Journal of
Mathematical
Inequalities
Volume 2, Number 2 (2008), 215–227

GENERAL WIENER–HOPF EQUATIONS AND NONEXPANSIVE MAPPINGS

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(*communicated by A. Čižmešija*)

Abstract. In this paper, we show that the general variational inequalities are equivalent to a new class of general Wiener-Hopf equations involving the nonexpansive mappings. Using this equivalence, we suggest and analyze an iterative method for finding the common elements of the solution set of the general variational inequalities and the solution set of the fixed-point of the nonexpansive mapping. We also consider the convergence criteria of the proposed method under some mild conditions. Since the general variational inequalities and the Wiener-Hopf equations include several classes of variational inequalities and Wiener-Hopf equations as special cases, our results continue to hold for these problems. Results obtained in this paper may be viewed as a refinement and improvement of the previously known results.

1. Introduction

General variational inequalities introduced by Noor [8] in 1988 are a very important and significant extension of the concept of variational inequalities. It has been shown that a wide class of both odd-order and even-order, nonsymmetric problems arising in various branches of mathematical and engineering sciences can be studied in the unified and general framework of general variational inequalities, see [3–29, 32]. Theory of variational inequalities combines theoretical and algorithmic advances with new and novel domain of applications. Analysis of these problems requires a blend of techniques from convex analysis, functional analysis and numerical analysis. As a result of interaction between these different branches of mathematical and engineering sciences, we have a variety of techniques for solving variational inequalities and related problems. Using the projection technique, one can show that the general variational inequalities are equivalent to the fixed-point problems. This equivalence has been used to develop some efficient and robust numerical techniques for solving variational inequalities and related optimization problems. Related to the variational inequalities, we have the problem of solving the Wiener-Hopf equations, which were introduced by Shi [27] in conjunction with variational inequalities. Essentially using the projection technique, one establishes the equivalance between the variational inequalities and the Wiener-Hopf equations. This alternative equivalent formulation has played an important and significant part in developing efficient and robust numerical techniques

Mathematics subject classification (2000): 49J40, 90C33.

Key words and phrases: Variational inequalities, fixed point problems, Wiener-Hopf equations, relaxed cocoercive operators, nonexpansive mappings convergence criteria.

Paper JMI-02-19

and sensitivity analysis framework for variational inequalities, see [9, 10, 13–18, 20, 21, 25–27] and the references therein.

Related to the variational inequalities and the Wiener-Hopf equations, we also have the problem of finding fixed points of the non-expansive mappings, which is the subject of current interest in functional analysis, see [30]. It is natural to consider a unified approach for these different problems. Motivated and inspired by the research going in this direction, Noor [16] and Noor and Huang [19] considered the problem of finding the common element of the set of the solutions of variational inequalities, the Wiener-Hopf equations and the set of the fixed points of the nonexpansive mappings. In this paper, we introduce and consider a new class of Wiener-Hopf equations involving two nonlinear operators and a nonexpansive mapping, which is called the general Wiener-Hopf equation involving mapping. Using the projection technique, we show that the general Wiener-Hopf equation involving a non-expansive mapping are equivalent to the general variational inequalities. We use this alternative equivalence to suggest and analyze an iterative scheme for finding the common solutions of the Winer-Hopf equations involving the nonexpansive mappings. We also prove the convergence criteria of these new iterative schemes under some mild conditions. Since Noor variational inequalities include variational inequalities, nonlinear complementarity problems and a class of quasi variational inequalities as special cases, results obtained in this paper continue to hold for these problems. Our results can be viewed as a significant and novel extension of the previously known results.

2. Premilinaries and Basic Results

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle .,. \rangle$ and $\|.\|$ respectively. Let K be a nonempty closed and convex set in H and $T, g : H \longrightarrow H$ be nonlinear operators.

