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ON WEIGHTED ČEBYŠEV–GRÜSS TYPE INEQUALITIES ON TIME SCALES

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Abstract. In this study, we establish weighted Čebyšev-Grüss type inequalities on time scales.

1. Introduction

In 1935, G. Grüss [8] proved the following inequality:

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx-\frac{1}{b-a}\int_a^b f(x)dx\frac{1}{b-a}\int_a^b g(x)dx\right| \leqslant \frac{1}{4}(\Phi-\varphi)(\Gamma-\gamma), \tag{1.1}$$

provided that f and g are two integrable functions on $[a,b]$ satisfying the condition

$$\varphi \leqslant f(x) \leqslant \Phi \text{ and } \gamma \leqslant g(x) \leqslant \Gamma \text{ for all } x \in [a,b]. \tag{1.2}$$

The constant $\frac{1}{4}$ is best possible.

In 1882, P. L. Čebyšev [6] gave the following inequality:

$$|T(f,g)| \leqslant \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty , \tag{1.3}$$

where $f,g : [a,b] \to R$ are absolutely continuous functions, whose first derivatives f' and g' are bounded,

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)g(x)dx - \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right) \tag{1.4}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|p\|_\infty = \operatorname*{ess\,sup}_{t\in[a,b]} |p(t)|$.

The following result of Grüss type was proved by Dragomir and Fedotov [7]:

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THEOREM 1. *Let $f,u:[a,b]\to\mathbb{R}$ be such that u is L-Lipschitzian on $[a,b]$, i.e.,*

$$|u(x)-u(y)|\leqslant L|x-y| \text{ for all } x\in[a,b], \tag{1.5}$$

f is Riemann integrable on $[a,b]$ and there exist the real numbers m,M so that

$$m\leqslant f(x)\leqslant M \text{ for all } x\in[a,b]. \tag{1.6}$$

Then we have the inequality,

$$\left|\int_a^b f(x)du(x)-\frac{u(b)-u(a)}{b-a}\int_a^b f(x)dx\right|\leqslant\frac{1}{2}L(M-m)(b-a).$$

From [11], if $f:[a,b]\to R$ is differentiable on $[a,b]$ with the first derivative f' integrable on $[a,b]$, then Montgomery identity holds:

$$f(x)=\frac{1}{b-a}\int_a^b f(t)dt+\int_a^b P(x,t)f'(t)dt, \tag{1.7}$$

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t)=\begin{cases}\dfrac{t-a}{b-a}, & a\leqslant t\leqslant x\\ \dfrac{t-b}{b-a}, & x<t\leqslant b.\end{cases}$$

In [12], Pachpatte established new inequalities of the Čebyšev type by using Pečarić's extension of the Montgomery identity [13].

The purpose of this paper is to establish the well-known weighted Čebyšev-Grüss type inequalities on time scales. To do this, we introduce the time scales calculus.

2. Preliminaries From Time Scale Calculus

The theory of time scales springs from 1988 doctoral dissertation of Hilger [9] that resulted in his seminal paper [10] in 1990. These works aimed to unify and generalize various mathematical concepts from the theories of discrete and continuous dynamical systems. Afterwards, the body of knowledge concerning time scales advanced monograph [2].

Many other information concerning time scales and dynamic equations on time scales can be found in the books ([2], [3]). We refer to the recently appeared works ([1]–[4], [15]–[17]).

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set $\mathbb{R}$ of the real numbers. We assume that any time scale has the topology that it inherits from the standard topology on $\mathbb{R}$. Since a time scale may or may not be connected, we need the concept of jump operators.

DEFINITION 1. Let $t \in \mathbb{T}$, where $\mathbb{T}$ is a time scale. Then the two mappings

$$\rho, \sigma : \mathbb{T} \to \mathbb{T}$$

satisfying

$$\rho(t) = \sup\{s \in \mathbb{T} : \ s < t\} \quad \text{and} \quad \sigma(t) = \inf\{s \in \mathbb{T} : \ s > t\}$$

are called the backward and forward jump operators on $\mathbb{T}$, respectively.

