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s–CONVEX EXTREMAL DISTRIBUTIONS WITH ARBITRARY DISCRETE SUPPORT

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Abstract. This paper considers the class of s-convex stochastic orderings for random variables valued in an arbitrary discrete subset of the half-positive real line. After having established a sufficient condition of crossing-type for these orderings, explicit expressions are derived for stochastic extrema in moment spaces. Some applications in actuarial science are discussed.

1. Introduction

Univariate stochastic orderings are partial orders defined on sets of distribution functions. Stochastic orderings allow for many interesting applications in probability; see, e.g., the books by SHAKED & SHANTHIKUMAR (1994) and DENUIT, DHAENE, GOOVAERTS & KAAS (2005) for overviews. For instance, stochastic orders can be used to compare complex models with more tractable ones which are "riskier", leading thus to more conservative decisions.

Stochastic orderings are defined on sets of distribution functions. In this paper, we consider classes of random variables to favor the intuitive contents of the results (the reader has to keep in mind that we do not compare the particular versions of the random variables but their respective distributions). Furthermore, all the random variables will be assumed to have a support bounded from below. Without loss of generality, these random variables will be assumed to take non-negative values.

Consider two random variables X and Y valued in a subset $\mathscr{S}$ of the half-positive real line $\mathbb{R}^+$. Many stochastic orderings, denoted here by $\preceq_*^{\mathscr{S}}$, can be defined by reference to some cone $\mathscr{U}_*^{\mathscr{S}}$ of measurable functions $f : \mathscr{S} \to \mathbb{R}$ by

$$X \preceq_*^{\mathscr{S}} Y \Leftrightarrow \mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)] \text{ for all } f \in \mathscr{U}_*^{\mathscr{S}},$$

provided that the expectations exist. Orderings defined in this way are generally referred to as *integral stochastic orderings*; see, e.g., DENUIT, DHAENE, GOOVAERTS & KAAS (2005).

In the notation "$\preceq_*^{\mathscr{S}}$", we made explicit the dependence of the integral stochastic ordering on the support $\mathscr{S}$ of the random variables X and Y to be compared. This dependence, usually ignored in the literature, can be fundamental as pointed out further.

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In fact, the structure of $\mathscr{S}$ may be exploited to propose more efficient orderings than those obtained by considering all the random variables as valued in $\mathbb{R}^+$.

Considering random variables valued in $\mathbb{R}^+$, and taking for $\mathscr{U}_*^{\mathbb{R}^+}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$ with non-negative first derivative $f^{1/}$ yields the well-known stochastic dominance $\preceq_{\mathrm{ST}}$. Taking for $\mathscr{U}_*^{\mathbb{R}^+}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$ with non-negative second derivative $f^{2/}$ yields the convex order $\preceq_{\mathrm{CX}}$. To generalize the orderings $\preceq_{\mathrm{ST}}$ and $\preceq_{\mathrm{CX}}$, DENUIT, LEFÈVRE & SHAKED (1998) introduced broad classes of univariate orderings, named the s-convex orders. These rely on the notion of convex functions with increasing degrees, such as introduced by POPOVICIU (1933). The convex functions of degree s (s being a positive integer) are well-known in interpolation theory where they are often called convex functions with respect to the Tchebycheff system $\{1, x, x^2, \ldots, x^{s-1}\}$ (see, e.g., KARLIN & NOVIKOFF (1963) as well as KARLIN & STUDDEN (1966)). It can be shown that the convex functions of degree 1 are the non-decreasing functions while the convex functions of degree 2 are the usual convex functions. Generally speaking, the convexity of degree s is usually characterized through sign properties of the divided difference operator. Considering random variables valued in $\mathbb{R}^+$, and taking for $\mathscr{U}_*^{\mathbb{R}^+}$ the class of the functions $f : \mathbb{R}^+ \to \mathbb{R}$ with non-negative sth derivative ($f^{s/} \geqslant 0$), yields the (continuous) s-convex order $\preceq_{s-cx}^{\mathbb{R}^+}$. As announced, the orderings with $s = 1$ and $s = 2$ reduce to $\preceq_{\mathrm{ST}}$ and $\preceq_{\mathrm{CX}}$, respectively.

In many situations, the stochastic orderings used are constructed for comparing *real* random variables. Classes of stochastic ordering specific to *discrete* random variables have received much less attention. FISHBURN & LAVALLE (1995) and LEFÈVRE & UTEV (1996) have introduced, independently, some classes of such stochastic orderings, in the context of economics and biology respectively. The s-orderings among discrete random variables valued in $\mathscr{D}_n = \{0, 1, \ldots, n\}$ are defined by means of forward differences. Recall that the forward difference operator Δ is defined for each function $f : \mathscr{D}_n \to \mathbb{R}$ by $\Delta f(i) = f(i+1) - f(i)$ for all $i = 0, \ldots, n-1$. The k-th forward difference operator Δ^k, $k = 1, 2, \ldots$, is defined recursively by $\Delta^k f(i) = \Delta^{k-1} f(i+1) - \Delta^{k-1} f(i)$ for all $i = 0, \ldots, n-k$ (by convention, $\Delta^0 f \equiv f$ and $\Delta^1 f \equiv \Delta f$). Taking for $\mathscr{U}_*^{\mathscr{D}_n}$ the class of the functions $f : \mathscr{D}_n \to \mathbb{R}$ such that $\Delta^s f(i) \geqslant 0$ for all $i = 0, \ldots, n-s$ yields the $\preceq_{s-cx}^{\mathscr{D}_n}$ order. The $\preceq_{s-cx}^{\mathscr{D}_n}$ orders have been studied in details in DENUIT, LEFÈVRE & MESFIOUI (1999) and COURTOIS, DENUIT & VAN BELLEGEM (2006).

Now, a more general situation is when the random variables take on values in an arbitrary (rather than equidistant) ordered finite grid of non-negative points, denoted by $\mathscr{E}_n = \{e_0, \ldots, e_n\}$ say. By convention, $e_0 < e_1 < \ldots < e_n$. Stochastic orderings specific for comparing such random variables have been proposed by DENUIT, LEFÈVRE & UTEV (1999). The s-convex orders on an arbitrary grid are of direct interest in various fields of applications, especially for problems of risky decision making, portfolio selection, insurance premium evaluation and of option pricing. We will come back to applications in Section 5. To motivate the results contained in this paper, and to illustrate the theoretical results derived in the next sections, let us consider the discrete claim severity distribution given in WALHIN & PARIS (1998) displayed in Table 1. This is the distribution of the amounts of claim made to an insurance company. The first three moments are $\mu_1 = 31.5$, $\mu_2 = 1401.8$ and $\mu_3 = 71879.1$. In such a case, we can use the continuous s-convex orders if we consider that the claim size is valued in $[0,67]$. We can also use the arithmetic s-convex orders considering the claim size as valued in $\mathscr{D}_{67} = \{0,1,\ldots,67\}$. As it will be argued below, it is more efficient to account for the particular form $\mathscr{E}_9 = \{0,7,12,17,23,28,39,46,53,67\}$ of the support.

Support points	0	7	12	17	21	23	28	39	46	53	67
Probability masses	0.05	0.1	0.15	0.05	0.05	0.05	0.1	0.1	0.1	0.15	0.1

Table 1. Claim Severity Distribution.

To deal with arbitrary support, we need the general approach to the s-convex orders that uses the concept of divided differences defined as follows. Let $f:\mathscr{S}\to\mathbb{R}$ and $x_0,x_1,\ldots,x_s\in\mathscr{S}$. Starting from

$$[x_i]f = f(x_i), \quad i=0,\ldots,s,$$

the sth divided differences are defined recursively by

$$[x_0,\ldots,x_s]f = \frac{[x_1,\ldots,x_s]f - [x_0,\ldots,x_{s-1}]f}{x_s-x_0} = \sum_{i=0}^{s}\frac{f(x_i)}{\prod_{j=0;j\neq i}^{s}(x_i-x_j)}.$$

The order $\preceq_{s-cx}^{\mathscr{S}}$ can then be defined by taking for $\mathscr{C}$ the class of all the s-convex function $f:\mathscr{S}\to\mathbb{R}$, i.e. the functions $f:\mathscr{S}\to\mathbb{R}$ such that $[x_0,\ldots,x_s]f\geqslant 0$ for any $x_0,x_1,\ldots,x_s\in\mathscr{S}$. This general approach works whatever the form of the support $\mathscr{S}$ of the random variables to be compared. Note that this definition reduces to the one in terms of derivative and forward differences given above when $\mathscr{S}$ is $\mathbb{R}^+$ and $\mathscr{D}_n$, respectively. Specifically, for $f:\mathscr{D}_n\to\mathbb{R}$, it can be shown that

$$[i,i+1,\ldots,i+s]f = \frac{\Delta^s f(i)}{s!}$$

so that the s-convex functions on $\mathscr{D}_n$ are those with positive s-th forward differences. Similarly, if $f:\mathbb{R}^+\to\mathbb{R}$ possesses an sth derivative then for any $x_0\in\mathbb{R}^+$, $h_0,h_1,\ldots,h_k\geqslant 0$

$$[x_0,x_0+h_1,\ldots,x_0+h_k]f = \int_{\xi_1=0}^{1}\int_{\xi_2=0}^{\xi_1}\cdots\int_{\xi_k=0}^{\xi_{k-1}} f^{k/}(\xi_k(h_k-h_{k-1})+\ldots + \xi_2(h_2-h_1)+\xi_1 h_1+x_0)d\xi_k\ldots d\xi_2 d\xi_1.$$

Since the functions x^k and $-x^k$ are both s-convex for $k=1,\ldots,s-1$, whatever $\mathscr{S}$, we see that

$$X\preceq_{s-cx}^{\mathscr{S}} Y \Rightarrow \mathbb{E}[X^k]=\mathbb{E}[Y^k] \text{ for } k=1,\ldots,s-1. \tag{1}$$

The relation $\preceq_{s-cx}^{\mathscr{S}}$ can therefore only be used to compare random variables with the same first $s-1$ moments. The relation $\preceq_{s-cx}^{\mathscr{S}}$ is thus restricted to moment spaces.