We now consider the problem of finding $u \in H : g(u) \in K$ such that

$$\langle Tu, g(v) - g(u) \rangle \geqslant 0, \quad \forall v \in H : g(v) \in K. \tag{2.1}$$

Problem (2.1) is called the *general variational inequality,* which was introduced and studied by Noor [8] in 1988. It has been shown that many diverse problems in science and engineering as well as in social sciences and ecology can be studied in the unified framework of the general variational inequalities, see [9-21] and the references therein. To convey an idea of the applications of general variational inequalities, we consider the third-order obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u''' \geqslant f(x) & \text{on } \Omega = [0,1] \\ u \geqslant \psi(x) & \text{on } \Omega = [0,1] \\ [-u''' - f(x)][u - \psi(x)] = 0 & \text{on } \Omega = [0,1] \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0 & \end{array}\right\}, \tag{2.2}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (2.2) in the framework of variational inequality approach. To do so, we first define the set K as

$$K = \{v : v \in H_0^2(\Omega) : v \geqslant \psi \quad \text{on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space, see [7]. One can easily show that the energy functional associated with the problem (2.2) is

$$\begin{aligned} I[v] &= -\int_0^1 \left(\frac{d^3v}{dx^3}\right)\left(\frac{dv}{dx}\right)dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx, \quad \text{for all } \frac{dv}{dx} \in K \\ &= \int_0^1 \left(\frac{d^2v}{dx^2}\right)^2 dx - 2\int_0^1 f(x)\left(\frac{dv}{dx}\right)dx \\ &= \langle Tv, g(v)\rangle - 2\langle f, g(v)\rangle \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} \langle Tu, g(v)\rangle &= \int_0^1 \left(\frac{d^2u}{dx^2}\right)\left(\frac{d^2v}{dx^2}\right)dx \\ \langle f, g(v)\rangle &= \int_0^1 f(x)\frac{dv}{dx}dx \end{aligned} \tag{2.4}$$

and $g = \dfrac{d}{dx}$ is the linear operator.

It is clear that the operator T defined by (2.4) is linear, g-symmetric and g-positive. Using the technique of Noor [15], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (2.3) associated with the problem (2.2) on the closed convex set K can be characterized by the inequality of the type

$$\langle Tu, g(v) - g(u)\rangle \geqslant \langle f, g(v) - g(u)\rangle, \quad \forall g(v) \in K,$$

which is exactly the general variational inequality (2.1). It is worth mentioning that a wide class of unrelated odd-order and nonsymmetric equilibrium problems arising in regional, physical, mathematical, engineering and applied sciences can be studied in the unified and general framework of the general variational inequalities (2.1), see [9–18] and the references therein.

For $g \equiv I$, where I is the identity operator, problem (2.1) is equivalent to finding $u \in K$ such that

$$\langle Tu, v - u\rangle \geqslant 0 \quad \forall v \in K, \tag{2.5}$$

which is known as the classical variational inequality introduced and studied by Stampacchia [29] in 1964. For recent state-of-the-art, see [1–32] and the references therein.

If $K^* = \{u \in H : \langle u, v\rangle \geqslant 0, \quad \forall v \in K\}$ is a polar (dual) cone of a convex cone K in H, then problem (2.1) is equivalent to finding $u \in H$ such that

$$g(u) \in K, \quad Tu \in K^* \quad \text{and} \quad \langle Tu, g(u)\rangle = 0, \tag{2.6}$$

which is known as the general complementarity problem. For $g(u) = m(u) + K$, where m is a point-to-point mapping, problem (2.6) is called the implicit (quasi) complementarity problem. If $g \equiv I$, then problem (2.6) is known as the generalized complementarity problem. Such problems have been studied extensively in the literature, see the references. For suitable and appropriate choice of the operators and spaces, one can obtain several classes of variational inequalities and related optimization problems.

We now recall the following well known results and concepts.

LEMMA 2.1. *For a given* $z \in H$, $u \in K$ *satisfies the inequality*

$$\langle u - z, v - u \rangle \geqslant 0, \quad \forall v \in K,$$

if and only if

$$u = P_K[z],$$

where P_K *is the projection of* H *onto* K. *Also, the projection operator* P_K *is nonexpansive.*

Related to the variational inequalities, we now consider the problem of solving the Wiener-Hopf equations. To be more precise, let $Q_K = I - SP_K$, where P_K is the projection of H onto the closed convex set K, I is the identity operator and S is the nonexpansive operator. For given nonlinear operators $T, g : H \longrightarrow H$, we consider the problem of finding $z \in H$ such that

$$Tg^{-1}SP_Kz + \rho^{-1}Q_Kz = 0, \tag{2.7}$$

which is called the Wiener-Hopf equation involving the nonexpansive operator S. For $S = I$, the identity operator, we obtain the original general Wiener-Hopf equation, introduced by Noor [9]. We also remark that, if $g = I$ the identity operator, then the General Wiener-Hopf equations are exactly the same as considered in Noor and Huang [23]. In passing, we would like to mention that the concept of Wiener-Hopf equations was introduced by Shi [27] in the context of variational inequalities. Using essentially the projection technique, one can show that the general Wiener-Hopf equations are equivalent to the variational inequalities. This equivalence has played a fundamental and basic role in developing some efficient and robust methods for solving variational inequalities, see [9,10, 13–15,20–23, 26,27] and the references therein for the applications and numerical methods.