These jump operators classify the points $\{t\}$ of a time scale $\mathbb{T}$ as right-dense, right-scattered, left-dense and left-scattered according to whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$ or $\rho(t) < t$, respectively, for $t \in \mathbb{T}$.

If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^k = \mathbb{T}$. The so-called graininess function is $\mu(t) := \sigma(t) - t$, $t \in \mathbb{T}$.

For $a, b \in \mathbb{T}$ with $a \leqslant b$ we define the interval $[a, b]$ in $\mathbb{T}$ by

$$[a, b] := \{t \in \mathbb{T} : \ a \leqslant t \leqslant b\}.$$

Open intervals and half-open intervals, etc. are defined accordingly.

DEFINITION 2. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we say that f has the delta derivative $f^{\Delta}(t) \in \mathbb{R}$ at t if for each $\varepsilon > 0$ there exists a neighborhood U of t such that provided it exists

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)[\sigma(t) - s] \right| \leqslant \varepsilon \left| \sigma(t) - s \right| \quad \text{for all } s \in U.$$

In this case, we say that f is delta differentiable at t.

For $\mathbb{T} = \mathbb{R}$, $f^{\Delta} = f'$, the usual derivative; for $\mathbb{T} = \mathbb{Z}$ the delta derivative is the forward difference operator, $f^{\Delta}(t) = f(t+1) - f(t)$. In the case $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k, k \in \mathbb{Z}\} \cup \{0\}$ so called q-time scales; with $q > 1$,

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}, \qquad f^{\Delta}(0) = \lim_{s \to 0} \frac{f(s) - f(0)}{s}.$$

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous or denoted by C_{rd} provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is rd-continuous if and only if f is continuous. It is known from Theorem 1.74 in [2] that if f is right-dense continuous, there is a functions F such that $F^{\Delta}(t) = f(t)$ and

$$\int_a^b f(t) \Delta t = F(b) - F(a).$$

Note that in the case $\mathbb{T} = \mathbb{R}$ we have

$$f^{\Delta}(t) = f'(t), \quad \int_a^b f(t) \Delta t = \int_a^b f(t) dt,$$

and in the case $\mathbb{T} = \mathbb{Z}$ we have

$$f^{\Delta}(t) = f(t+1) - f(t),$$

$$\int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t)$$

where $a, b \in \mathbb{T}$ with $a \leqslant b$.

THEOREM 2. ([2], Theorem 1.77) *If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{rd}$, then*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

(ii) $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

(iii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

(v) $\int_a^b \left[f(\sigma(t))g^{\Delta}(t)\right]\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t$;

(vi) $\int_a^b \left[f(t)g^{\Delta}(t)\right]\Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t$;

(vii) $\int_a^a f(t)\Delta t = 0$.

THEOREM 3. (Substitution, [2], Theorem 1.98) *Assume $\nu : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \to \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$,*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} \left(f \circ \nu^{-1}\right)(s)\widetilde{\Delta}s.$$

THEOREM 4. (Cauchy-Schwarz inequality, [2], Theorem 6.15) *Let $a, b \in \mathbb{T}$. For rd-continuous $f, g : [a, b] \to \mathbb{R}$, we have*

$$\int_a^b |f(x)g(x)|\Delta x \leqslant \sqrt{\left(\int_a^b |f(x)|^2 \Delta x\right)\left(\int_a^b |g(x)|^2 \Delta x\right)}.$$

Throughout this paper, we suppose that $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval means the intersection of real interval with the given time scale.

