{(x-a)^{n+p}}$ and taking (iii) into account, it follows that

$$\left(\frac{\xi-a}{x-a}\right)^p=\binom{n+p}{n}^{-1}+o(1) \qquad (x\to a),$$

whence (3) holds. □

3. Asymptotic behavior of the intermediate point in Pawlikowska's mean value theorem

T. M. Flett [7] established a mean value theorem which is similar to the classical (Lagrange) mean value theorem: if $f : [a,b]\to\mathbb{R}$ is differentiable on $[a,b]$ and $f'(a)=f'(b)$, then there exists a point $\eta\in(a,b)$ such that

$$f(\eta)-f(a)=f'(\eta)(\eta-a).$$

P. K. Sahoo and T. Riedel [10] removed the boundary hypothesis on the derivative. They proved that if $f : [a,b]\to\mathbb{R}$ is differentiable on $[a,b]$, then there exists a point $\eta\in(a,b)$ such that

$$f(\eta)-f(a)=f'(\eta)(\eta-a)-\frac{1}{2}\frac{f'(b)-f'(a)}{b-a}(\eta-a)^2.$$

A nice generalization of these results was obtained by I. Pawlikowska [8] (see also [3]). Namely, she proved that if $f : [a,b]\to\mathbb{R}$ possesses a derivative of order n on $[a,b]$, then there exists a point $\eta\in(a,b)$ such that (compare with (1))

$$f(a)=T_n(f;\eta)(a)+\frac{1}{(n+1)!}\frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}(a-\eta)^{n+1}.$$

Now let I be an open interval in $\mathbb{R}$, let a be a fixed point in I, let $n\in\mathbb{N}$, and let $f : I\to\mathbb{R}$ be a function whose derivative $f^{(n)}$ exists on I. According to Pawlikowska's result, for any other point x in I one has

$$f(a)=T_n(f;\eta)(a)+\frac{1}{(n+1)!}\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a}(a-\eta)^{n+1}, \tag{7}$$

with intermediate point (or points) η strictly between a and x. The main purpose of this section is to derive for η a complete asymptotic expansion similar to that obtained by U. Abel for the intermediate point ξ in the Taylor–Lagrange formula. This asymptotic expansion is contained in the following theorem.

THEOREM 2. *Let I be an open interval in $\mathbb{R}$, let a be an arbitrary point in I, let n, $p \geqslant 2$ and q be positive integers, and let $f : I \to \mathbb{R}$ be a function fulfilling the following conditions:*

(i) *the derivative $f^{(n+p+q)}$ exists on I and it is continuous at a,*

(ii) *$f^{(n+j)}(a) = 0$ for $2 \leqslant j < p$,*

(iii) *$f^{(n+p)}(a) \neq 0$.*

Then the intermediate point $\eta = \eta(x)$ in (7) admits the asymptotic expansion

$$\eta(x) = a + \sum_{k=1}^{q-1} \frac{c_k}{k!}(x-a)^k + O\left(|x-a|^q\right) \qquad (x \to a).$$

The coefficients c_k are given by the recurrence formula

$$c_1 = \left(\frac{n+p}{p(n+1)}\right)^{1/(p-1)}, \qquad c_{k+1} = \mathbf{R}_k(c_1, \ldots, c_k) \qquad (k = 1, \ldots, q-1), \quad (8)$$

with

$$\begin{aligned} \mathbf{R}_k(c_1, \ldots, c_k) &= (k+1)c_1 \sum_{i=1}^{k} \binom{\frac{1}{p-1}}{i} i!\, \mathbf{B}_{k,i}[f_\nu] \qquad (9) \\ &\quad - \sum_{j=1}^{k} (j+1)\, \mathbf{B}_{k+1,j+1}(c_1, \ldots, c_{k-j+1}) \sum_{i=1}^{j} \binom{\frac{1}{p-1}}{i} i!\, \mathbf{B}_{j,i}[\widetilde{f}_\nu], \end{aligned}$$