At this stage, we could wonder whether there is anything to gain by considering the specific form of the support of the random variables to be compared (instead of viewing all of them valued in $\mathbb{R}^+$). For $s=1,2$ the form of the support of the random variables to be compared is not relevant, in the sense that they can all be seen as valued in $\mathbb{R}^+$: $\preceq_{1-cx}^{\mathscr{S}} \Leftrightarrow \preceq_{1-cx}^{\mathbb{R}^+} \Leftrightarrow \preceq_{\mathrm{ST}}$ and $\preceq_{2-cx}^{\mathscr{S}} \Leftrightarrow \preceq_{2-cx}^{\mathbb{R}^+} \Leftrightarrow \preceq_{\mathrm{CX}}$ for any $\mathscr{S} \subseteq \mathbb{R}^+$. The equivalences for $s = 1$ and $s = 2$ follows from the fact that it is always possible to continue any non-decreasing function or any convex function on $\mathscr{S}$ as a function with the same shape on (a larger subset of) $\mathbb{R}$, using a piecewise linear function for instance. For $s \geqslant 3$, however, this is no more necessarily true, and the structure of the support matters. For instance, having two random variables valued in $\mathscr{S}$, the implication

$$X \preceq_{s-cx}^{\mathscr{S}} Y \Rightarrow X \preceq_{s-cx}^{\mathscr{T}} Y$$

always holds true for $\mathscr{S} \subset \mathscr{T}$, but the reciprocal is false in general. Various counterexamples can be found in DENUIT, LEFÈVRE & UTEV (1999); see also FISHBURN & LAVALLE (1995). We thus get finer stochastic inequalities taking into account the particular form of the support. For example, in the context of decision analysis, if the decision-maker's preferences agree with some s-convex ordering, when comparing two alternatives, it is safer to consider them valued in a smaller set of outcomes rather than in a larger one (because any such comparison can be extended to a larger set but not reciprocally).

In this paper, we consider random variables valued in an arbitrary subset $\mathscr{E}_n = \{e_0, e_1, \ldots, e_n\}$ of the half positive real line. Having two random variables X and Y valued in $\mathscr{E}_n$, we have that $X \preceq_{s-cx}^{\mathscr{E}_n} Y \Rightarrow X \preceq_{s-cx}^{\mathbb{R}^+} Y$, and this implication is strict for $s \geqslant 3$. The order $\preceq_{s-cx}^{\mathscr{E}_n}$ studied in the present paper is thus stronger than the order $\preceq_{s-cx}^{\mathbb{R}^+}$ defined and studied in DENUIT, LEFÈVRE & SHAKED (1998). We first prove that the sufficient condition of crossing type established in DENUIT, LEFÈVRE & SHAKED (1998) for $\preceq_{s-cx}^{\mathbb{R}^+}$ is also sufficient for $\preceq_{s-cx}^{\mathscr{E}_n}$. This result is exploited in the second part of this paper to get the extrema with respect to $\preceq_{s-cx}^{\mathscr{E}_n}$. The paper is organized as follows. To begin with, we give in Section 2 a new characterization of $\preceq_{s-cx}^{\mathscr{E}_n}$. In Section 3, a sufficient condition for $\preceq_{s-cx}^{\mathscr{E}_n}$ is obtained in terms of the number of crossing points of the respective distribution functions. Section 4 is devoted to the construction of the extrema with respect to $\preceq_{s-cx}^{\mathscr{E}_n}$. Finally, Section 5 discusses some applications.

2. Characterizations of $\preceq_{s-cx}^{\mathscr{E}_n}$

Note that it suffices to check the sign of the divided differences of order s on consecutive points of $\mathscr{E}_n$ to prove that $f : \mathscr{E}_n \to \mathbb{R}$ is a s-convex function on $\mathscr{E}_n$. Indeed, for any $x_0, \ldots, x_s$ in $\mathscr{E}_n$, there exist coefficients $a_0, \ldots, a_{n-s}$ non-negative, of sum 1 and independent of f such that

$$[x_0, \ldots, x_s]f = \sum_{i=0}^{n-s} a_i [e_i, \ldots, e_{i+s}]f. \tag{2}$$

Hence, f is s-convex on $\mathscr{E}_n$ if $[e_i, \ldots, e_{i+s}]f \geqslant 0$ for $i = 0, \ldots, n-s$. This provides an efficient test for the s-convex property on $\mathscr{E}_n$.

A question of practical importance is how to check the validity of $\preceq_{s-cx}^{\mathscr{E}_n}$ for a given pair of random variables X and Y. It is indeed expensive to check whether

$\mathbb{E}[f(X)] \leqslant \mathbb{E}[f(Y)]$ holds for all the s-convex functions $f : \mathscr{E}_n \to \mathbb{R}$. Therefore, we would like to have a set of test functions (as small as possible). The following characterization has been established in DENUIT, LEFÈVRE & UTEV (1999). It provides a simple iterative procedure to check the validity of $X \preceq_{s-cx}^{\mathscr{E}_n} Y$. Specifically, $X \preceq_{s-cx}^{\mathscr{E}_n} Y$ holds if, and only if,

$$\sum_{i=k}^{n} \Big(\Pr[Y = e_i] - \Pr[X = e_i] \Big) (e_i - e_0) \dots (e_i - e_{k-1}) = 0 \text{ for } k = 0, \dots, s-1,$$

and

$$\sum_{i=k+s}^{n} \Big(\Pr[Y = e_i] - \Pr[X = e_i] \Big) (e_i - e_{k+1}) \dots (e_i - e_{k+s-1}) \geqslant 0 \text{ for } k = 0, \dots, n-s.$$

For any $x_0 < x_1 < \cdots < x_n$ fixed in $\mathscr{E}_n$, let us consider the functions

$$\widetilde{w}_{j;k} \equiv \widetilde{w}_{j;k}(x_j, \dots, x_{j+k}|\cdot) : \mathscr{E}_n \to \mathbb{R}, \quad \text{for } j = 0, 1, \dots, n-k,$$

which are defined by

$$\widetilde{w}_{j;k}(x_j, \dots, x_{j+k}|x) = \begin{cases} (x - x_j) \dots (x - x_{j+k}) & \text{if } x \geqslant x_{j+k+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, it can be checked that

$$\begin{aligned} [x_i, \dots, x_{i+k+1}]\widetilde{w}_{j;k} &= 0 \text{ if } i \leqslant j-1 \\ [x_i, \dots, x_{i+k+1}]\widetilde{w}_{j;k} &= 1 \text{ if } i \geqslant j \\ [x_i, \dots, x_{i+k+2}]\widetilde{w}_{j;k} &= 0 \text{ if } i \neq j-1 \\ [x_{j-1}, \dots, x_{j+k+1}]\widetilde{w}_{j;k} &> 0. \end{aligned}$$

Henceforth, the function $\widetilde{w}_{j;k}(e_j, \dots, e_{j+k}|\cdot) : \mathscr{E}_n \to \mathbb{R}$ ($j = 0, 1, \dots, n-k$) will be simply denoted as $\widetilde{w}_{j;k}(\cdot)$.

Let us now establish a useful characterization of $\preceq_{s-cx}^{\mathscr{E}_n}$ that extends Proposition 3.3. in DENUIT & LEFÈVRE (1997).

