SOME INEQUALITIES ON STATISTICAL SUMMABILITY  \( (C, 1) \)

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Abstract. We prove some inequalities related to the concepts of \( C_1(st) \)-conservative matrices, \( C_1(st) \)-lim sup and \( C_1(st) \)-lim inf which are natural analogues of \( (c, st \cap l_\infty) \)-matrices, \( st \)-lim sup and \( st \)-lim inf respectively.

1. Introduction

Let \( l_\infty \) and \( c \) be the Banach spaces of bounded and convergent sequences of real numbers with the usual supremum norm. Let \( A = (a_{nk}) \), \( n, k \in \mathbb{N} \), be an infinite matrix of real numbers, and let \( x = (x_k) \) be a sequence of real numbers. We write \( Ax = (A_n(x)) \) if \( A_n(x) = \sum_k a_{nk}x_k \) converges for each \( n \). Let \( X \) and \( Y \) be any two sequence spaces. If \( x \in X \) implies \( Ax \in Y \), then we say that the matrix \( A \) maps \( X \) into \( Y \). We denote the class of all matrices \( A \) which map \( X \) into \( Y \) by \( (X, Y) \). If \( X \) and \( Y \) are equipped with \( X \)-lim and \( Y \)-lim, \( A \in (X, Y) \) and \( Y \)-lim \( Ax = X \)-lim \( x \) for all \( x \in X \), then we write \( A \in (X, Y)_{\text{reg}} \).

It is known that \( A \in (c, c) \), that is, \( A \) is conservative if and only if

(i) \(||A|| = \sup_n \sum_k |a_{nk}| < \infty\),
(ii) \( a_k = \lim n a_{nk} \), for each \( k \),
(iii) \( a = \lim n \sum_k a_{nk} \).

If \( A \) is conservative, the number \( \chi = \chi(A) = a - \sum_k a_k \) is called the characteristic of \( A \). \( A \) is said to be regular if and only if (i), (ii) with \( a_k = 0 \) for all \( k \); and (iii) with \( a = 1 \) hold.

Let \( E \subseteq \mathbb{N} \). Natural density \( \delta \) of \( E \) is defined by

\[
\delta(E) = \lim_n \frac{1}{n} |\{k \leq n : k \in E\}|,
\]

where the vertical bars indicate the number of elements in the enclosed set. The sequence \( x = (x_k) \) is said to be statistically convergent to \( L \) if for every \( \epsilon > 0 \),
\[ k : |x_k - L| \geq \epsilon \} = 0 \text{ (cf. Fast [5] and Steinhauss [11]). Statistical convergence for double sequences has been defined and studied by Mursaleen and Edely [9].}

The idea of A-statistical convergence was defined by Kolk [7] and Duman et al [4] used A-statistical convergence for approximating operators.

Recently, Moricz [8] defined the concept of statistical \((C, 1)\) summability as follows. Let \(\sigma_k\) denote the (first) arithmetic means of a sequence \(x = (x_k)\), that is,

\[
\sigma_k = \sigma_k(x) = \frac{1}{k+1} \sum_{j=0}^{k} x_j, \quad k = 0, 1, 2, \ldots
\]

We say that \(x = (x_k)\) is statistically summable \((C, 1)\) to \(L\) if the sequence \(\sigma = (\sigma_k)\) is statistically convergent to \(L\), i.e. \(st\)-lim \(\sigma = L\). We denote by \(C_1(st)\) the set of all sequences which are statistically summable \((C, 1)\).

In this paper we prove some inequalities related to the concepts of \(C_1(st)\)-conservative matrices, \(C_1(st)\)-lim sup and \(C_1(st)\)-lim inf which are natural analogues of \((c, st\cap l_\infty)\)-matrices (cf. Kolk [7]), \(st\)-lim sup and \(st\)-lim inf respectively (cf. Fridy and Orhan [6]). Such type of inequalities are also considered by Çoşkun and Çakan [2], and Çakan and Altay [1].

### 2. Main Result

The following lemma is a consequence of Theorem 1 of Kolk [7].