Using Lemma 2.1, we can show that the general variational inequality (2.1) is equivalent to the fixed point problem. This result is mainly due to Noor [8].

LEMMA 2.2. *The function* $u \in H : g(u) \in K$ *satisfies the general variational inequality (2.1), if and only if,* $u \in H$ *satisfies the relation*

$$g(u) = P_K[g(u) - \rho Tu], \tag{2.8}$$

where $\rho > 0$ *is a constant.*

It is clear from Lemma 2.2 that the general variational inequalities (2.1) and the fixed point problems (2.8) are equivalent. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

It is convenient to write (2.8) in the following form which is very useful in obtaining our results.

$$u = u - g(u) + P_K[g(u) - \rho Tu].$$

Let S be a nonexpansive mapping. We denote the set of the fixed points of S by $F(S)$ and the set of the solutions of the general variational inequalities (2.1) by $GVI(K,T,g)$. We now characterize the problem. If $u \in F(S) \cap GVI(K,T,g)$, then $u \in F(S)$ and $u \in GVI(K,T,g)$. Thus from Lemma 2.2, it follows that

$$\begin{aligned} u = Su &= u - g(u) + P_K[g(u) - \rho Tu] \\ &= S\{u - g(u) + P_K[g(u) - \rho Tu]\}. \end{aligned} \tag{2.9}$$

where $\rho > 0$ is a constant.

This fixed point formulation has been [18] used to suggest the following iterative method for finding a common element of two different sets of solutions of the fixed points of the nonexpansive mappings and the variational inequalities.

ALGORITHM 2.1. For a given $x_0 \in H$, compute the approximate solution x_n by the iterative schemes

$$x_{n+1} = (1 - a_n)x_n + a_n S\{x_n - g(x_n) + P_K[g(x_n) - \rho Tx_n]\},$$

where $a_n \in [0,1]$ for all $n \geqslant 0$ and S is the nonexpansive operator. For $S = I$, the identity operator, Algorithm 2.1 is essentially due to Noor [11].

DEFINITION 2.1. A mapping $T : H \to H$ is called μ-Lipschitzian if there exists a constant $\mu > 0$, such that

$$||Tx - Ty|| \leqslant \mu||x - y||, \quad \forall x, y \in H.$$

DEFINITION 2.2. A mapping $T : H \to H$ is called α-inverse strongly monotonic if there exists a constant $\alpha > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geqslant \alpha||Tx - Ty||^2, \quad \forall x, y \in H.$$

DEFINITION 2.3. A mapping $T : H \to H$ is called r-strongly monotonic if there exists a constant $r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geqslant r||x - y||^2, \quad \forall x, y \in H.$$

DEFINITION 2.4. A mapping $T : H \to H$ is called relaxed (γ, r)-cocoercive if there exists constants $\gamma > 0, r > 0$, such that

$$\langle Tx - Ty, x - y\rangle \geqslant -\gamma||Tx - Ty||^2 + r||x - y||^2, \quad \forall x, y \in H.$$

REMARK 2.2. Clearly a r-strongly monotonic mapping or a γ-inverse strongly monotonic mapping must be a relaxed (γ, r)-cocoercive mapping, but the converse is not true. Therefore the class of the relaxed (γ, r)-cocoercive mappings is the most general class, and hence definition 2.4 includes both the definition 2.2 and the definition 2.3 as special cases.

LEMMA 2.3. [31] *Suppose* $\{\delta_k\}_{k=0}^{\infty}$ *is a nonnegative sequence satisfying the following inequality:*

$$\delta_{k+1} \leqslant (1 - \lambda_k)\delta_k + \sigma_k, \ k \geqslant 0$$

with $\lambda_k \in [0,1]$, $\sum_{k=0}^{\infty} \lambda_k = \infty$, *and* $\sigma_k = o(\lambda_k)$. *Then* $\lim_{k\to\infty} \delta_k = 0$.