THEOREM 2.1. *X and Y being two random variables valued in $\mathscr{E}_n$, $X \preceq_{s-cx}^{\mathscr{E}_n} Y$ if, and only if, the two conditions below are satisfied:*

$$\mathbb{E}[\widetilde{w}_{0;k-1}(X)] = \mathbb{E}[\widetilde{w}_{0;k-1}(Y)] \textit{ for } k = 1, 2, \dots, s-1,$$

and

$$\mathbb{E}[\widetilde{w}_{k;s-2}(X)] \leqslant \mathbb{E}[\widetilde{w}_{k;s-2}(Y)] \textit{ for } k = 1, 2, \dots, n-s+1.$$

Proof. Let us first consider the "$\Leftarrow$"-part. Starting from the following expansion formula for a function $f : \mathscr{E}_n \to \mathbb{R}$:

$$\begin{aligned} f(e_k) &= f(e_0) + (e_k - e_0)_+[e_0, e_1]f + (e_k - e_0)_+(e_k - e_1)_+[e_0, e_1, e_2]f \\ &\quad + \cdots + (e_k - e_0)_+ \cdots (e_k - e_{s-2})_+[e_0, e_1, \dots, e_{s-1}]f \\ &\quad + \sum_{i=1}^{n-s+1} (e_k - e_i)_+ \cdots (e_k - e_{i+s-2})_+(e_{i+s-1} - e_{i-1})[e_{i-1}, e_i, \dots, e_{i+s-1}]f \end{aligned}$$

we get

$$\begin{aligned}\mathbb{E}[f(Y)] - \mathbb{E}[f(X)] &= \Big(\mathbb{E}[\widetilde{w}_{0;0}(Y)] - \mathbb{E}[\widetilde{w}_{0;0}(X)]\Big)[e_0, e_1]f \\ &+ \Big(\mathbb{E}[\widetilde{w}_{0;1}(Y)] - \mathbb{E}[\widetilde{w}_{0;1}(X)]\Big)[e_0, e_1, e_2]f \\ &+ \cdots + \Big(\mathbb{E}[\widetilde{w}_{0;s-2}(Y)] - \mathbb{E}[\widetilde{w}_{0;s-2}(X)]\Big)[e_0, e_1, \ldots, e_{s-1}]f \\ &+ \sum_{i=1}^{n-s+1} \Big(\mathbb{E}[\widetilde{w}_{i;s-2}(Y)] - \mathbb{E}[\widetilde{w}_{i;s-2}(X)]\Big)[e_{i-1}, e_i, \ldots, e_{i+s-1}]f.\end{aligned}$$

Therefore, $\mathbb{E}[f(Y)] - \mathbb{E}[f(X)]$ is nonnegative if f is s-convex.

The "$\Rightarrow$"-part is obvious since the functions $x \mapsto \pm\widetilde{w}_{0;k-1}(x)$ ($k = 1, 2, \ldots, s-1$) and $x \mapsto \widetilde{w}_{k;s-2}(x)$ ($k = 1, 2, \ldots, n-s+1$) are s-convex on $\mathscr{E}_n$. □

3. Sufficient conditions for $\preceq_{s-cx}^{\mathscr{E}_n}$

Let f be any real-valued function defined on a subset $\mathscr{S}$ of $\mathbb{R}^+$. The operator S^-, when applied to f, counts the number of sign changes of f over its domain $\mathscr{S}$. More precisely,

$$S^-(f) = \sup S^-\big(f(x_1), f(x_2), \ldots, f(x_n)\big),$$

where the supremum is extended over all $x_1 < x_2 < \ldots < x_n \in \mathscr{S}$, n is arbitrary but finite and $S^-\big(y_1, y_2, \ldots, y_n\big)$ denotes the number of sign changes of the indicated sequence $\{y_1, y_2, \ldots, y_n\}$, zero terms being discarded. The functions f_1 and f_2 are said to cross each other k times if $S^-(f_1 - f_2) = k$.

If X and Y are random variables valued in $\mathscr{E}_n$, we say that $F_X \geqslant F_Y$ near e_n if $F_X(e_k) \geqslant F_Y(e_k)$ for all $k \geqslant k_0$, with $k_0 \leqslant n-1$. We show in the next result that the crossing condition given by DENUIT, LEFÈVRE & SHAKED (1998) for $\preceq_{s-cx}^{\mathbb{R}^+}$ is also sufficient for $\preceq_{s-cx}^{\mathscr{E}_n}$.

PROPOSITION 3.1. *Let X and Y be two random variables valued in $\mathscr{E}_n$, such that $\mathbb{E}[X^k] = \mathbb{E}[Y^k]$ for $k = 1, \ldots, s-1$. Then $S^-(F_X - F_Y) \leqslant s-1$ together with $F_X \geqslant F_Y$ near e_n $\Rightarrow$ $X \preceq_{s-cx}^{\mathscr{E}_n} Y$.*

Proof. For $1 \leqslant i, j \leqslant n$, let

$$\Psi_j(i) = \mathbb{E}[\widetilde{w}_{i-j+1;j-2}(X)] - \mathbb{E}[\widetilde{w}_{i-j+1;j-2}(Y)].$$

From Theorem 2.1, we have to prove that $S^-(\Psi_s) = 0$ and $\Psi_s(k) \leqslant 0$ for all $k = 0, 1, \ldots, n$. By hypothesis, as $\Psi_1(k) = F_Y(e_{k-1}) - F_X(e_{k-1})$, we have $S^-(\Psi_1) \leqslant s-1$ and $\Psi_1(k) \leqslant 0$ for $k \geqslant k_0$ ($k_0 \leqslant n-1$). Let us now consider $s = 2$. If $S^-(\Psi_1) = 0$, the result is trivial; else Ψ_1 exhibits opposite signs on the consecutive intervals I_1 and I_2. Then, as $\Psi_2(k) = -\sum_{i=k}^{n}(e_i - e_{i-1})(F_X - F_Y)(e_i) = \sum_{i=k}^{n}(e_i - e_{i-1})\Psi_1(i)$, Ψ_2 is monotonic on each of these intervals and is negative on I_2. Consequently, Ψ_2 exhibits at most one sign change (on I_1). Moreover, a sign change on I_1 would imply that $\Psi_2(0) \neq 0$ which is not possible. So this yields $S^-(\Psi_2) = 0$ which suffices because Ψ_2 is negative on I_2.

Now, let us assume that the result holds for $s-1$ and let us establish it for $s \geqslant 3$. Since for a random variable X valued in $\mathscr{E}_n$,

$$\mathbb{E}[\widetilde{w}_{k-s+1;s-2}(X)] = \begin{cases} (e_n - e_{k-1}) - \sum_{i=k}^{n}(e_i - e_{i-1})F_X(e_{i-1}) & \text{for } s = 2, \\ \sum_{i=k}^{n}(e_i - e_{i-s+1})\mathbb{E}[\widetilde{w}_{i-s+2;s-3}(X)] & \text{for } s \geqslant 3. \end{cases}$$

we have the relation $\Psi_s(k) = \sum_{i=k}^{n}(e_i - e_{i-s+1})\Psi_{s-1}(i)$. Let us suppose that $S^-(\Psi_{s-1}) = i \geqslant 1$, Ψ_{s-1} exhibiting opposite signs on consecutive intervals $I_1, I_2, \ldots, I_{i+1}$. Then Ψ_s is monotonic on each of these intervals and is negative on I_{i+1}, so that Ψ_s exhibits at most i sign changes, one over each I_j, $j = 1, 2, \ldots, i$. Moreover, a sign change on I_1 would imply that $\Psi_s(0) \neq 0$ which is not possible, yielding $S^-(\Psi_s) \leqslant i-1$. Since $S^-(\Psi_{s-1}) = 0 \Rightarrow S^-(\Psi_s) = 0$, we get that $S^-(\Psi_s) \leqslant \max[0, S^-(\Psi_{s-1}) - 1]$. □

Remark that the sufficient conditions given by DENUIT & LEFÈVRE (1997) in their Section 4 (Lemma 4.2., Lemma 4.3. and Corollary 4.5.) for $\preceq_{s-cx}^{\mathscr{D}_n}$ are easily extendable for $\preceq_{s-cx}^{\mathscr{E}_n}$.

4. Extrema with respect to $\preceq_{s-cx}^{\mathscr{E}_n}$

As pointed out in (1), the s-convex orders can only be used to compare random variables sharing the same first $s-1$ moments. This means that these orders can only be used inside moment spaces.

Let us denote as $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ the set of all the (distribution functions of) random variables valued in $\mathscr{E}_n$ and with prescribed first $s-1$ moments $\mu_k = \mathbb{E}[X^k]$, $k = 1, \ldots, s-1$. This set is usually referred to as a moment space in the literature. Henceforth, the moment sequence $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ is supposed to be such that $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ is non void. The following condition for such a space to be non void can be found in KARLIN & STUDDEN (1966): $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ is non void if, and only if, the point $(\mu_1, \ldots, \mu_{s-1})$ is an element of the convex hull of $\{(\theta, \theta^2, \ldots, \theta^{s-1}) | \theta \in \mathscr{E}_n\}$. The conditions for $s = 2, 3, 4$ are

- $s = 2$: $e_0 \leqslant \mu_1 \leqslant e_n$;
- $s = 3$: $\mu_2 \leqslant (e_0 + e_n)\mu_1 - e_0 e_n$ and $\mu_2 \geqslant (e_i + e_{i+1})\mu_1 - e_i e_{i+1}$ ($i = 0, \ldots, n-1$);
- $s = 4$: $\mu_3 \leqslant e_i e_{i+1} e_n - (e_i e_{i+1} + e_i e_n + e_{i+1} e_n)\mu_1 + (e_i + e_{i+1} + e_n)\mu_2$ ($i = 0, \ldots, n-2$) and $\mu_3 \geqslant e_0 e_j e_{j+1} - (e_0 e_j + e_0 e_{j+1} + e_j e_{j+1})\mu_1 + (e_0 + e_j + e_{j+1})\mu_2$ ($j = 1, \ldots, n-1$).