3. Main Results

THEOREM 5. *Let $f, g \in C_{rd}$ and $f, g : [a,b] \to \mathbb{R}$ are two Δ-integrable functions on $[a,b]$. Then for*

$$\varphi \leqslant f(x) \leqslant \Phi \text{ and } \gamma \leqslant g(x) \leqslant \Gamma \text{ for all } x \in [a,b], \tag{3.1}$$

we have

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)\Delta x - \frac{1}{b-a} \int_a^b f(x)\Delta x \frac{1}{b-a} \int_a^b g(x)\Delta x \right| \leqslant \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma). \tag{3.2}$$

Proof. This theorem can be seen by similar way Theorem 3.1 is proved in [1]. □

Let us also state that the weighted version of (3.2), that is, with condition (3.1) we have the following generalization (3.2):

$$|D(f,g;w)| \leqslant \frac{1}{4}(\Phi - \varphi)(\Gamma - \gamma), \tag{3.3}$$

where

$$D(f,g;w) = A(f,g;w) - A(f;w)A(g;w),$$

$$A(f;w) = \frac{1}{\int_a^b w(x)\Delta x} \int_a^b w(x)f(x)\Delta x \text{ and } A(f,g;w) = \frac{1}{\int_a^b w(x)\Delta x} \int_a^b w(x)f(x)g(x)\Delta x.$$

THEOREM 6. *Let f, $u, w \in C_{rd}$ and f, $u : [a,b] \to \mathbb{R}$ be such that f is Δ-integrable on $[a,b]$ and u is L-Lipschitzian on $[a,b]$, i.e. (1.5) holds true. If $w : [a,b] \to \mathbb{R}$ is a positive weight function, then*

$$|T(f,u;w)| \leqslant L \int_a^b w(x) \left| f(x) - A(f;w) \right| \Delta x, \tag{3.4}$$

where

$$T(f,u;w) = \int_a^b w(x)f(x)\Delta u(x) - \frac{1}{\int_a^b w(x)\Delta x} \int_a^b w(x)\Delta u(x) \int_a^b w(x)f(x)\Delta x. \tag{3.5}$$

Moreover, if there exist the real numbers m, M such that (1.6) is valid, then

$$|T(f,u;w)| \leqslant L(M-m) \int_a^b w(x)\Delta x. \tag{3.6}$$

Proof. As in [7], we have

$$\begin{aligned} |T(f,u;w)| &= \left| \int_a^b w(x)\left[f(x) - A(f;w)\right]\Delta u(x) \right| \\ &\leqslant \frac{L}{2}\int_a^b w(x)\left|f(x) - A(f;w)\right|\Delta x. \end{aligned}$$

That is, (3.4) is valid. Furthermore, from an application of Cauchy-Schwarz inequality, we get

$$|T(f,u;w)| \leqslant L\left(\int_a^b w(x)\Delta x\right)^{\frac{1}{2}}\left(\int_a^b w(x)\left(f(x) - A(f;w)\right)^2\Delta x\right)^{\frac{1}{2}}. \tag{3.7}$$

Thus, we obtain

$$|T(f,u;w)| \leqslant L\left(D(f,f;w)\right)^{\frac{1}{2}}\int_a^b w(x)\Delta x.$$

From (3.3) for $g \equiv f$ we get:

$$\left(D(f,f;w)\right)^{\frac{1}{2}} \leqslant \frac{1}{2}(\Phi - \varphi). \tag{3.8}$$

Now, (3.7) and (3.8) give (3.6). □

THEOREM 7. *Let $f,u,w \in C_{rd}$ and f, $u : [a,b] \to \mathbb{R}$ be M-Lipschitzian and L-Lipschitzian on $[a,b]$, respectively. If $w : [a,b] \to \mathbb{R}$ is a positive weight function, then*

$$|T(f,u;w)| \leqslant \frac{LM}{\int_a^b w(y)\Delta y}\int_a^b\int_a^b w(x)w(y)\left|x - y\right|\Delta x\Delta y. \tag{3.9}$$

Proof. From (3.4), we obtain the following result:

$$\begin{aligned} |T(f,u;w)| &\leqslant L\int_a^b w(x)\left|\frac{\int_a^b w(y)\left[f(x) - f(y)\right]\Delta y}{\int_a^b w(y)\Delta y}\right|\Delta x \\ &\leqslant \frac{L}{\int_a^b w(y)\Delta y}\int_a^b w(x)\int_a^b w(y)\left|f(x) - f(y)\right|\Delta y\Delta x \\ &\leqslant \frac{LM}{\int_a^b w(y)\Delta y}\int_a^b\int_a^b w(x)w(y)\left|x - y\right|\Delta x\Delta y. \end{aligned}$$

