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SOME INEQUALITIES ON STATISTICAL SUMMABILITY $(C,1)$

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Abstract. We prove some inequalities related to the concepts of $C_1(st)$-conservative matrices, $C_1(st)$-$\limsup$ and $C_1(st)$-$\liminf$ which are natural analogues of $(c, st\cap l_\infty)$-matrices, st-$\limsup$ and st-$\liminf$ respectively.

1. Introduction

Let l_∞ and c be the Banach spaces of bounded and convergent sequences of real numbers with the usual supremum norm. Let $A=(a_{nk})$, $n,k\in\mathbb{N}$, be an infinite matrix of real numbers, and let $x=(x_k)$ be a sequence of real numbers. We write $Ax=(A_n(x))$ if $A_n(x)=\sum_k a_{nk}x_k$ converges for each n. Let X and Y be any two sequence spaces. If $x\in X$ implies $Ax\in Y$, then we say that the matrix A maps X into Y. We denote the class of all matrices A which map X into Y by (X,Y). If X and Y are equipped with X-$\lim$ and Y-$\lim$, $A\in(X,Y)$ and Y-$\lim Ax = X$-$\lim x$ for all $x\in X$, then we write $A\in(X,Y)_{reg}$.

It is known that $A\in(c,c)$, that is, A is *conservative* if and only if

(i) $||A||=\sup_n\sum_k|a_{nk}|<\infty$,

(ii) $a_k=\lim_n a_{nk}$, for each k,

(iii) $a=\lim_n\sum_k a_{nk}$.

If A is conservative, the number $\chi=\chi(A)=a-\sum_k a_k$ is called the *characteristic* of A. A is said to be *regular* if and only if (i), (ii) with $a_k=0$ for all k; and (iii) with $a=1$ hold.

Let $E\subseteq\mathbb{N}$. Natural density δ of E is defined by

$$\delta(E)=\lim_n\frac{1}{n}|\{k\leqslant n: k\in E\}|,$$

where the vertical bars indicate the number of elements in the enclosed set. The sequence $x=(x_k)$ is said to be *statistically convergent* to L if for every $\epsilon>0$,

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$\delta\{k : |x_k - L| \geqslant \epsilon\} = 0$ (cf. Fast [5] and Steinhauss [11]). Statistical convergence for double sequences has been defined and studied by Mursaleen and Edely [9].
The idea of A-statistical convergence was defined by Kolk [7] and Duman et al [4] used A-statistical convergence for approximating operators.

Recently, Moricz [8] defined the concept of statistical $(C,1)$ summability as follows. Let σ_k denote the (first) arithmetic means of a sequence $x = (x_k)$, that is,

$$\sigma_k = \sigma_k(x) = \frac{1}{k+1}\sum_{j=0}^{k} x_j, \quad k = 0, 1, 2....$$

We say that $x = (x_k)$ is *statistically summable* $(C,1)$ to L if the sequence $\sigma = (\sigma_k)$ is statistically convergent to L, i.e. $st\text{-}\lim \sigma = L$. We denote by $C_1(st)$ the set of all sequences which are statistically summable $(C,1)$.

In this paper we prove some inequalities related to the concepts of $C_1(st)$-conservative matrices, $C_1(st)\text{-}\limsup$ and $C_1(st)\text{-}\liminf$ which are natural analogues of $(c, st\cap l_\infty)$-matrices (cf. Kolk [7]), $st\text{-}\limsup$ and $st\text{-}\liminf$ respectively (cf. Fridy and Orhan [6]). Such type of inequalities are also considered by Çoşkun and Çakan [2], and Çakan and Altay [1].

2. Main Result

The following lemma is a consequence of Theorem 1 of Kolk [7].

LEMMA 2.1. *$A \in (c, C_1(st) \cap l_\infty)$ if and only if*

$$\sup_n \sum_k |a_{nk}| < \infty,$$

$$C_1(st)\text{-}\lim_n a_{nk} = \alpha_k \ \text{ for every } k, \ \text{ and}$$

$$C_1(st)\text{-}\lim_n \sum_k a_{nk} = \alpha.$$

We call such matrices as $C_1(st)$-conservative matrices, and in this case

$$\kappa = \alpha - \sum_k \alpha_k$$

is defined which is known as the $C_1(st)$-characteristic of A. This number is analogous to the number χ_{st} defined by Çoşkun and Çakan [2].