The purpose of this section is to derive extremal random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$, i.e. elements of $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ such that the stochastic inequalities

$$X_{\min}^{(s)} \preceq_{s-cx}^{\mathscr{E}_n} X \preceq_{s-cx}^{\mathscr{E}_n} X_{\max}^{(s)} \text{ for all } X \in \mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}) \qquad (3)$$

hold true. Such extrema are very useful in numerous applications. As we will see in Section 5, these stochastic extrema will furnish useful numerical bounds on quantities that are otherwise hard to compute.

The theory of discrete Tchebycheff systems, described in KARLIN & STUDDEN (1966), may be used to solve this problem. Here, however, we will derive the extrema from the sufficient conditions that we obtain in Section 3.

Let us quote that the concept of extrema in some classes of random variables with respect to a given stochastic ordering is properly said not new (see, e.g., HARRIS (1959, 1962), ROLSKI (1976) and STOYAN (1983, Section 1.9)). It originated in the analytic problem of moments (see, e.g., KARLIN & SHAPLEY (1953) and KARLIN AND STUDDEN (1966)). In DENUIT, LEFÈVRE & SHAKED (1998), DENUIT & LEFÈVRE (1997), DENUIT, LEFÈVRE & MESFIOUI (1999) and COURTOIS, DENUIT & VAN BELLEGEM (2006) a similar problem is discussed within the classes $\mathcal{M}_s([a,b];\mu_1,\mu_2,\ldots,\mu_{s-1})$ or $\mathcal{M}_s(\mathcal{D}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$ of all the random variables valued in the interval $[a,b]$ or in the grid $\mathcal{D}_n=\{0,1,\ldots,n\}$ with prescribed first $s-1$ moments. The extrema built here are better than those obtained for $\mathcal{M}_s([e_0,e_n];\mu_1,\mu_2,\ldots,\mu_{s-1})$ (since the latter do not take into account the fact that the support of X is just $\mathcal{E}_n$ and not the whole interval $[e_0,e_n]$).

Before stating the next property, let us mention that the results given in Propositions 3.7. and 3.8. of DENUIT, LEFÈVRE & SHAKED (1998) for $\preceq_{s-cx}^{\mathbb{R}^+}$ can be extended to $\preceq_{s-cx}^{\mathcal{E}_n}$. We have the following result.

PROPERTY 4.1. *(i) The support of* $X_{\max}^{(s)}$ *is the set*

$$\left\{e_i\in\mathcal{E}_n|e_i^s=c_0+c_1\cdot e_i+c_2\cdot e_i^2+\cdots+c_{s-1}\cdot e_i^{s-1}\right\} \tag{4}$$

where the c_i *'s are real constants such that*

$$e_i^s\leqslant c_0+c_1\cdot e_i+c_2\cdot e_i^2+\cdots+c_{s-1}\cdot e_i^{s-1},\ \textit{for all}\ e_i\in\mathcal{E}_n.$$

(ii) The support of $X_{\min}^{(s)}$ *is the set*

$$\left\{e_i\in\mathcal{E}_n|e_i^s=c_0+c_1\cdot e_i+c_2\cdot e_i^2+\cdots+c_{s-1}\cdot e_i^{s-1}\right\} \tag{5}$$

where the c_i *'s are real constants such that*

$$e_i^s\geqslant c_0+c_1\cdot e_i+c_2\cdot e_i^2+\cdots+c_{s-1}\cdot e_i^{s-1},\ \textit{for all}\ e_i\in\mathcal{E}_n.$$

Proof. We only prove (i); the proof for (ii) is similar. Let us establish the "$\Leftarrow$"-part of (i). Let X be a random variable in $\mathcal{M}_s(\mathcal{E}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$, i.e. such that

$$\sum_{i=0}^{n}\Pr[X=i]e_i^k=\mu_k\qquad,k=0,1,\ldots,s-1.$$

Assume further that the support of X is (4). Let Z be any random variable in $\mathcal{M}_s(\mathcal{E}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$. We then have

$$\begin{aligned}\mathbb{E}[X^s] &= \sum_{i=0}^{n}\Pr[X=i]e_i^s=\sum_{i=0}^{n}\Pr[X=i]\sum_{k=0}^{s-1}c_ke_i^k=\sum_{k=0}^{s-1}c_k\sum_{i=0}^{n}\Pr[X=i]e_i^k\\ &= \sum_{k=0}^{s-1}c_k\mu_k=\sum_{k=0}^{s-1}c_k\sum_{i=0}^{n}\Pr[Z=i]e_i^k=\sum_{i=0}^{n}\Pr[Z=i]\sum_{k=0}^{s-1}c_ke_i^k\\ &\geqslant \sum_{i=0}^{n}\Pr[Z=i]e_i^s=\mathbb{E}[Z^s].\end{aligned}$$

Hence, X achieves the maximum of $\mathbb{E}[Z^s]$ over $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$. Since the function x^s is s-convex on $\mathscr{E}_n$, X must be the s-convex maximum in $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$.

Let us now prove the "$\Rightarrow$"-part of (i). It is obvious that $\mathbb{E}[(X_{\max}^{(s)})^s] \geqslant \mathbb{E}[Z^s]$ for all Z in $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$. Now, if the support of $X_{\max}^{(s)}$ is not of the form (4) then for any random variable Z in $\mathscr{M}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ with such a support, we have $\mathbb{E}[(X_{\max}^{(s)})^s] \leqslant \mathbb{E}[Z^s]$, which ends the proof □

In the next section, we use the stochastic extrema with up to three moments known (i.e. for s up to 4). This is why we derive hereafter analytic expressions for the distribution of $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ for $s = 1, 2, 3, 4$.

The extrema with respect to $\preceq_{1-cx}^{\mathscr{E}_n}$ in $\mathscr{M}_1(\mathscr{E}_n)$ are easily derived. Note that $\mathscr{M}_1(\mathscr{E}_n)$ does not assign any condition on the moments. Therefore, it is obvious that with respect to $\preceq_{1-cx}^{\mathscr{E}_n}$, $X_{\min}^{(1)} = e_0$ and $X_{\max}^{(1)} = e_n$ almost surely. For the claim distribution in Table 1, we thus have $X_{\min}^{(1)} = 0$ and $X_{\max}^{(1)} = 67$ almost surely. Needless to say that these extrema only yield trivial bounds on the quantities of interest.

Let us now derive the extrema with respect to $\preceq_{2-cx}^{\mathscr{E}_n}$ in $\mathscr{M}_2(\mathscr{E}_n; \mu_1)$. It is natural to expect that the maximum with respect to $\preceq_{2-cx}^{\mathscr{E}_n}$ will concentrate all the probability mass on the extreme points e_0 and e_n, whereas the the minimum will concentrate all the probability mass around the mean. This will indeed be the case, as demonstrated in the next result.

PROPOSITION 4.2. *For fixed μ_1, let $\xi \in \{0, 1, \ldots, n-1\}$ be the integer such that $e_\xi < \mu_1 \leqslant e_{\xi+1}$. Then, with respect to $\preceq_{2-cx}^{\mathscr{E}_n}$,*

$$X_{\min}^{(2)} = \begin{cases} e_\xi & \text{with probability} \quad r_1 = \frac{e_{\xi+1} - \mu_1}{e_{\xi+1} - e_\xi}, \\ e_{\xi+1} & \text{with probability} \quad r_2 = \frac{\mu_1 - e_\xi}{e_{\xi+1} - e_\xi}, \end{cases} \tag{6}$$

and

$$X_{\max}^{(2)} = \begin{cases} e_0 & \text{with probability} \quad t_1 = \frac{e_n - \mu_1}{e_n - e_0}, \\ e_n & \text{with probability} \quad t_2 = \frac{\mu_1 - e_0}{e_n - e_0}. \end{cases} \tag{7}$$

Proof. The numbers r_1 and r_2 in (6) are probabilities since $r_1 + r_2 = 1$ and $r_1, r_2 \geqslant 0$ by definition of e_ξ. Moreover, $\mathbb{E}[X_{\min}^{(2)}] = \mu_1$ and it is directly seen that the distribution function of $X_{\min}^{(2)}$ and any $X \in \mathscr{M}_2(\mathscr{E}_n; \mu_1)$ intersect at most once. Thus, applying Proposition 3.1 yields the result. The analysis for $X_{\max}^{(2)}$ is similar. □

Considering the claim distribution in Table 1, we thus have

$$X_{\min}^{(2)} = \begin{cases} 28 & \text{with probability } 0.6818, \\ 39 & \text{with probability } 0.3182, \end{cases}$$

and

$$X_{\max}^{(2)} = \begin{cases} 0 & \text{with probability } 0.5299, \\ 67 & \text{with probability } 0.4701. \end{cases}$$

It is interesting to compare these results with the extrema in $\mathscr{M}_2(\mathscr{D}_{67};\mu_1)$ and in $\mathscr{M}_2([0,67];\mu_1)$. The maximum is the same in all the cases, but the minima are

$$Y_{\min}^{(2)} = \begin{cases} 31 & \text{with probability } 0.5, \\ 32 & \text{with probability } 0.5, \end{cases}$$

in $\mathscr{M}_2(\mathscr{D}_{67};\mu_1)$ and $Z_{\min}^{(2)} = 31.5$ almost surely in $\mathscr{M}_2([0,67];\mu_1)$. Clearly, $Z_{\min}^{(2)} \preceq_{2-cx}^{[0,67]} Y_{\min}^{(2)} \preceq_{2-cx}^{\mathscr{D}_{67}} X_{\min}^{(2)}$ so that taking the particular structure of the support into account improves the lower bound in the 2-convex sense.