**Lemma 2.1.** \(A \in (c, C_1(st) \cap l_\infty)\) if and only if

\[
\sup_n \sum_k |a_{nk}| < \infty,
\]

\(C_1(st)\)-lim \(a_{nk} = \alpha_k\) for every \(k\), and

\(C_1(st)\)-lim \(\sum_k a_{nk} = \alpha\).

We call such matrices as \(C_1(st)\)-conservative matrices, and in this case

\[
\kappa = \alpha - \sum_k \alpha_k
\]

is defined which is known as the \(C_1(st)\)-characteristic of \(A\). This number is analogous to the number \(\chi_{st}\) defined by Çoşkun and Çakan [2].

**Theorem 2.1.** Let \(A\) be conservative and \(x \in l_\infty\). Then

\[
\limsup_n \sum_k (a_{nk} - a_k)x_k \leq \frac{\lambda + \chi}{2} \beta(x) + \frac{\lambda - \chi}{2} \alpha(-x) \tag{2.1.1}
\]

for some constant \(\lambda \geq |\chi|\), if and only if

\[
\limsup_n \sum_k |a_{nk} - a_k| \leq \lambda, \tag{2.1.2}
\]
\[
\lim_n \sum_{k \in E} |a_{nk} - a_k| = 0
\]  
(2.1.3)

for every \( E \subseteq \mathbb{N} \) with \( \delta(E) = 0 \), where \( \beta(x) = C_1(st) \cdot \limsup x \) and \( \alpha(x) = C_1(st) \cdot \liminf x \).

Proof. Necessity. Let \( L(x) = \limsup x \) and \( l(x) = \liminf x \). Since \( \beta(x) \leq L(x) \) and \( \alpha(-x) \leq -l(x) \) for all \( x \in l\infty \), we have

\[
\limsup_n \sum_k (a_{nk} - a_k)x_k \leq \frac{\lambda + \chi L(x) - \lambda - \chi l(x)}{2},
\]

and the necessity of (2.1.2) follows from Theorem 1 of Das [3]. Define the matrix \( B = (b_{nk}) \) by

\[
b_{nk} = \begin{cases} 
a_{nk} - a_k & \text{for } k \in E, \\
0 & \text{for } k \notin E.
\end{cases}
\]

Since \( A \) is conservative, the matrix \( B \) satisfies the conditions of Corollary 12 of Simons [10]. Hence there exists a \( y \in l\infty \) such that \( ||y|| \leq 1 \) and

\[
\limsup_n \sum_k |b_{nk}| = \limsup_n \sum_k b_{nk}y_k.
\]  
(2.1.4)

Now, let \( y = (y_k) \) be defined by

\[
y_k = \begin{cases} 
1 & \text{for } k \in E, \\
0 & \text{for } k \notin E.
\end{cases}
\]

So that, \( C_1(st) \cdot \lim y = \beta(y) = \alpha(y) = 0 \); and by (2.1.1) and (2.1.4) we have

\[
\limsup_n \sum_{k \in E} |a_{nk} - a_k| \leq \frac{\lambda + \chi \beta(y) - \lambda - \chi \alpha(-y)}{2} = 0
\]

and we get (2.1.3).

Sufficiency. Let \( x \in l\infty \). Write \( E_1 = \{ k : \sigma_k > \beta(x) + \epsilon \} \) and \( E_2 = \{ k : \sigma_k < \alpha(x) - \epsilon \} \). Then we have \( \delta(E_1) = \delta(E_2) = 0 \); and hence \( \delta(E) = 0 \) for \( E = E_1 \cap E_2 \).

We can write

\[
\sum_k (a_{nk} - a_k)x_k = \sum_{k \in E} (a_{nk} - a_k)x_k + \sum_{k \notin E} (a_{nk} - a_k)^+ x_k - \sum_{k \notin E} (a_{nk} - a_k)^- x_k,
\]

where \( \lambda^+ = \max\{0, \lambda\}, \lambda^- = \max\{-\lambda, 0\} \). Hence

\[
\limsup_n \sum_k (a_{nk} - a_k)x_k \leq \limsup_n \sum_{k \in E} |a_{nk} - a_k||x_k| + \limsup_n \sum_{k \notin E} (a_{nk} - a_k)^+ \sigma_k
\]