□

We assume that weight function $w:[a,b]\to[0,\infty)$ is some probability density function, i.e. $\int_a^b w(x)\Delta x=1,$ and set

$$W(t)=\begin{cases}\int_a^t w(s)\Delta s, & a\leqslant t\leqslant b\\ 0, & t<a \text{ and } t>b.\end{cases}$$

Let $\psi:[0,1]\to\mathbb{R}$ be a Δ-differentiable function on $[0,1]$, with $\psi(0)=0$, $\psi(1)\neq 0$ and ψ^{Δ} be a Δ-integrable on $[0,1]$. To simplify the notation, we set

$$\begin{aligned} T(f,g,\psi^{\Delta}) &= \int_a^b\left[\psi\left(\int_a^x w(t)\Delta t\right)\right]^{\Delta} f^{\sigma}(x)g^{\sigma}(x)\Delta x\\ &\quad-\frac{1}{\psi(1)}\left(\int_a^b\left[\psi\left(\int_a^x w(t)\Delta t\right)\right]^{\Delta} f^{\sigma}(x)\Delta x\right)\\ &\quad\times\left(\int_a^b\left[\psi\left(\int_a^x w(t)\Delta t\right)\right]^{\Delta} g^{\sigma}(x)\Delta x\right). \end{aligned}\tag{3.10}$$

THEOREM 8. *Let $f:[a,b]\to\mathbb{R}$ be Δ-differentiable and f^{Δ} be a Δ-integrable on $[a,b]$, then*

$$f^{\sigma}(x)=\frac{1}{\psi(1)}\int_a^b\left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t+\frac{1}{\psi(1)}\int_a^b P_{w,\psi}(x,t)f^{\Delta}(\sigma(t))\Delta t,\tag{3.11}$$

where $P_{w,\psi}$ is a generalization of the weighted Peano kernel defined by:

$$P_{w,\psi}(x,t)=\begin{cases}\psi\left(W(t)\right), & a\leqslant t<x\\ \psi\left(W(t)\right)-\psi\left(1\right), & x\leqslant t\leqslant b.\end{cases}\tag{3.12}$$

Proof. Using $P_{w,\psi}(x,t)$ kernel and applying partial integration method, we have

$$\begin{aligned}\int_a^b P_{w,\psi}(x,t)f^{\Delta}(t)\Delta t &= \int_a^x \psi\left(W(t)\right)f^{\Delta}(t)\Delta t+\int_x^b(\psi\left(W(t)\right)-\psi\left(1\right))f^{\Delta}(\sigma(t))\Delta t\\ &=\int_a^b \psi\left(W(t)\right)f^{\Delta}(\sigma(t))\Delta t-\psi\left(1\right)\int_x^b f^{\Delta}(\sigma(t))\Delta t\\ &=\psi\left(W(t)\right)f^{\sigma}(t)\Big|_a^b-\int_a^b\left[\psi\left(W(t)\right)\right]^{\Delta}f^{\sigma}(t)\Delta t\\ &\quad-\psi\left(1\right)\left[f^{\sigma}(b)-f^{\sigma}(x)\right]\end{aligned}$$

$$= \psi(1) f^{\sigma}(x) - \int_a^b [\psi(W(t))]^{\Delta} f^{\sigma}(t)\Delta t.$$

Multiplying both sides by $\frac{1}{\psi(1)}$, we obtain (3.11). This completes the proof. □

THEOREM 9. *Let $f, g : [a,b] \to \mathbb{R}$ be Δ-differentiable on $[a,b]$ and f^{Δ}, g^{Δ} be Δ-integrable on $[a,b]$, then*

$$\left|T(f,g,\psi^{\Delta})\right| \leqslant \frac{1}{\psi^2(1)} \left\|f^{\Delta}\right\|_{L_{\Delta}^{\infty}} \left\|g^{\Delta}\right\|_{L_{\Delta}^{\infty}} \int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta} H^2(x)\Delta x,$$

where $H(x) = \int_a^b \left|P_{w,\psi}(x,t)\right| \Delta t$ and $\|.\|_{L_{\Delta}^{\infty}}$ denotes the norm in $L_{\Delta}^{\infty}([a,b])$ defined as $\|q\|_{L_{\Delta}^{\infty}} = ess \sup_{t\in[a,b]} |q(t)|$.