The extrema with respect to $\preceq_{3-cx}^{\mathscr{E}_n}$ in $\mathscr{M}_2(\mathscr{E}_n;\mu_1,\mu_2)$ are given in the next result.

PROPOSITION 4.3. *For fixed* μ_1 *and* μ_2, *let* $\xi_1 \in \{1,2,\dots,n-1\}$ *and* $\xi_2 \in \{0,1,\dots,n-2\}$ *such that* $e_{\xi_1} < \frac{\mu_2-\mu_1 e_0}{\mu_1-e_0} \leqslant e_{\xi_1+1}$ *and* $e_{\xi_2} < \frac{\mu_1 e_n-\mu_2}{e_n-\mu_1} \leqslant e_{\xi_2+1}$.

$$X_{\min}^{(3)} = \begin{cases} e_0 & \textit{with probability} \quad p_1 = 1-p_2-p_3, \\ e_{\xi_1} & \textit{with probability} \quad p_2 = \frac{-\mu_2+\mu_1(e_0+e_{\xi_1+1})-e_0 e_{\xi_1+1}}{(e_{\xi_1}-e_0)(e_{\xi_1+1}-e_{\xi_1})}, \\ e_{\xi_1+1} & \textit{with probability} \quad p_3 = \frac{\mu_2-\mu_1(e_0+e_{\xi_1})+e_0 e_{\xi_1}}{(e_{\xi_1+1}-e_0)(e_{\xi_1+1}-e_{\xi_1})}, \end{cases} \tag{8}$$

and

$$X_{\max}^{(3)} = \begin{cases} e_{\xi_2} & \textit{with probability} \quad q_1 = \frac{\mu_2-\mu_1(e_{\xi_2+1}+e_n)+e_{\xi_2+1}e_n}{(e_{\xi_2+1}-e_{\xi_2})(e_n-e_{\xi_2})}, \\ e_{\xi_2+2} & \textit{with probability} \quad q_2 = \frac{-\mu_2+\mu_1(e_{\xi_2}+e_n)-e_{\xi_2}e_n}{(e_{\xi_2+1}-e_{\xi_2})(e_n-e_{\xi_2+1})}, \\ e_n & \textit{with probability} \quad q_3 = 1-q_2-q_3. \end{cases} \tag{9}$$

Proof. Using the cut-criterion, it can be verified that the possible structure of the supports of the 3-convex discrete extrema takes the form $\{e_0, e_{\xi_1}, e_{\xi_1+1},\}$ or $\{e_{\xi_2}, e_{\xi_2+1}, e_n\}$. Property 4.1 is then used to derive the conditions on the support points ξ_1 and ξ_2 so that the random variable corresponding to such support has moments μ_1 and μ_2.

To that end, we just compute the polynomials $p(e_i) = c_0 + c_1 e_i + c_2 e_i^2$ of degree 2 (i.e. c_0, c_1 and $c_2 \in \mathbb{R}$) such that $X_{\max}^{(3)} \in \mathscr{M}_3(\mathscr{E}_n;\mu_1,\mu_2)$ (*resp.* $X_{\min}^{(3)}$) is concentrated on the set

$$\begin{aligned} \{e_i \in \mathscr{E}_n | e_i^3 = c_0 + c_1 e_i + c_2 e_i^2\} \quad = \quad & \{e_{\xi_2}, e_{\xi_2+1}, e_n\} \, (0 \leqslant \xi_2 \leqslant n-2) \\ & \textit{resp.} \\ & \{e_0, e_{\xi_1}, e_{\xi_1+1}\} \, (1 \leqslant \xi_1 < \xi_1 + 1 \leqslant n-1) \end{aligned}$$

and $e_i^3 \leqslant c_0 + c_1 e_i + c_2 e_i^2$ for all $i \in \{0,1,\dots,n\}$ (*resp.* $\geqslant$).

The only polynomial of degree 2 that fulfills the conditions

$$\begin{aligned} e_{\xi_2}^3 &= c_0 + c_1 e_{\xi_2} + c_2 e_{\xi_2}^2 \\ e_{\xi_2+1}^3 &= c_0 + c_1 e_{\xi_2+1} + c_2 e_{\xi_2+1}^2 \\ e_n^3 &= c_0 + c_1 e_n + c_2 e_n^2 \end{aligned}$$

is

$$p(e_i) = \left(e_{\xi_2} + e_{\xi_2+1} + e_n\right) e_i^2 - \left(e_{\xi_2} e_{\xi_2+1} + e_{\xi_2} e_n + e_{\xi_2+1} e_n\right) e_i + e_{\xi_2} e_{\xi_2+1} e_n.$$

The zeros of the polynomial $x^3 - p(x)$ are of course e_{ξ_2}, e_{ξ_2+1} and e_n and $x^3 - p(x)$ is always negative on $\mathscr{E}_n$. So, as we have checked that $e_i^3 \leqslant p(e_i)$ on $\mathscr{E}_n$, the random variable with support $\{e_{\xi_2}, e_{\xi_2+1}, e_n\}$ $(0 \leqslant \xi_2 \leqslant n-2)$ has to be $X^{(3)}_{\max}$.

The only polynomial of degree 2 that fulfills the conditions

$$\begin{aligned} e_0^3 &= c_0 + c_1 e_0 + c_2 e_0^2 \\ e_{\xi_1}^3 &= c_0 + c_1 e_{\xi_1} + c_2 e_{\xi_1}^2 \\ e_{\xi_1+1}^3 &= c_0 + c_1 e_{\xi_1+1} + c_2 e_{\xi_1+1}^2 \end{aligned}$$

is

$$p(e_i) = \left(e_0 + e_{\xi_1} + e_{\xi_1+1}\right) e_i^2 - \left(e_0 e_{\xi_1} + e_0 e_{\xi_1+1} + e_{\xi_1} e_{\xi_1+1}\right) e_i + e_0 e_{\xi_1} e_{\xi_1+1}.$$

The zeros of the polynomial $x^3 - p(x)$ are of course e_0, e_{ξ_1} and e_{ξ_1+1} and $x^3 - p(x)$ is always positive on $\mathscr{E}_n$. So, as we have checked that $e_i^3 \geqslant p(e_i)$ on $\mathscr{E}_n$, the random variable with support $\{e_0, e_{\xi_1}, e_{\xi_1+1}\}$ $(1 \leqslant \xi_1 \leqslant n-1)$ has to be $X^{(3)}_{\min}$.

Finally, we have to fix conditions on the support points to assure the non-negativity of their associated probabilities. The conditions on the support points of $X^{(3)}_{\min}$ and $X^{(3)}_{\max}$ are respectively

$$\begin{aligned} &0 < \xi_1 < \xi_1 + 1 \leqslant n \\ &\mu_2 - \mu_1 \left(e_{\xi_1} + e_{\xi_1+1}\right) + e_{\xi_1} e_{\xi_1+1} \geqslant 0 \\ &\mu_2 - \mu_1 \left(e_0 + e_{\xi_1+1}\right) + e_0 e_{\xi_1+1} \leqslant 0 \\ &\mu_2 - \mu_1 \left(e_0 + e_{\xi_1}\right) + e_0 e_{\xi_1} \geqslant 0 \end{aligned}$$

and

$$\begin{aligned} &0 \leqslant \xi_2 < \xi_2 + 1 < n \\ &\mu_2 - \mu_1 \left(e_{\xi_2+1} + e_n\right) + e_{\xi_2+1} e_n \geqslant 0 \\ &\mu_2 - \mu_1 \left(e_{\xi_2} + e_n\right) + e_{\xi_2+1} e_n \leqslant 0 \\ &\mu_2 - \mu_1 \left(e_{\xi_2} + e_{\xi_2+1}\right) + e_{\xi_2} e_{\xi_2+1} \geqslant 0 \end{aligned}$$

Moreover, because we have $e_i^2 - \left(e_{\xi_2} + e_{\xi_2+1}\right) e_i + e_{\xi_2} e_{\xi_2+1} \geqslant 0$ on $\mathscr{E}_n$, the first condition of the minimum and the last of the maximum are respectively always verified and the system of conditions reduces to the one wanted. □