THEOREM 2.1. *Let A be conservative and $x \in l_\infty$. Then*

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \frac{\lambda + \chi}{2}\beta(x) + \frac{\lambda - \chi}{2}\alpha(-x) \tag{2.1.1}$$

for some constant $\lambda \geqslant |\chi|$, if and only if

$$\limsup_n \sum_k |a_{nk} - a_k| \leqslant \lambda, \tag{2.1.2}$$

$$\lim_n \sum_{k\in E} |a_{nk} - a_k| = 0 \tag{2.1.3}$$

for every $E \subseteq \mathbb{N}$ *with* $\delta(E) = 0$, *where* $\beta(x) = C_1(st)$ - $\limsup x$ *and* $\alpha(x) = C_1(st)$ - $\liminf x$.

Proof. Necessity. Let $L(x) = \limsup x$ and $l(x) = \liminf x$. Since $\beta(x) \leqslant L(x)$ and $\alpha(-x) \leqslant -l(x)$ for all $x \in l_\infty$, we have

$$\limsup_n \sum_k (a_{nk} - a_k)x_k \leqslant \frac{\lambda+\chi}{2} L(x) - \frac{\lambda-\chi}{2} l(x),$$

and the necessity of (2.1.2) follows from Theorem 1 of Das [3]. Define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} a_{nk} - a_k & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

Since A is conservative, the matrix B satisfies the conditions of Corollary 12 of Simons [10]. Hence there exists a $y \in l_\infty$ such that $||y|| \leqslant 1$ and

$$\limsup_n \sum_k |b_{nk}| = \limsup_n \sum_k b_{nk} y_k. \tag{2.1.4}$$

Now, let $y = (y_k)$ be defined by

$$y_k = \begin{cases} 1 & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

So that, $C_1(st)$- $\lim y = \beta(y) = \alpha(y) = 0$; and by (2.1.1) and (2.1.4) we have

$$\limsup_n \sum_{k\in E} |a_{nk} - a_k| \leqslant \frac{\lambda+\chi}{2}\beta(y) + \frac{\lambda-\chi}{2}\alpha(-y) = 0$$

and we get (2.1.3).

Sufficiency. Let $x \in l_\infty$. Write $E_1 = \{k : \sigma_k > \beta(x) + \epsilon\}$ and $E_2 = \{k : \sigma_k < \alpha(x) - \epsilon\}$. Then we have $\delta(E_1) = \delta(E_2) = 0$; and hence $\delta(E) = 0$ for $E = E_1 \cap E_2$. We can write

$$\sum_k (a_{nk} - a_k)x_k = \sum_{k\in E} (a_{nk} - a_k)x_k + \sum_{k\notin E} (a_{nk} - a_k)^+ x_k - \sum_{k\notin E} (a_{nk} - a_k)^- x_k,$$

where $\lambda^+ = \max\{0,\lambda\}, \lambda^- = \max\{-\lambda, 0\}$. Hence

$$\begin{aligned} \limsup_n \sum_k (a_{nk} - a_k)x_k &\leqslant \limsup_n \sum_{k\in E} |a_{nk} - a_k||x_k| + \limsup_n \sum_{k\notin E} (a_{nk} - a_k)^+ \sigma_k \\ &\quad + \limsup_n \left[- \sum_{k\notin E} (a_{nk} - a_k)^- \sigma_k \right] \\ &= I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

From condition (2.1.3), we have $I_1(x) = 0$. Let $\epsilon > 0$, then there is a set E as defined above such that for $k \notin E$,

$$\alpha(x) - \epsilon \leqslant \sigma_k \leqslant \beta(x) + \epsilon, \quad \beta(-x) - \epsilon \leqslant -\sigma_k \leqslant \alpha(-x) + \epsilon. \tag{2.1.5}$$

Therefore from conditions (2.1.2) and (2.1.5) and Lemma 1 of Das [3], we get

$$I_2(x) \leqslant \frac{\lambda + \chi}{2}(\beta(x) + \epsilon)$$
$$I_3(x) \geqslant \frac{\lambda - \chi}{2}(\alpha(-x) + \epsilon).$$

Hence we get

$$\limsup_n \sum_k (a_{nk} - a_k)x_k < \frac{\lambda + \chi}{2}\beta(x) + \frac{\lambda - \chi}{2}\alpha(-x) + \lambda\epsilon,$$
$$\leqslant \frac{\lambda + \chi}{2}\beta(x) + \frac{\lambda - \chi}{2}\alpha(-x),$$

since ϵ was arbitrary. This completes the proof of the theorem. □

To prove our next theorem, we need the following lemma which is $C_1(st)$-analogue of a result of Çoşkun and Çakan [2].