3. Main Results

In this section, we use the general Wiener-Hopf equations to suggest and analyze an iterative method for finding the common element of the nonexpansive mappings and the general variational inequalities $GVI(K,T,g)$. For this purpose, we need the following result, which can be proved by using Lemma 2.2. However, for the sake of completeness, we include its proof.

LEMMA 3.1. *The element $u \in H : g(u) \in K$ is a solution of $GVI(K,T,g)$ if and only if $z \in H$ satisfies the Wiener-Hopf equation (2.7), where*

$$\begin{aligned} g(u) &= SP_K z, && (3.1)\\ z &= g(u) - \rho T u, && (3.2) \end{aligned}$$

where $\rho > 0$ is a constant.

Proof. Let $u \in H : g(u) \in K$ be a solution of $GVI(K,T)$. Then, from Lemma 2.2 and (2.9), we have

$$g(u) = SP_K[g(u) - \rho T u]. \tag{3.3}$$

Let

$$z = g(u) - \rho T u. \tag{3.4}$$

Form (3.3) and (3.4), we have

$$\begin{aligned} g(u) &= SP_K z\\ z &= g(u) - \rho T u, \end{aligned}$$

from which, we have

$$z = SP_K z - \rho T g^{-1} SP_K z,$$

which is exactly the general Wiener-Hopf equation (2.7), the required result. □

From Lemma 3.1, it follows that the general variational inequality (2.1) and the general Wiener-Hopf equation (2.7) are equivalent. We denote the set of the solutions of the Wiener-Hopf equations by $GWHE(H,T,g,S)$.

Using Lemma 3.1 and (2.9), we now suggest and analyze a new iterative algorithm for finding the common element of the solution sets of the general variational inequalities and nonexpansive mappings S and this is the main motivation of this paper.

ALGORITHM 3.1. For a given $z_0 \in H$ and some sequence a_n, $a_n \in [0,1]$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= SP_K z_n && (3.5)\\ z_{n+1} &= (1-a_n)z_n + a_n\{g(u_n) - \rho T u_n\} && (3.6) \end{aligned}$$

where S is a non-expansive operator. For $S = I$, the identity operator, Algorithm 3.1 reduces to the following iterative method for solving variational inequalities (2.1) and appears to be a new one.

ALGORITHM 3.2. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= P_K z_n \\ z_{n+1} &= (1-a_n)z_n + a_n\{g(u_n) - \rho T u_n\}. \end{aligned}$$

For $a_n = 1$ and $S = I$, the identity operator, Algorithm 3.1 collapses to the following iterative method for solving variational inequalities (1), which is mainly due to Noor [9].

ALGORITHM 3.3. For a given $z_0 \in H$, compute the approximate solution z_{n+1} by the iterative schemes

$$\begin{aligned} g(u_n) &= P_K z_n \\ z_{n+1} &= g(u_n) - \rho T u_n. \end{aligned}$$

For the convergence analysis and applications of Algorithm 3.3, see Noor [9] and the references therein.

We consider convergence analysis of Algorithm 3.1 under some mild conditions and this is the main motivation of next result.

THEOREM 3.1. *Let K be a closed convex subset of a real Hilbert space H. Let T be a relaxed (γ, r)-cocoercive and μ-Lipschitzian mapping Let g be a relaxed (γ_1, r_1)-cocoercive and μ_1-Lipschitzian mapping of H into H and S be a nonexpansive mapping of such that $F(S) \cap GWHE(H,T,g,S) \neq \emptyset$. Let $\{z_n\}$ be a sequence defined by Algorithm 3.1, for any initial point $z_0 \in K$, with conditions*

$$\left\| \rho - \frac{r - \gamma\mu^2}{\mu^2} \right\| < \frac{\sqrt{(r-\gamma\mu)^2 - \mu^2 k(2-k)}}{\mu^2}, \tag{3.7}$$
$$r > \gamma\mu^2 + \mu\sqrt{k(2-k)}, \quad k < 1,$$

where

$$k = 2\sqrt{1 + 2\gamma_1\mu_1^2 - 2r_1 + \mu_1^2}, \tag{3.8}$$

$a_n \in [0,1]$ and $\sum_{n=0}^{\infty} a_n = \infty$, then z_n obtained from Algorithm 3.1 converges strongly to $z^* \in F(S) \cap GWHE(H,T,g,S)$.