Considering the claim distribution in Table 1, we thus have

$$X^{(3)}_{\min} = \begin{cases} 0 & \text{with probability } 0.2889, \\ 39 & \text{with probability } 0.1729, \\ 46 & \text{with probability } 0.5382, \end{cases}$$

and

$$X^{(3)}_{\max} = \begin{cases} 0 & \text{with probability } 0.1840, \\ 21 & \text{with probability } 0.5717, \\ 67 & \text{with probability } 0.2443. \end{cases}$$

It is interesting to compare these results with the extrema in $\mathscr{M}_3(\mathscr{D}_{67}; \mu_1, \mu_2)$ and in $\mathscr{M}_3([0,67]; \mu_1, \mu_2)$. In $\mathscr{M}_3(\mathscr{D}_{67}; \mu_1, \mu_2)$, we have

$$Y^{(3)}_{\min} = \begin{cases} 0 & \text{with probability } 0.2921, \\ 44 & \text{with probability } 0.3568, \\ 45 & \text{with probability } 0.3511, \end{cases}$$

and

$$Y_{\max}^{(3)} = \begin{cases} 19 & \text{with probability } 0.0271, \\ 20 & \text{with probability } 0.7277, \\ 67 & \text{with probability } 0.2453. \end{cases}$$

In $\mathscr{M}_3([0,67];\mu_1,\mu_2)$, we have

$$Z_{\min}^{(3)} = \begin{cases} 0 & \text{with probability } 0.2922, \\ 44.5016 & \text{with probability } 0.7078, \end{cases}$$

and

$$Z_{\max}^{(3)} = \begin{cases} 19.9634 & \text{with probability } 0.7547, \\ 67 & \text{with probability } 0.2453. \end{cases}$$

We now have that $X_{\max}^{(3)} \preceq_{3-cx}^{[0,67]} Y_{\max}^{(3)} \preceq_{3-cx}^{\mathscr{D}_{67}} Z_{\max}^{(3)}$ and $Z_{\min}^{(3)} \preceq_{3-cx}^{[0,67]} Y_{\min}^{(3)} \preceq_{3-cx}^{\mathscr{D}_{67}} X_{\min}^{(3)}$ so that we get finer extrema when the particular form of the support is exploited.

Let us now consider the case where three moments are known. The extrema with respect to $\preceq_{4-cx}^{\mathscr{E}_n}$ in $\mathscr{M}_4(\mathscr{E}_n;\mu_1,\mu_2,\mu_3)$ are given in the next result.

PROPOSITION 4.4. *If* η_1, η_2 *and* $\zeta \in \{0,1,\ldots,n-1\}$ *are such that* $0 \leqslant \eta_1 < \eta_1+1 < \eta_2 < \eta_2+1 \leqslant n$, $0 < \zeta < \zeta+1 < n$ *and* $e_\zeta < \frac{\mu_3-\mu_2(e_0+e_n)+\mu_1 e_0 e_n}{\mu_2-\mu_1(e_0+e_n)+e_0 e_n} \leqslant e_{\zeta+1}$ *and define*

$$\begin{aligned} \alpha_1 &:= -\mu_3+\mu_2(e_{\eta_1+1}+e_{\eta_2}+e_{\eta_2+1})-\mu_1(e_{\eta_1+1}e_{\eta_2}+e_{\eta_1+1}e_{\eta_2+1}+e_{\eta_2}e_{\eta_2+1})+e_{\eta_1+1}e_{\eta_2}e_{\eta_2+1} \\ \alpha_2 &:= \mu_3-\mu_2(e_{\eta_1}+e_{\eta_2}+e_{\eta_2+1})+\mu_1(e_{\eta_1}e_{\eta_2}+e_{\eta_1}e_{\eta_2+1}+e_{\eta_2}e_{\eta_2+1})-e_{l_1}e_{\eta_2}e_{\eta_2+1} \\ \alpha_3 &:= -\mu_3+\mu_2(e_{\eta_1}+e_{\eta_1+1}+e_{\eta_2+1})-\mu_1(e_{\eta_1}e_{\eta_1+1}+e_{\eta_1}e_{\eta_2+1}+e_{\eta_1+1}e_{\eta_2+1})+e_{\eta_1}e_{\eta_1+1}e_{\eta_2+1} \\ \alpha_4 &:= \mu_3-\mu_2(e_{\eta_1}+e_{\eta_1+1}+e_{\eta_2})+\mu_1(e_{\eta_1}e_{\eta_1+1}+e_{\eta_1}e_{\eta_2}+e_{\eta_1+1}e_{\eta_2})-e_{\eta_1}e_{\eta_1+1}e_{\eta_2} \end{aligned} \tag{10}$$

that are positive, then, with respect to $\preceq_{4-cx}^{\mathscr{E}_n}$,

$$X_{\min}^{(4)} = \begin{cases} e_{\eta_1} & \textit{with probability} \quad v_1 = \frac{\alpha_1}{(e_{\eta_1+1}-e_{\eta_1})(e_{\eta_2}-e_{\eta_1})(e_{\eta_2+1}-e_{\eta_1})}, \\ e_{\eta_1+1} & \textit{with probability} \quad v_2 = \frac{\alpha_2}{(e_{\eta_1+1}-e_{\eta_1})(e_{\eta_2}-e_{\eta_1+1})(e_{\eta_2+1}-e_{\eta_1+1})}, \\ e_{\eta_2} & \textit{with probability} \quad v_3 = \frac{\alpha_3}{(e_{\eta_2}-e_{\eta_1})(e_{\eta_2}-e_{\eta_1+1})(e_{\eta_2+1}-e_{\eta_2})}, \\ e_{\eta_2+1} & \textit{with probability} \quad v_4 = \frac{\alpha_4}{(e_{\eta_2+1}-e_{\eta_1})(e_{\eta_2+1}-e_{\eta_1+1})(e_{\eta_2+1}-e_{\eta_2})}, \end{cases} \tag{11}$$

and

$$X_{\max}^{(4)} = \begin{cases} e_0 & \textit{with probability} \quad w_1 = 1-w_2-w_3-w_4, \\ e_\zeta & \textit{with probability} \quad w_2 = \frac{\mu_3-\mu_2(e_0+e_{\zeta+1}+e_n)+\mu_1(e_0e_{\zeta+1}+e_0e_n+e_{\zeta+1}e_n)-e_0e_{\zeta+1}e_n}{(e_\zeta-e_0)(e_{\zeta+1}-e_\zeta)(e_n-e_\zeta)}, \\ e_{\zeta+1} & \textit{with probability} \quad w_3 = \frac{-\mu_3+\mu_2(e_0+e_\zeta+e_n)-\mu_1(e_0e_\zeta+e_0e_n+e_\zeta e_n)+e_0e_\zeta e_n}{(e_{\zeta+1}-e_0)(e_{\zeta+1}-e_\zeta)(e_n-e_{\zeta+1})}, \\ e_n & \textit{with probability} \quad w_4 = \frac{\mu_3-\mu_2(e_0+e_\zeta+e_{\zeta+1})+\mu_1(e_0e_\zeta+e_0e_{\zeta+1}+e_\zeta e_{\zeta+1})-e_0e_\zeta e_{\zeta+1}}{(e_n-e_0)(e_n-e_\zeta)(e_n-e_{\zeta+1})}. \end{cases} \tag{12}$$

Proof. Using the cut-criterion, it can be verified that the possible structure of the supports of the 4-convex discrete extrema takes the form $\{e_{\eta_1}, e_{\eta_1+1}, e_{\eta_2}, e_{\eta_2+1}\}$ or $\{e_0, e_\zeta, e_{\zeta+1}, e_n\}$. Property 4.1 is then used to derive the conditions on the support points η_1, η_2 and ζ so that the random variable corresponding to such support has moments μ_1, μ_2 and μ_3.

To that end, we just compute the polynomials $p(e_i) = c_0 + c_1 e_i + c_2 e_i^2 + c_3 e_i^3$ of degree 3 (i.e. c_0, c_1, c_2 and $c_3 \in \mathbb{R}$) such that $X_{\max}^{(4)} \in \mathscr{M}_4(\mathscr{E}_n; \mu_1, \mu_2, \mu_3)$ (*resp.* $X_{\min}^{(4)}$) is concentrated on the set

$$\left\{i \in \mathscr{E}_n : e_i^4 = c_0 + c_1 e_i + c_2 e_i^2 + c_3 e_i^3\right\} = \left\{e_0, e_\zeta, e_{\zeta+1}, e_n\right\} \ (1 \leqslant \zeta \leqslant n-2)$$

resp.

$$\{e_{\eta_1}, e_{\eta_1+1}, e_{\eta_2}, e_{\eta_2+1}\} \ (0 \leqslant \eta_1 < \eta_1 + 1 < \eta_2 < \eta_2 + 1 \leqslant n)$$

and $e_i^4 \leqslant c_0 + c_1 e_i + c_2 e_i^2 + c_3 e_i^3$ for all $i \in \{0, 1, \ldots, n\}$ (*resp.* $\geqslant$).