Proof. Since the functions f and g satisfy the hypothesis of Theorem 8, the following identities hold:

$$f^{\sigma}(x) = \frac{1}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t + \frac{1}{\psi(1)} \int_a^b P_{w,\psi}(x,t) f^{\Delta}(\sigma(t))\Delta t, \tag{3.13}$$

and

$$g^{\sigma}(x) = \frac{1}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t + \frac{1}{\psi(1)} \int_a^b P_{w,\psi}(x,t) g^{\Delta}(\sigma(t))\Delta t. \tag{3.14}$$

Using (3.13) and (3.14), we have

$$\begin{aligned} &\left[f^{\sigma}(x) - \frac{1}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t\right] \\ &\qquad \times \left[g^{\sigma}(x) - \frac{1}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t\right] \\ &\quad = \frac{1}{\psi^2(1)} \left[\int_a^b P_{w,\psi}(x,t) f^{\Delta}(\sigma(t))\Delta t\right] \left[\int_a^b P_{w,\psi}(x,t) g^{\Delta}(\sigma(t))\Delta t\right]. \end{aligned}$$

Consequently, we write

$$
\begin{aligned}
& f^{\sigma}(x)g^{\sigma}(x) - \frac{f^{\sigma}(x)}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t \\
& \quad - \frac{g^{\sigma}(x)}{\psi(1)} \int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t \\
& \quad + \frac{1}{\psi^2(1)} \left[\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t\right] \left[\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t\right] \\
& = \frac{1}{\psi^2(1)} \left[\int_a^b P_{w,\psi}(x,t) f^{\Delta}(\sigma(t))\Delta t\right] \left[\int_a^b P_{w,\psi}(x,t) g^{\Delta}(\sigma(t))\Delta t\right].
\end{aligned}
\tag{3.15}
$$

Multiplying both sides of (3.15) by $\left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta}$ and then integrating the resulting identity with respect to x from a to b, we have

$$
\begin{aligned}
& \int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(x)g^{\sigma}(x)\Delta x \\
& \quad - \frac{1}{\psi(1)} \left(\int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(x)\Delta x\right) \left(\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t\right) \\
& \quad - \frac{1}{\psi(1)} \left(\int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(x)\Delta x\right) \left(\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t\right) \\
& \quad + \frac{1}{\psi^2(1)} \left(\int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^{\Delta} \Delta x\right) \left[\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} f^{\sigma}(t)\Delta t\right] \\
& \quad \times \left[\int_a^b \left[\psi\left(\int_a^t w(s)\Delta s\right)\right]^{\Delta} g^{\sigma}(t)\Delta t\right] \\
& = \frac{1}{\psi^2(1)} \left[\int_a^b P_{w,\psi}(x,t) f^{\Delta}(\sigma(t))\Delta t\right] \left[\int_a^b P_{w,\psi}(x,t) g^{\Delta}(\sigma(t))\Delta t\right].
\end{aligned}
$$

From (3.10) with $\psi\left(\int_a^b w(s)\Delta s\right) = \psi(1)$, we get

$$T(f,g,\psi^\Delta) = \frac{1}{\psi^2(1)} \int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^\Delta \left[\int_a^b P_{w,\psi}(x,t) f^\Delta(\sigma(t))\Delta t\right] \times \left[\int_a^b P_{w,\psi}(x,t) g^\Delta(\sigma(t))\Delta t\right] \Delta x$$

Thus, we obtain

$$\left|T(f,g,\psi^\Delta)\right| \leqslant \frac{1}{\psi^2(1)} \left\|f^\Delta\right\|_{L_\Delta^\infty} \left\|g^\Delta\right\|_{L_\Delta^\infty} \int_a^b \left[\psi\left(\int_a^x w(s)\Delta s\right)\right]^\Delta H^2(x)\Delta x.$$

□

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