Proof. Let $z^* \in H$ be a solution of $F(S) \cap GWHE(H,T,g,S)$. Then

$$g(u^*) = SP_K z^* \tag{3.9}$$
$$z^* = (1-a_n)z^* + a_n\{g(u^*) - \rho T u^*\}, \tag{3.10}$$

where $a_n \in [0,1]$ and $u^* \in H$ is a solution of the general variational inequality. To prove the result, we need first to evaluate $||z_{n+1} - z^*||$ for all $n \geqslant 0$. From (3.10), (3.6) and the nonexpansive properties of the projection P_K and the nonexpansive mapping S, we have

$$\begin{aligned}||z_{n+1}-z^*|| &= ||(1-a_n)z_n+a_n\{g(u_n)-\rho Tu_n\}-(1-a_n)z^*-a_n\{g(u^*)-\rho Tu^*\}||\\ &\leqslant (1-a_n)||z_n-z^*||+a_n||g(u_n)-g(u^*)-\rho(Tu_n-Tu^*)||\\ &\leqslant (1-a_n)\|z_n-z^*\|+a_n\|u_n-u^*-\rho(Tu_n-Tu^*)\|\\ &\quad +a_n\|u_n-u^*-(g(u_n)-g(u^*))\|. \end{aligned} \tag{3.11}$$

From the relaxed (γ, r)-cocoercive and μ-Lipschitzian definition on $T,$ we have

$$\begin{aligned}&||u_n-u^*-\rho(Tu_n-Tu^*)||^2\\ &\quad =||u_n-u^*||^2-2\rho\langle Tu_n-Tu^*,u_n-u^*\rangle+\rho^2||Tu_n-Tu^*||^2\\ &\quad \leqslant ||u_n-x^*||^2-2\rho[-\gamma||Tu_n-Tu^*||^2+r||u_n-u^*||^2]+\rho^2||Tu_n-Tu^*||^2\\ &\quad \leqslant ||u_n-u^*||^2+2\rho\gamma\mu^2||u_n-u^*||^2-2\rho r||u_n-u^*||^2+\rho^2\mu^2||u_n-u^*||^2\\ &\quad =[1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2]||u_n-u^*||^2\\ &\quad =\theta_1^2\|u_n-u^*\|^2, \end{aligned} \tag{3.12}$$

where

$$\theta_1=\sqrt{1+2\rho\gamma\mu^2-2\rho r+\rho^2\mu^2}. \tag{3.13}$$

In a similar way, using the relaxed (γ_1, r_1)-cocoercivity and μ_1 Lipschitzian of the operator $g,$ we have

$$\|u_n-u^*-(g(u_n)-g(u^*))\|\leqslant \frac{k}{2}\|u_n-u^*\|, \tag{3.14}$$

where k is defined by (3.8).

Combining (3.11), (3.11) and (3.14), we have

$$\|z_{n+1}-z^*\|\leqslant (1-a_n)\|z_n-z^*\|+a_n(\theta_1+\frac{k}{2})\|z_n-z^*\|. \tag{3.15}$$

Also from (3.5) and (3.9), we have

$$\begin{aligned}\|u_n-u^*\| &\leqslant \|u_n-u^*-(g(u_n)-g(u^*))\|+\|SP_Kz_n-SP_Kz^*\|\\ &\leqslant \frac{k}{2}\|u_n-u^*\|+\|z_n-z^*\|\end{aligned}$$

from which, we have

$$\|u_n-u^*\|\leqslant \frac{1}{1-\frac{k}{2}}\|z_n-z^*\|. \tag{3.16}$$

From (3.15) and (3.16), we obtain that

$$\begin{aligned}||z_{n+1}-z^*|| &\leqslant (1-a_n)||z_n-z^*||+a_n\theta||z_n-z^*||\\ &= \{1-a_n(1-\theta)\}\,||z_n-z^*||,\end{aligned}$$

where

$$\theta = \frac{\frac{k}{2} + \theta_1}{1 - \frac{k}{2}}.$$

From (3.7), it follows that $\theta < 1$. and consequently by Lemma 2.3, we have $\lim_{n\to\infty} ||z_n - z^*|| = 0$, completing the proof. □

REMARK 3.1. For $g = I$, the identity operator, Theorem 3.1 reduces to a result of Noor and Huang [23] for the variational inequalities and nonexpansive mappings.