The only polynomial of degree 3 that fulfills the conditions

$$\begin{aligned} e_0^4 &= c_0 \\ e_\zeta^4 &= c_0 + c_1 e_\zeta + c_2 e_\zeta^2 + c_3 e_\zeta^3 \\ e_{\zeta+1}^4 &= c_0 + c_1 e_{\zeta+1} + c_2 e_{\zeta+1}^2 + c_3 e_{\zeta+1}^3 \\ e_n^4 &= c_0 + c_1 e_n + c_2 e_n^2 + c_3 e_n^3 \end{aligned}$$

is

$$\begin{aligned} p(e_i) = {} & \left(e_0 + e_\zeta + e_{\zeta+1} + e_n\right) e_i^3 \\ & - \left(e_0 e_\zeta + e_0 e_{\zeta+1} + e_0 e_n + e_\zeta e_{\zeta+1} + e_\zeta e_n + e_{\zeta+1} e_n\right) e_i^2 \\ & + \left(e_0 e_\zeta e_{\zeta+1} + e_0 e_\zeta e_n + e_0 e_{\zeta+1} e_n + e_\zeta e_{\zeta+1} e_n\right) e_i \\ & - e_0 e_\zeta e_{\zeta+1} e_n. \end{aligned}$$

The zeros of the polynomial $x^4 - p(x)$ are of course e_0, e_ζ, $e_{\zeta+1}$ and e_n and $x^4 - p(x)$ is always negative on $\mathscr{E}_n$. So, as we have checked that $e_i^4 \leqslant p(e_i)$ on $\mathscr{E}_n$, the random variable with support $\{e_0, e_\zeta, e_{\zeta+1}, e_n\}$ $(1 \leqslant \zeta \leqslant n-2)$ has to be $X_{\max}^{(4)}$.

The only polynomial of degree 3 that fulfills the conditions

$$\begin{aligned} e_{\eta_1}^4 &= c_0 + c_1 e_{\eta_1} + c_2 e_{\eta_1}^2 + c_3 e_{\eta_1}^3 \\ e_{\eta_1+1}^4 &= c_0 + c_1 e_{\eta_1+1} + c_2 e_{\eta_1+1}^2 + c_3 e_{\eta_1+1}^3 \\ e_{\eta_2}^4 &= c_0 + c_1 e_{\eta_2} + c_2 e_{\eta_2}^2 + c_3 e_{\eta_2}^3 \\ e_{\eta_2+1}^4 &= c_0 + c_1 e_{\eta_2+1} + c_2 e_{\eta_2+1}^2 + c_3 e_{\eta_2+1}^3 \end{aligned}$$

is

$$\begin{aligned} p(e_i) = {} & (e_{\eta_1} + e_{\eta_1+1} + e_{\eta_2} + e_{\eta_2+1})\, e_i^3 \\ & - (e_{\eta_1} e_{\eta_1+1} + e_{\eta_1} e_{\eta_2} + e_{\eta_1} e_{\eta_2+1} + e_{\eta_1+1} e_{\eta_2} + e_{\eta_1+1} e_{\eta_2+1} + e_{\eta_2} e_{\eta_2+1})\, e_i^2 \\ & + (e_{\eta_1} e_{\eta_1+1} e_{\eta_2} + e_{\eta_1} e_{\eta_1+1} e_{\eta_2+1} + e_{\eta_1} e_{\eta_2} e_{\eta_2+1} + e_{\eta_1+1} e_{\eta_2} e_{\eta_2+1})\, e_i \\ & - e_{\eta_1} e_{\eta_1+1} e_{\eta_2} e_{\eta_2+1}. \end{aligned}$$

The zeros of the polynomial $x^4 - p(x)$ are of course e_{η_1}, e_{η_1+1}, e_{η_2} and e_{η_2+1} and $x^4 - p(x)$ is always positive on $\mathscr{E}_n$. So, as we have checked that $e_i^4 \geqslant p(e_i)$ on $\mathscr{E}_n$, the random variable with support $\{e_{\eta_1}, e_{\eta_1+1}, e_{\eta_2}, e_{\eta_2+1}\}$ $(0 \leqslant \eta_1 < \eta_1 + 1 < \eta_2 < \eta_2 + 1 \leqslant n)$ has to be $X_{\min}^{(4)}$.

Finally, we have to fix conditions on the support points to assure the non-negativity of their associated probabilities. The conditions on the support points of $X_{\max}^{(4)}$ are

$$\begin{array}{l} 0 < \zeta < \zeta + 1 < n \\ -\mu_3 + \mu_2 \left(e_\zeta + e_{\zeta+1} + e_n\right) - \mu_1 \left(e_\zeta e_{\zeta+1} + e_\zeta e_n + e_{\zeta+1} e_n\right) + e_\zeta e_{\zeta+1} e_n \geqslant 0 \\ \mu_3 - \mu_2 \left(e_0 + e_{\zeta+1} + e_n\right) + \mu_1 \left(e_0 e_{\zeta+1} + e_0 e_n + e_{\zeta+1} e_n\right) - e_0 e_{\zeta+1} e_n \geqslant 0 \\ -\mu_3 + \mu_2 \left(e_0 + e_\zeta + e_n\right) - \mu_1 \left(e_0 e_\zeta + e_0 e_n + e_\zeta e_n\right) + e_0 e_\zeta e_n \geqslant 0 \\ \mu_3 - \mu_2 \left(e_0 + e_\zeta + e_{\zeta+1}\right) + \mu_1 \left(e_0 e_\zeta + e_0 e_{\zeta+1} + e_\zeta e_{\zeta+1}\right) - e_0 e_\zeta e_{\zeta+1} \geqslant 0 \end{array}$$

and because we have $\left(e_\zeta + e_{\zeta+1} + e_n\right) e_i^2 - \left(e_\zeta e_{\zeta+1} + e_\zeta e_n + e_{\zeta+1} e_n\right) e_i + e_\zeta e_{\zeta+1} e_n \geqslant e_i^3$ on $\mathscr{E}_n$ (cfr. 3-convex maximum) and $\left(e_0 + e_\zeta + e_{\zeta+1}\right) e_i^2 - \left(e_0 e_\zeta + e_0 e_{\zeta+1} + e_\zeta e_{\zeta+1}\right) e_i + e_0 e_\zeta e_{\zeta+1} \leqslant e_i^3$ on $\mathscr{E}_n$ (cfr. 3-convex minimum), the first and the last conditions are respectively always verified and the system of conditions reduces to

$$0 < \zeta < \zeta + 1 < n \text{ and } e_\zeta < \frac{\mu_3 - \mu_2 \left(e_0 + e_n\right) + \mu_1 e_0 e_n}{\mu_2 - \mu_1 \left(e_0 + e_n\right) + e_0 e_n} \leqslant e_{\zeta+1}.$$

The conditions on the support points of $X_{\min}^{(4)}$ are given by

$$\alpha_1 \geqslant 0, \alpha_2 \geqslant 0, \alpha_3 \geqslant 0 \text{ and } \alpha_4 \geqslant 0. \tag{13}$$

□

Note that the solution $\{e_{\eta_1}, e_{\eta_1+1}, e_{\eta_2}, e_{\eta_2+1}\}$ of (10) cannot be obtained explicitly. In practice, each admissible pair $\{e_{\eta_1}, e_{\eta_2}\}$ of $\mathscr{E}_n$ has to be tested.

Considering the claim distribution in Table 1, we thus have

$$X_{\min}^{(3)} = \begin{cases} 12 & \text{with probability } 0.3918, \\ 17 & \text{with probability } 0.1607, \\ 53 & \text{with probability } 0.4225, \\ 67 & \text{with probability } 0.0250, \end{cases}$$

and

$$X_{\max}^{(3)} = \begin{cases} 0 & \text{with probability } 0.1821, \\ 28 & \text{with probability } 0.4660, \\ 39 & \text{with probability } 0.1830, \\ 67 & \text{with probability } 0.1689. \end{cases}$$

It is interesting to compare these results with the extrema in $\mathscr{M}_4(\mathscr{D}_{67}; \mu_1, \mu_2, \mu_3)$ and in $\mathscr{M}_4([0,67]; \mu_1, \mu_2, \mu_3)$. In $\mathscr{M}_4(\mathscr{D}_{67}; \mu_1, \mu_2, \mu_3)$, we have

$$Y_{\min}^{(3)} = \begin{cases} 13 & \text{with probability } 0.2894, \\ 14 & \text{with probability } 0.2683, \\ 54 & \text{with probability } 0.3463, \\ 55 & \text{with probability } 0.0960, \end{cases}$$

and

$$Y_{\max}^{(3)} = \begin{cases} 0 & \text{with probability } 0.1897, \\ 31 & \text{with probability } 0.5707, \\ 32 & \text{with probability } 0.0641, \\ 67 & \text{with probability } 0.1755. \end{cases}$$

In $\mathscr{M}_4([0,67];\mu_1,\mu_2,\mu_3)$, we have

$$Z_{\min}^{(3)} = \begin{cases} 13.4722 & \text{with probability } 0.5576, \\ 54.2177 & \text{with probability } 0.4424, \end{cases}$$

and

$$Z_{\max}^{(3)} = \begin{cases} 0 & \text{with probability } 0.1897, \\ 31.1013 & \text{with probability } 0.6348, \\ 67 & \text{with probability } 0.1755. \end{cases}$$

We now have that $X_{\max}^{(4)} \preceq_{4-cx}^{[0,67]} Y_{\max}^{(4)} \preceq_{4-cx}^{\mathscr{D}_{67}} Z_{\max}^{(4)}$ and $Z_{\min}^{(4)} \preceq_{4-cx}^{[0,67]} Y_{\min}^{(4)} \preceq_{4-cx}^{\mathscr{D}_{67}} X_{\min}^{(4)}$ so that we get finer extrema when the particular form of the support is exploited.