where $\mathbf{B}_{k,j}[x_\nu] = \mathbf{B}_{k,j}(x_1, \ldots, x_{k-j+1})$ denote the exponential partial Bell polynomials in the variables $x_1, x_2, \ldots$, and

$$\begin{aligned} f_j &= \binom{p+j}{j}^{-1} \frac{f^{(n+p+j)}(a)}{f^{(n+p)}(a)}, \\ \widetilde{f}_j &= \frac{n+p}{n+p+j} \binom{p+j-1}{j}^{-1} \frac{f^{(n+p+j)}(a)}{f^{(n+p)}(a)}, \qquad j = 1, \ldots, q-1. \end{aligned}$$

Before passing to the proof, let us recall that the exponential partial Bell polynomials are the polynomials $\mathbf{B}_{n,k}[x_\nu] = \mathbf{B}_{n,k}(x_1, x_2, \ldots)$ in an infinite number of variables $x_1, x_2, \ldots$, defined by the formal series expansion

$$\frac{1}{k!}\left(\sum_{m \geqslant 1} x_m \frac{t^m}{m!}\right)^k = \sum_{n=k}^{\infty} \mathbf{B}_{n,k}[x_\nu] \frac{t^n}{n!}.$$

In fact, the polynomial $\mathbf{B}_{n,k}[x_\nu]$ depends only on $x_1, \ldots, x_{n-k+1}$ and it has integral coefficients. Its exact expression is

$$\mathbf{B}_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum \frac{n!}{i_1!\, i_2! \cdots (1!)^{i_1} (2!)^{i_2} \cdots} x_1^{i_1} x_2^{i_2} \cdots,$$

where the summation extends over all nonnegative integers $i_1, i_2, \ldots$ satisfying

$$i_1 + i_2 + i_3 + \cdots = k, \qquad i_1 + 2i_2 + 3i_3 + \cdots = n.$$

We note that $\mathbf{B}_{0,0}[x_\nu] = 1$, $\mathbf{B}_{n,0}[x_\nu] = 0$, $\mathbf{B}_{n,1}[x_\nu] = x_n$, $\mathbf{B}_{n,n}[x_\nu] = x_1^n$ for every positive integer n (see the book by L. Comtet [5, pp. 133–137]).

Proof of Theorem 2. Remark first that if f is a polynomial of degree at most $n+1$, then every point η in I satisfies

$$\frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} = f^{(n+1)}(\eta),$$

so (7) may be rewritten in the equivalent form $f(a) = T_{n+1}(f;\eta)(a)$. But this equality is nothing else but Taylor's formula and it holds for every point η in I. Thus, if f is a polynomial of degree at most $n+1$, then every point η in I satisfies (7). By an eventual substraction from f of a polynomial of degree $n+1$, we may assume without losing the generality that, in addition to (ii), f fulfils the condition

(ii') $f^{(j)}(a) = 0$ for $0 \leqslant j \leqslant n+1$.

Consequently, we may put (7) in the equivalent form

$$\frac{(-1)^n (n+1)!}{(\eta - a)^{n+1}} \sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\eta)(\eta - a)^k = \frac{f^{(n)}(x)}{x-a}. \tag{10}$$

By virtue of the Lagrange–Taylor formula, we have

$$f^{(n)}(x) = \sum_{j=0}^{q-1} \frac{f^{(n+p+j)}(a)}{(p+j)!} (x-a)^{p+j} + \frac{f^{(n+p+q)}(\xi_n^*)}{(p+q)!} (x-a)^{p+q}, \tag{11}$$

with ξ_n^* strictly between a and x. Using again the Lagrange–Taylor formula, for each $k \in \{0, 1, \ldots, n\}$ we have

$$f^{(k)}(\eta) = \sum_{j=0}^{q-1} \frac{f^{(n+p+j)}(a)}{(n+p+j-k)!} (\eta - a)^{n+p+j-k} + \frac{f^{(n+p+q)}(\xi_k)}{(n+p+q-k)!} (\eta - a)^{n+p+q-k}, \tag{12}$$

with ξ_k strictly between a and η. Letting $S := \sum_{k=0}^{n} \frac{(-1)^k f^{(n+p+q)}(\xi_k)}{k!\,(n+p+q-k)!}$, from (12) we deduce that