Now we prove the strong convergence theorem for Algorithm 3.1 under the α-inverse strong monotonicity.

THEOREM 3.3. *Let K be a closed convex subset of a real Hilbert space H. Let $\alpha > 0$ and $\alpha_1 > 0$. Let T be an α-inverse strongly monotonic mapping of H into H. Let g be an α_1-inverse strongly monotonic mapping of H into H. and S be a nonexpansive mapping such that $F(S) \cap GWHE(H,T,g,S) \neq \emptyset$. If*

$$|\rho - \alpha| \leqslant \alpha(1-k), \tag{3.17}$$

where

$$\nu = 2\left(\frac{\alpha_1 - 1}{\alpha_1}\right), \tag{3.18}$$

then the approximate solution z_n obtained from Algorithm 3.1 converges strongly to $z^* \in F(S) \cap GWHE(H,T,g,S)$.

Proof. It is well known that if T is α-inverse strongly monotonic with the constant $\alpha > 0$, then T is $\frac{1}{\alpha}$–Lipschitzian continuous [29, page 419]. Consider

$$\begin{aligned}
&||u_n - u^* - \rho[Tu_n - Tu^*]||^2 \\
&\quad = ||u_n - u^*||^2 + \rho^2||Tu_n - Tu^*||^2 - 2\rho\langle Tu_n - Tx^*, u_n - u^*\rangle \\
&\quad \leqslant ||u_n - u^*||^2 + \rho^2||Tu_n - Tu^*||^2 - 2\rho\alpha||Tu_n - Tu^*||^2 \\
&\quad = ||x_n - x^*||^2 + (\rho^2 - 2\rho\alpha)||Tu_n - Tu^*||^2 \\
&\quad \leqslant ||u_n - u^*||^2 + (\rho^2 - 2\rho\alpha)\cdot\frac{1}{\alpha^2}||u_n - u^*||^2 \\
&\quad = \left(1 + \frac{(\rho^2 - 2\rho\alpha)}{\alpha^2}\right)||u_n - u^*||^2.
\end{aligned} \tag{3.19}$$

In a similar way, using the α_1-inverse strongly monotonicity of g, we have

$$\|u_n - u^* - (g(u_n) - g(u^*))\| \leqslant \nu\|u_n - u^*\|, \tag{3.20}$$

where ν is given by (3.18).

From (3.5), (3.9), (3.19) and (3.20), we have

$$\begin{aligned}
||z_{n+1} - z^*|| &\leqslant (1 - a_n)||z_n - z^*|| + a_n||g(u_n) - g(u^*) - \rho(Tu_n - Tu^*)\| \\
&= [1 - a_n(1 - \theta_2)]||z_n - z^*||.
\end{aligned}$$

where

$$\theta_2 = \sqrt{1 + \frac{\rho^2 - 2\rho\alpha}{\alpha^2}} + \nu. \tag{3.21}$$

From (3.17), it follows that $\theta_2 < 1$ and consequently using Lemma 2.3, we have $\lim_{n\to\infty} ||z_n - z^*|| = 0,$ we obtain the required result. □

4. Applications

In this section we show that the results obtained in Section 3 can be extended for a class of quasi variational inequalities. If the convex set K depends upon the solution explicitly or implicitly, then variational inequality problem is known as the quasi variational inequality. For a given operator $T : H \longrightarrow H,$ and a point-to-set mapping $K : u \longrightarrow K(u),$ which associates a closed convex-valued set $K(u)$ with any element u of $H,$ we consider the problem of finding $u \in K(u)$ such that

$$\langle Tu, v - u\rangle \quad \geqslant 0, \quad \forall v \in K(u). \tag{4.1}$$

Inequality of type (4.1) is called the quasi variational inequality. To convey an idea of the applications of the quasi variational inequalities, we consider the second-order implicit obstacle boundary value problem of finding u such that

$$\left.\begin{array}{ll} -u'' \geqslant f(x) & \text{on } \Omega = [a,b] \\ u \geqslant M(u) & \text{on } \Omega = [a,b] \\ [-u'' - f(x)][u - M(u)] = 0 & \text{on } \Omega = [a,b] \\ u(a) = 0, \quad u(b) = 0. & \end{array}\right\} \tag{4.2}$$

where $f(x)$ is a continuous function and $M(u)$ is the cost (obstacle) function. The prototype encountered is