\[
+ \limsup_n \left[ -\sum_{k \notin E} (a_{nk} - a_k)^- \sigma_k \right]
\]

\[
= I_1(x) + I_2(x) + I_3(x).
\]
From condition (2.1.3), we have $I_1(x) = 0$. Let $\epsilon > 0$, then there is a set $E$ as defined above such that for $k \not\in E$,
\[ \alpha(x) - \epsilon \leq \sigma_k \leq \beta(x) + \epsilon, \quad \beta(-x) - \epsilon \leq -\sigma_k \leq \alpha(-x) + \epsilon. \] (2.1.5)
Therefore from conditions (2.1.2) and (2.1.5) and Lemma 1 of Das [3], we get
\[ I_2(x) \leq \frac{\lambda + \chi}{2} (\beta(x) + \epsilon) \]
\[ I_3(x) \geq \frac{\lambda - \chi}{2} (\alpha(-x) + \epsilon). \]
Hence we get
\[ \limsup_n \sum_k (a_{nk} - a_k)x_k \leq \frac{\lambda + \chi}{2} \beta(x) + \frac{\lambda - \chi}{2} \alpha(-x) + \lambda \epsilon, \]
since $\epsilon$ was arbitrary. This completes the proof of the theorem. \[\square\]

To prove our next theorem, we need the following lemma which is $C_1(st)$-analogue of a result of Çoşkun and Çakan [2].

**Lemma 2.2.** Let $\|A\| < \infty$ and $C_1(st)$-$\lim_n |a_{nk}| = 0$. Then there exists a $y \in l_\infty$ such that $\|y\| \leq 1$ and
\[ C_1(st) -$\limsup_n \sum_k a_{nk}y_k = C_1(st) -$\limsup_n \sum_k |a_{nk}|. \]

The following lemma is derived by replacing the functional $st$-$\lim$ by $C_1(st)$-$\lim$ in Lemma 2.3 of Çoşkun and Çakan [2].

**Lemma 2.3.** Let $A$ be $C_1(st)$-conservative and $\lambda > 0$. Then
\[ C_1(st) -$\limsup_n \sum_k |a_{nk} - \alpha_k| \leq \frac{\lambda + \kappa}{2} \text{ if and only if } C_1(st) -$\limsup_n \sum_k (a_{nk} - \alpha_k)^+ \leq \frac{\lambda + \kappa}{2} \text{ and } C_1(st) -$\limsup_n \sum_k (a_{nk} - \alpha_k)^- \leq \frac{\lambda - \kappa}{2}. \]

**Theorem 2.2.** Let $A$ be $C_1(st)$-conservative. Then, for some constant $\lambda \geq |\kappa|$ and for all $x \in l_\infty$,
\[ C_1(st) -$\limsup_n \sum_k (a_{nk} - \alpha_k)x_k \leq \frac{\lambda + \kappa}{2} L(x) - \frac{\lambda - \kappa}{2} l(x) \] (2.2.1)
if and only if
\[ C_1(st) -$\limsup_n \sum_k |a_{nk} - \alpha_k| \leq \lambda. \] (2.2.2)
Proof. Necessity. If we define the matrix \( B = (b_{nk}) \) by \( b_{nk} = a_{nk} - \alpha_k \) for all \( n, k \), then, since \( A \) is \( C_1 \text{(st)} \)-conservative, the matrix \( B \) satisfies the hypothesis of Lemma 2.2. Hence for a \( y \in l_\infty \) such that \( ||y|| \leq 1 \) we have

\[
C_1 \text{(st)} - \limsup \sum_n b_{nk} = C_1 \text{(st)} - \limsup \sum_k b_{nk} y_k.
\]

Using (2.2.1), we get

\[
C_1 \text{(st)} - \lim sup \sum_n |b_{nk}| \leq \frac{\lambda + \kappa}{2} L(y) - \frac{\lambda - \kappa}{2} l(y)
\]

\[
\leq \left( \frac{\lambda + \kappa}{2} + \frac{\lambda - \kappa}{2} \right) ||y||
\]

\[
\leq \lambda, \quad \text{since} \ ||y|| \leq 1.
\]

Hence (2.2.2) holds.