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\eta)(\eta - a)^k = \sum_{j=0}^{q-1} \frac{f^{(n+p+j)}(a)(\eta - a)^{n+p+j}}{(n+p+j)!} S_{n+p+j,n} + (\eta - a)^{n+p+q} S,$$

where

$$S_{r,s} := \sum_{k=0}^{s} (-1)^k \binom{r}{k} = (-1)^s \binom{r-1}{s}.$$

Therefore, it holds

$$\sum_{k=0}^{n} \frac{(-1)^k}{k!} f^{(k)}(\eta)(\eta - a)^k = \frac{(-1)^n}{n!} \sum_{j=0}^{q-1} \frac{f^{(n+p+j)}(a)(\eta - a)^{n+p+j}}{(n+p+j)(p+j-1)!} + (\eta - a)^{n+p+q} S.$$

Replacing (11) and the left hand side of the last equality into (10) we obtain

$$(n+1)\sum_{j=0}^{q-1}\frac{f^{(n+p+j)}(a)}{(n+p+j)(p+j-1)!}(\eta-a)^{p+j-1}+(\eta-a)^{p+q-1}S$$
$$=\sum_{j=0}^{q-1}\frac{f^{(n+p+j)}(a)}{(p+j)!}(x-a)^{p+j-1}+\frac{f^{(n+p+q)}(\xi_n^*)}{(p+q)!}(x-a)^{p+q-1}.$$

Using f_j and $\widetilde{f_j}$ and taking into account that $|\eta-a|<|x-a|$, by (i) we conclude that

$$1+\sum_{j=0}^{q-1}f_j\frac{(x-a)^j}{j!} = \left(\frac{\eta-a}{x-a}\right)^{p-1}\frac{p(n+1)}{n+p}\left[1+\sum_{j=0}^{q-1}\widetilde{f_j}\frac{(\eta-a)^j}{j!}\right]$$
$$+O\left(|x-a|^q\right) \qquad (x\to a).$$

By proceeding like U. Abel [1] in his proof of Theorem 1, from this equality one can deduce that

$$\sum_{k=0}^{m-1}\frac{(x-a)^{k+1}}{k!}\sum_{i=0}^{k}\binom{\frac{1}{p-1}}{i}i!\,\mathbf{B}_{k,i}[f_\nu] \tag{13}$$
$$=\left(\frac{p(n+1)}{n+p}\right)^{\frac{1}{p-1}}\sum_{j=0}^{m-1}\frac{(\eta-a)^{j+1}}{j!}\sum_{i=0}^{j}\binom{\frac{1}{p-1}}{i}i!\,\mathbf{B}_{j,i}[\widetilde{f_\nu}]$$
$$+O\left(|x-a|^{m+1}\right)$$

as $x\to a$. The rest of the proof coincides with that of Theorem 1 in [1] (see also the proof of Theorem 1 in [2]) and we omit it. □

If the function f is analytic at a, then $\eta(x)$ can be expanded in a power series around a.

THEOREM 3. *Let I be an open interval in $\mathbb{R}$, let a be an arbitrary point in I, let n and $p\geqslant 2$ be positive integers, and let $f: I\to\mathbb{R}$ be a function fulfilling the following conditions:*

(i) *f is analytic at a,*

(ii) *$f^{(n+j)}(a)=0$ for $2\leqslant j<p$,*

(iii) *$f^{(n+p)}(a)\neq 0$.*

Then there exists a real interval J around a such that the intermediate point $\eta=\eta(x)$ in (7) can be represented as the sum of a power series around a,

$$\eta(x)=a+\sum_{k=1}^{\infty}\frac{c_k}{k!}(x-a)^k.$$

The coefficients are given by (8) and (9).