$$M(u) = k + \inf_i \{u^i\}. \tag{4.3}$$

In (4.3), k represents the switching cost. It is positive when the unit is turned on and equal to zero when the unit is turned off. Note that the operator M provides the coupling between the unknowns $u = (u^1, u^2, \ldots, u^i),$ see [2]. We study the problem (4.2) in the framework of variational inequalities. To do so, we first define the set K as

$$K(u) = \{v : v \in H_0^1(\Omega) : v \geqslant M(u), \quad \text{on } \Omega\},$$

which is a closed convex-valued set in $H_0^1(\Omega)$, where $H_0^1(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (4.2) is

$$\begin{aligned} I[v] &= -\int_a^b \left(\frac{d^2v}{dx^2}\right) v dx - 2\int_a^b f(x)(v)\,dx, \quad \forall v \in K(u) \\ &= \int_a^b \left(\frac{dv}{dx}\right)^2 dx - 2\int_a^b f(x)(v)\,dx \\ &= \langle Tv, v\rangle - 2\langle f, v\rangle \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} \langle Tu, v\rangle &= \int_a^b \left(\frac{d^2u}{dx^2}\right)(v)\,dx = \int_a^b \frac{du}{dx}\frac{dv}{dx}dx \\ \langle f, v\rangle &= \int_a^b f(x)(v)dx. \end{aligned} \tag{4.5}$$

It is clear that the operator T defined by (4.5) is linear, symmetric and positive. Using the technique of Noor [13], one can show that the minimum of the functional $I[v]$ defined by (4.4) associated with the problem (4.2) on the closed convex-valued set $K(u)$ can be characterized by the inequality of the type

$$\langle Tu, v-u\rangle \geqslant \langle f, v-u\rangle, \quad \forall v \in K(u), \tag{4.6}$$

which is exactly the quasi variational inequality (4.1). See also [4–6, 13, 18] for the formulation, applications, numerical methods and sensitivity analysis of the quasi variational inequalities.

Using Lemma 2.1, one can show that the quasi variational inequality (4.1) is equivalent to finding $u \in K(u)$ such that

$$u = P_{K(u)}[u - \rho Tu]. \tag{4.7}$$

In many important applications [2], the convex-valued set $K(u)$ is of the form

$$K(u) = m(u) + K, \tag{4.8}$$

where m is a point-to-point mapping and K is a closed convex set.

From (4.7) and (4.8), we see that problem (4.1) is equivalent to

$$\begin{aligned} u &= P_{K(u)}[u - \rho Tu] = P_{m(u)+K}[u - \rho Tu] \\ &= m(u) + P_K[u - m(u) - \rho Tu] \end{aligned}$$

which implies that

$$g(u) = P_K[g(u) - \rho Tu] \quad \text{with} \quad g(u) = u - m(u),$$

which is equivalent to the general variational inequality (2.1) by an application of Lemma 2.1. We have shown that the quasi variational inequalities (4.1) with the convex-valued set $K(u)$ defined by (4.8) are equivalent to the general variational inequalities (2.1). Using Lemma 3.1, one can show that the quasi variational inequalities are also equivalent to a class of the implicit (quasi) Wiener-Hopf equations, see Noor [9]. Thus all the logorithmic results obtained in this paper continue to hold for quasi variational inequalities (4.1) with $K(u)$ defined by (4.8).

5. Outlook and Conclusions

In this paper, we have shown that the variational inequalities are equivalent to a new class of Wiener-Hopf equations involving the nonexpansive operator. This equivalence is used to suggest and analyze an iterative method for finding the common element of set of the solutions of the general variational inequalities and the set of the fixed-points of the nonexpansive operator. It is worth mentioning that a special case of Algorithm 3.1 has been used by Pitonyak, Shi and Schiller [26] to find the numerical solutions of the obstacle problems. The results are encouraging and perform better than other methods. In addition, Noor, Wang and Xiu [24] have developed a very efficient and robust method using the technique of the Wiener-Hopf equations for solving the variational inequalities. It is interesting to use the technique and idea of this paper to develop some new iterative methods for solving the variational inequalities involving the nonexpansive operators. This is another direction for future work.

Acknowledgement. Authors would like to thank Dr. S. M. Junaid Zaidi, Rector, CIIT, for providing excellent research facilities and the referee for his/her useful comments and suggestions.

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(Received March 21, 2007)
(Revised January 20, 2008)

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