Sufficiency. As in Theorem 2.1, for some \( k_0 \in \mathbb{N} \ (k > k_0) \), we can write

\[
\sum (a_{nk} - a_k) x_k = \sum_{k \leq k_0} (a_{nk} - a_k) x_k + \sum_{k > k_0} (a_{nk} - a_k)^+ x_k - \sum_{k > k_0} (a_{nk} - a_k)^- x_k.
\]

Since for any \( \epsilon > 0 \), \( l(x) - \epsilon < x_k < L(x) + \epsilon \); and \( A \) is \( C_1 \text{(st)} \)-conservative, we get by Lemma 2.3 that

\[
C_1 \text{(st)} - \lim sup \sum_n (a_{nk} - a_k) x_k \leq (L(x) + \epsilon) \left( \frac{\lambda + \kappa}{2} \right) - (l(x) - \epsilon) \left( \frac{\lambda - \kappa}{2} \right)
\]

\[
= \frac{\lambda + \kappa}{2} L(x) - \frac{\lambda - \kappa}{2} l(x) + \lambda \epsilon,
\]

which gives (2.2.1), since \( \epsilon \) was arbitrary.

This completes the proof of the theorem.

\[\square\]

THEOREM 2.3.. Let \( A \) be \( C_1 \text{(st)} \)-conservative. Then, for some constant \( \lambda \geq |\kappa| \) and for all \( x \in l_\infty \),

\[
C_1 \text{(st)} - \lim sup \sum_k (a_{nk} - a_k) x_k \leq \frac{\lambda + \kappa}{2} \beta(x) + \frac{\lambda - \kappa}{2} \alpha(-x)
\]

if and only if (2.2.2) holds and

\[
C_1 \text{(st)} - \lim \sum_{k \in E} |a_{nk} - a_k| = 0
\]

for every \( E \subseteq \mathbb{N} \) with \( \delta(E) = 0 \).

Proof. Necessity. Let (2.3.1) hold. Since \( \beta(x) \leq L(x) \) and \( \alpha(-x) \leq -l(x) \), (2.2.2) follows from Theorem 2.2. Now let us show the necessity of (2.3.2). For any
$E \subseteq \mathbb{N}$ with $\delta(E) = 0$, let us define a matrix $B = (b_{nk})$ as follows

$$b_{nk} = \begin{cases} \alpha_k & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

Then, it is clear that $B$ satisfies the conditions of Lemma 2.2 and hence there exists a $y \in l_\infty$ such that $\|y\| \leq 1$ and

$$C_1(st) - \limsup_n \sum_k b_{nk}y_k = C_1(st) - \limsup_n \sum_k |b_{nk}|.$$

Let us define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} 1 & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

Using the fact that $C_1(st) - \lim y = \beta(y) = \alpha(y) = 0$ and (2.3.1), we get

$$C_1(st) - \limsup_n \sum_{k \in E} |a_{nk} - \alpha_k| \leq \frac{\lambda + \kappa}{2} \beta(y) + \frac{\lambda - \kappa}{2} \alpha(-y) = 0,$$

and hence we get (2.3.2).

**Sufficiency.** Let (2.2.2) and (2.3.2) hold and $x \in l_\infty$. As in Theorem 2.1, we can write

$$\sum_k (a_{nk} - \alpha_k)x_k = \sum_{k \in E} (a_{nk} - \alpha_k)x_k + \sum_{k \notin E} (a_{nk} - \alpha_k)^+x_k - \sum_{k \notin E} (a_{nk} - \alpha_k)^-x_k.$$

Using Lemma 2.3 and $C_1(st)$ - conservativeness of $A$, we have

$$C_1(st) - \limsup_n \sum_k (a_{nk} - \alpha_k)x_k \leq \frac{\lambda + \kappa}{2} \beta(x) + \frac{\lambda - \kappa}{2} \alpha(-x) + \lambda \epsilon.$$

But $\epsilon$ was arbitrary, so (2.3.1) holds.

This completes the proof of theorem.

\[\Box\]

**REFERENCES**


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