Proof. Reasoning as in the proof of Theorem 2, we arrive at

$$\sum_{j=0}^{\infty}\frac{f^{(n+p+j)}(a)}{(p+j)!}(x-a)^{p+j-1}=(n+1)\sum_{j=0}^{\infty}\frac{f^{(n+p+j)}(a)}{(n+p+j)(p+j-1)!}(\eta-a)^{p+j-1}.$$

Instead of (13) we have now $F(x,\eta) = g(x) - h(\eta) = 0$, where

$$g(x) = \sum_{k=0}^{\infty} \frac{(x-a)^{k+1}}{k!} \sum_{i=0}^{k} \binom{\frac{1}{p-1}}{i} i!\, \mathbf{B}_{k,i}[f_\nu],$$

$$h(\eta) = \left(\frac{p(n+1)}{n+p}\right)^{\frac{1}{p-1}} \sum_{j=0}^{\infty} \frac{(\eta-a)^{j+1}}{j!} \sum_{i=0}^{j} \binom{\frac{1}{p-1}}{i} i!\, \mathbf{B}_{j,i}[\widetilde{f_\nu}].$$

Since $F'_\eta(a,a) = -h'(a) = -\left(\frac{p(n+1)}{n+p}\right)^{\frac{1}{p-1}}$, by applying the implicit function theorem and following the method used in the proof of Theorem 2 in [1] (see also the proof of Theorem 5 in [2]) we obtain the conclusion. □

4. A Cauchy version of Pawlikowska's mean value theorem and asymptotic behavior of its intermediate point

Pawlikowska's mean value theorem possesses a Cauchy version.

THEOREM 4. *Let $n \in \mathbb{N}$ and let $f,g : [a,b] \to \mathbb{R}$ be functions whose derivatives $f^{(n)}$ and $g^{(n)}$ exist both on $[a,b]$. If $g^{(n)}(a) \neq g^{(n)}(b)$, then there exists a point $\eta \in (a,b)$ such that*

$$f(a) - T_n(f;\eta)(a) = \frac{f^{(n)}(b) - f^{(n)}(a)}{g^{(n)}(b) - g^{(n)}(a)} \left[g(a) - T_n(g;\eta)(a)\right].$$

Proof. Let $\lambda := \frac{f^{(n)}(b)-f^{(n)}(a)}{g^{(n)}(b)-g^{(n)}(a)}$ and let $h : [a,b] \to \mathbb{R}$ be the function defined by $h(x) := f(x) - \lambda g(x)$. Since $h^{(n)}(a) = h^{(n)}(b)$, by virtue of a result established by I. Pawlikowska [8, Lemma 2.2] it follows that there exists a point $\eta \in (a,b)$ such that $h(a) - T_n(h;\eta)(a) = 0$, whence the conclusion. □

Now let I be an open interval in $\mathbb{R}$, let a be an arbitrary point in I, and let $n \in \mathbb{N}$. Further, let $f,g : I \to \mathbb{R}$ be functions whose derivatives $f^{(n)}$ and $g^{(n+1)}$ exist both on I. If, in addition, $g^{(n+1)}$ does not vanish in I, then for every $x \neq a$ in I one has $g^{(n)}(a) \neq g^{(n)}(x)$, whence

$$\left[g^{(n)}(x) - g^{(n)}(a)\right] \left[f(a) - T_n(f;\eta)(a)\right] = \left[f^{(n)}(x) - f^{(n)}(a)\right] \left[g(a) - T_n(g;\eta)(a)\right], \tag{14}$$

with intermediate point (or points) η strictly between a and x. The following theorem is similar to Theorem 1 and it provides the asymptotic behavior of η in (14) when the interval whose endpoints are a and x shrinks to zero.