We refer the reader to the appendix for a general procedure allowing to get the extrema for $s \geqslant 5$.

5. Applications

In this section, we derive bounds for the eventual ruin probability in the compound Poisson risk process. In this classical model, the discrete claim amounts $X_1, X_2, \dots$ recorded by an insurance company are assumed to be independent and identically distributed with common distribution function F, such that $F(0) = 0$. The number of claims in the time interval $[0,t]$ is assumed to be independent of the individual claim amounts and to form a Poisson process $\{N(t), t \geqslant 0\}$ with constant rate λ. Let also the premium rate $c > 0$ be such that the inequality $c > \lambda\mathbb{E}[X_1]$ holds (i.e. the premium received in each period is larger than the net premium). Further, let $\psi(\kappa)$ be the ultimate ruin probability with an initial capital κ; that is, the probability that the process $Z(t) = \kappa + ct - \sum_{i=1}^{N(t)} X_i$, $t \geqslant 0$, describing the wealth of the insurance company, ever falls below zero. If the moment generating function of X exists, the Lundberg's inequality provides an exponential upper bound on ψ, namely $\psi(\kappa) \leqslant \mathrm{e}^{-z\kappa}$, where z is the Lundberg's adjustment coefficient satisfying the integral equation $\mathbb{E}[\mathrm{e}^{zX}] = 1 + \frac{cz}{\lambda}$. Remark that in storage theory, $\sum_{i=1}^{N(t)} X_i$ can be seen as the input process to a storage system when its content is positive. Then $1-\psi$ is known to be the limiting contents distribution. In queueing theory, $1-\psi$ can be viewed as the waiting time distribution for an $M/G/1$ queue with arrival rate λ and service time distribution F.

Assume that the X_i's are valued in $\mathscr{E}_n$. As the function $f(x) = \mathrm{e}^{zx}$ is s-convex, we have $\mathbb{E}[\mathrm{e}^{zX_{\min}^{(s)}}] \leqslant \mathbb{E}[\mathrm{e}^{zX}] \leqslant \mathbb{E}[\mathrm{e}^{zX_{\max}^{(s)}}]$, with $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ the stochastic extrema with respect to $\preceq_{s-cx}^{\mathscr{E}_n}$. This provides bounds $z_{\min}^{(s)} \leqslant z \leqslant z_{\max}^{(s)}$ on the Lundberg's coefficient where $z_{\min}^{(s)}$ and $z_{\max}^{(s)}$ are respectively the roots of the equations $\mathbb{E}[\mathrm{e}^{zX_{\max}^{(s)}}] = 1 + \frac{cz}{\lambda}$ and $\mathbb{E}[\mathrm{e}^{zX_{\min}^{(s)}}] = 1 + \frac{cz}{\lambda}$.

To illustrate our results, we use the discrete claim severity distribution given in WALHIN & PARIS (1998) displayed in Table 1. For this special distributions, we thus have $n = 67$, $\mu_1 = 31.5$, $\mu_2 = 1401.8$ and $\mu_3 = 71879.1$. We take furthermore $\lambda = 10$ and $c = 400$ (so that $c > \lambda\mu_1$). Table 2 depicts the bounds on the Lundberg's coefficient in case two ($s = 3$) or three ($s = 4$) moments are considered. The bounds were computed using the continuous s-convex extrema on $[0,67]$, the discrete s-convex extrema on the equidistant support $\mathscr{D}_{67} = \{0,1,\ldots,67\}$ and the discrete s-convex extrema on the arbitrary grid $\mathscr{E}_9 = \{0,7,12,17,23,28,39,46,53,67\}$. The explicit expression of these extrema have been given in the preceding section. The third method gives better results than the two other ones. The accuracy of the bounds is remarkable.

	$s=3$	$s=4$
$z^{(3)}_{\min}$, $[0,67]$	0.00992428	0.01009877
$z^{(3)}_{\min}$, $\mathscr{D}_{67}$	0.00992431	0.01009877
$z^{(3)}_{\min}$, $\mathscr{E}_9$	0.00992664	0.01009981
$z^{(3)}_{\max}$, $\mathscr{E}_9$	0.01033457	0.01011517
$z^{(3)}_{\max}$, $\mathscr{D}_{67}$	0.01034111	0.01011622
$z^{(3)}_{\max}$, $[0,67]$	0.01034132	0.01011625

Table 2. Bounds on the Lundberg's coefficient z with $c = 400$ and $\lambda = 10$.

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Appendix: Derivation of the extrema with respect to $\preceq^{\mathscr{E}_n}_{s-cx}$ in $\mathscr{M}_s(\mathscr{E}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$ for $s \geqslant 5$

The method proposed can be extended to the general case $s \geqslant 5$. It is done in the following way. Using the cut-criterion and Property 4.1, it can be seen that the most general form for the supports of the s-convex extrema, denoted by $\mathrm{Supp}_{X^{(s)}_{\min}}$ and $\mathrm{Supp}_{X^{(s)}_{\max}}$, are given as follows: for $s = 2m$, we have $\mathrm{Supp}_{X^{(s)}_{\min}} = \{e_{\xi_1}, e_{\xi_1+1}, \ldots, e_{\xi_m}, e_{\xi_m+1}\}$ $(e_0 \leqslant e_{\xi_1} < e_{\xi_1+1} < \ldots < e_{\xi_m} < e_{\xi_m+1} \leqslant e_n)$ and $\mathrm{Supp}_{X^{(s)}_{\max}} = \{e_0, e_{\zeta_1}, e_{\zeta_1+1}, \ldots, e_{\zeta_{m-1}}, e_{\zeta_{m-1}+1}, e_n\}$ $(0 < e_{\zeta_1} < e_{\zeta_1+1} < \ldots < e_{\zeta_{m-1}} < e_{\zeta_{m-1}+1} < e_n)$ while for $s = 2m+1$, we have $\mathrm{Supp}_{X^{(s)}_{\min}} = \{e_0, e_{\xi_1}, e_{\xi_1+1}, \ldots, e_{\xi_m}, e_{\xi_m+1}\}$ $(e_0 < e_{\xi_1} < e_{\xi_1+1} < \ldots < e_{\xi_m} < e_{\xi_m+1} \leqslant e_n)$ and $\mathrm{Supp}_{X^{(s)}_{\max}} = \{e_{\zeta_1}, e_{\zeta_1+1}, \ldots, e_{\zeta_m}, e_{\zeta_m+1}, e_n\}$ $(e_0 \leqslant e_{\zeta_1} < e_{\zeta_1+1} < \ldots < e_{\zeta_{m-1}} < e_{\zeta_{m-1}+1} < e_n)$.

Then, to express the conditions on the support points so that $X^{(s)}_{\min}$ and $X^{(s)}_{\max}$ have the required moments $\mu_1, \mu_2, \ldots, \mu_{s-1}$, we just have to compute the probabilities associated to the support points and to check that they are positive. We get the resulting probabilities using that

$$X \in \mathscr{D}_s(\mathscr{E}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}) \text{ with } \mathrm{Supp}_X = \left\{e_{j_1}, e_{j_2}, \ldots, e_{j_k}\right\}$$

$$\Rightarrow \Pr[X = e_{j_i}] = \frac{\mathbb{E}\left[\prod_{l \neq i}(X - e_{j_l})\right]}{\prod_{l \neq i}(e_{j_i} - e_{j_l})} \; (i = 0, 1, \ldots, k).$$

The solution $\left(e_{\xi_1}, \ldots, e_{\xi_{s/2}}, e_{\zeta_1}, \ldots, e_{\zeta_{(s/2)-1}}\right)$ (s even) (*resp.* $\left(e_{\xi_1}, \ldots, e_{\xi_{(s-1)/2}}, e_{\zeta_1}, \ldots, e_{\zeta_{(s-1)/2}}\right)$ (s odd)) cannot be obtained explicitly. Nevertheless, it is easily obtained by testing each admissible sequence $\left(e_{\xi_1}, \ldots, e_{\xi_{s/2}}, e_{\zeta_1}, \ldots, e_{\zeta_{(s/2)-1}}\right)$ (*resp.* $\left(e_{\xi_1}, \ldots, e_{\xi_{(s-1)/2}}, e_{\zeta_1}, \ldots, e_{\zeta_{(s-1)/2}}\right)$) of $\mathscr{E}_n$.

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