THEOREM 5. *Under the above assumptions let* p *and* q *be positive integers and suppose that* f *and* g *fulfil the following conditions:*

(i) *the derivatives* $f^{(n+p+q)}$ *and* $g^{(n+p+q)}$ *exist on* I *and they are both continuous at* a,

(ii) $f^{(n+j)}(a)g^{(n+1)}(a) = f^{(n+1)}(a)g^{(n+j)}(a)$ *for* $2 \leqslant j < p$,

(iii) $f^{(n+p)}(a)g^{(n+1)}(a) \neq f^{(n+1)}(a)g^{(n+p)}(a)$.

Then the point η *in* (14) *satisfies*

$$\eta = \eta(x) = a + \left(\frac{n+p}{p(n+1)}\right)^{\frac{1}{p-1}} (x-a) + o(|x-a|) \qquad (x \to a). \tag{15}$$

Proof. Since $g^{(n+1)}(a) \neq 0$, by (ii) it follows that

$$f^{(n+k)}(a)g^{(n+j)}(a) = f^{(n+j)}(a)g^{(n+k)}(a) \qquad \text{for all } 1 \leqslant j,k < p. \tag{16}$$

Using (i), by virtue of the Lagrange–Taylor formula we have

$$f^{(n)}(x) - f^{(n)}(a) = \sum_{k=1}^{p+q-1} \frac{f^{(n+k)}(a)}{k!}(x-a)^k + O\left(|x-a|^{p+q}\right), \tag{17}$$

$$g^{(n)}(x) - g^{(n)}(a) = \sum_{k=1}^{p+q-1} \frac{g^{(n+k)}(a)}{k!}(x-a)^k + O\left(|x-a|^{p+q}\right), \tag{18}$$

as $x \to a$. On the other hand, we have

$$f(\eta) - f(a) = \sum_{j=1}^{n+p+q-1} \frac{f^{(j)}(a)}{j!}(\eta-a)^j + O\left(|\eta-a|^{n+p+q}\right)$$

and

$$f^{(k)}(\eta) = \sum_{j=0}^{n+p+q-k-1} \frac{f^{(k+j)}(a)}{j!}(\eta-a)^j + O\left(|\eta-a|^{n+p+q-k}\right)$$

as $x \to a$, for every $k \in \{1,\ldots,n\}$. From these equalities we deduce that

$$\begin{aligned} f(a) - T_n(f;\eta)(a) &= f(a) - f(\eta) - \sum_{k=1}^{n} \frac{(-1)^k}{k!} f^{(k)}(\eta)(\eta-a)^k \\ &= -\sum_{j=1}^{n+p+q-1} \frac{f^{(j)}(a)}{j!}(\eta-a)^j - \sum_{k=1}^{n}\sum_{j=0}^{n+p+q-k-1} \frac{(-1)^k f^{(k+j)}(a)}{k!\,j!}(\eta-a)^{k+j} \\ &\qquad + O\left(|\eta-a|^{n+p+q}\right) \\ &= -\sum_{j=1}^{n+p+q-1} \frac{f^{(j)}(a)}{j!}(\eta-a)^j - \sum_{r=1}^{n+p+q-1} S_r \frac{f^{(r)}(a)}{r!}(\eta-a)^r + O\left(|\eta-a|^{n+p+q}\right) \end{aligned}$$

as $x \to a$, with $S_r := \sum (-1)^k \frac{(k+j)!}{k!\,j!}$, where the summation extends for all integers k and j satisfying

$$1 \leqslant k \leqslant n, \qquad 1 \leqslant j \leqslant n+p+q-k-1 \qquad \text{and} \qquad j+k=r.$$

It is easily seen that

$$S_r = \sum_{k=1}^{\min\{n,r\}} (-1)^k \binom{r}{k} = S_{r,\min\{n,r\}} - 1$$

(see the notation used in the proof of Theorem 2). Therefore

$$S_r = (-1)^{\min\{n,r\}} \binom{r-1}{\min\{n,r\}} - 1 = \begin{cases} -1 & \text{if} \quad r \leqslant n, \\ (-1)^n \binom{r-1}{n} - 1 & \text{if} \quad r \geqslant n+1. \end{cases}$$

Thus, we conclude that

$$\begin{aligned} & f(a) - T_n(f;\eta)(a) \\ &\quad = \sum_{j=1}^{p+q-1} \frac{(-1)^{n+1} \binom{n+j-1}{n} f^{(n+j)}(a)}{(n+j)!} (\eta - a)^{n+j} + O\left(|\eta - a|^{n+p+q}\right) \end{aligned} \tag{19}$$

as $x \to a$. Analogously, one has

$$\begin{aligned} & g(a) - T_n(g;\eta)(a) \\ &\quad = \sum_{j=1}^{p+q-1} \frac{(-1)^{n+1} \binom{n+j-1}{n} g^{(n+j)}(a)}{(n+j)!} (\eta - a)^{n+j} + O\left(|\eta - a|^{n+p+q}\right) \end{aligned} \tag{20}$$

as $x \to a$. Substituting (17), (18), (19) and (20) into (14) and taking into account (16) as well as $|\eta - a| < |x - a|$, we deduce that

$$\begin{aligned} & \frac{(x-a)^{n+p}(\eta-a)^{n+1}}{p!\,(n+1)!} \left[f^{(n+p)}(a) g^{(n+1)}(a) - f^{(n+1)}(a) g^{(n+p)}(a) \right] \\ &\quad = \frac{(x-a)^{n+1}(\eta-a)^{n+p}}{(n+p)(p-1)!\,n!} \left[f^{(n+p)}(a) g^{(n+1)}(a) - f^{(n+1)}(a) g^{(n+p)}(a) \right] \\ &\qquad + o\left(|x-a|^{n+1} |\eta - a|^{n+p}\right) \qquad (x \to a). \end{aligned}$$

Dividing both sides by

$$\frac{f^{(n+p)}(a) g^{(n+1)}(a) - f^{(n+1)}(a) g^{(n+p)}(a)}{p!\,(n+1)!} (x-a)^{n+1} (\eta - a)^{n+p},$$

we get

$$\left(\frac{x-a}{\eta - a} \right)^{p-1} = \frac{p(n+1)}{n+p} + o(1) \qquad (x \to a),$$

hence (15) holds. $\square$

REFERENCES

[1] U. ABEL, *On the Lagrange remainder of the Taylor formula*, Amer. Math. Monthly, **110** (2003), 627–633.
[2] U. ABEL AND M. IVAN, *The differential mean value of divided differences*, J. Math. Anal. Appl. **325** (2007), 560–570.
[3] U. ABEL, M. IVAN AND T. RIEDEL, *The mean value theorem of Flett and divided differences*, J. Math. Anal. Appl. **295** (2004), 1–9.
[4] A. G. AZPEITIA, *On the Lagrange remainder of the Taylor formula*, Amer. Math. Monthly **89** (1982), 311–312.
[5] L. COMTET, *Advanced Combinatorics*, Reidel, Dordrecht, 1974.
[6] D. I. DUCA AND O. POP, *On the intermediate point in Cauchy's mean-value theorem*, Math. Inequal. Appl. **9** (2006), 375–389.
[7] T. M. FLETT, *A mean value theorem*, Math. Gazette **42** (1958), 38–39.
[8] I. PAWLIKOWSKA, *An extension of a theorem of Flett*, Demonstratio Math. **32** (1999), 281–286.
[9] R. C. POWERS, T. RIEDEL AND P. K. SAHOO, *Limit properties of differential mean values*, J. Math. Anal. Appl. **227** (1998), 216–226.
[10] P. K. SAHOO AND T. RIEDEL, *Mean Value Theorems and Functional Equations*, World Scientific, River Edge, NJ, 1998.

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