LEMMA 2.2. *Let* $||A|| < \infty$ *and* $C_1(st)\text{-}\lim_n |a_{nk}| = 0$. *Then there exists a* $y \in l_\infty$ *such that* $||y|| \leqslant 1$ *and*

$$C_1(st)\text{-}\limsup \sum_k a_{nk}y_k = C_1(st)\text{-}\limsup \sum_k |a_{nk}|.$$

The following lemma is derived by replacing the functional st-lim by $C_1(st)$-lim in Lemma 2.3 of Çoşkun and Çakan [2].

LEMMA 2.3. *Let* A *be* $C_1(st)$*-conservative and* $\lambda > 0$. *Then* $C_1(st)\text{-}\limsup_n \sum_k |a_{nk} - \alpha_k| \leqslant \lambda$ *if and only if* $C_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)^+ \leqslant \frac{\lambda+\kappa}{2}$ *and* $C_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)^- \leqslant \frac{\lambda-\kappa}{2}$.

THEOREM 2.2. *Let* A *be* $C_1(st)$*-conservative. Then, for some constant* $\lambda \geqslant |\kappa|$ *and for all* $x \in l_\infty$,

$$C_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k)x_k \leqslant \frac{\lambda + \kappa}{2}L(x) - \frac{\lambda - \kappa}{2}l(x) \tag{2.2.1}$$

if and only if

$$C_1(st)\text{-}\limsup_n \sum_k |a_{nk} - \alpha_k| \leqslant \lambda. \tag{2.2.2}$$

Proof. Necessity. If we define the matrix $B=(b_{nk})$ by $b_{nk}=a_{nk}-\alpha_k$ for all n,k, then, since A is $C_1(st)$-conservative, the matrix B satisfies the hypothesis of Lemma 2.2. Hence for a $y\in l_\infty$ such that $||y||\leqslant 1$ we have

$$C_1(st)\text{-}\limsup_n \sum_k |b_{nk}| = C_1(st)\text{-}\limsup_n \sum_k b_{nk}y_k.$$

Using (2.2.1), we get

$$\begin{aligned} C_1(st)\text{-}\limsup_n \sum_k |b_{nk}| &\leqslant \frac{\lambda+\kappa}{2}L(y)-\frac{\lambda-\kappa}{2}l(y)\\ &\leqslant \left(\frac{\lambda+\kappa}{2}+\frac{\lambda-\kappa}{2}\right)||y||\\ &\leqslant \lambda, \quad \text{since } ||y||\leqslant 1. \end{aligned}$$

Hence (2.2.2) holds.

Sufficiency. As in Theorem 2.1, for some $k_0\in\mathbb{N}$ $(k>k_0)$, we can write

$$\sum_k (a_{nk}-a_k)x_k=\sum_{k\leqslant k_0}(a_{nk}-a_k)x_k+\sum_{k>k_0}(a_{nk}-a_k)^+x_k-\sum_{k>k_0}(a_{nk}-a_k)^-x_k.$$

Since for any $\epsilon>0$, $l(x)-\epsilon<x_k<L(x)+\epsilon$; and A is $C_1(st)$-conservative, we get by Lemma 2.3 that

$$\begin{aligned} C_1(st)\text{-}\limsup_n \sum_k (a_{nk}-\alpha_k)x_k &\leqslant (L(x)+\epsilon)\left(\frac{\lambda+\kappa}{2}\right)-(l(x)-\epsilon)\left(\frac{\lambda-\kappa}{2}\right)\\ &=\frac{\lambda+\kappa}{2}L(x)-\frac{\lambda-\kappa}{2}l(x)+\lambda\epsilon, \end{aligned}$$

which gives (2.2.1), since ϵ was arbitrary.

This completes the proof of the theorem. □

THEOREM 2.3.. *Let A be $C_1(st)$-conservative. Then, for some constant $\lambda\geqslant|\kappa|$ and for all $x\in l_\infty$,*

$$C_1(st)\text{-}\limsup_n \sum_k (a_{nk}-\alpha_k)x_k \leqslant \frac{\lambda+\kappa}{2}\beta(x)+\frac{\lambda-\kappa}{2}\alpha(-x) \tag{2.3.1}$$

if and only if (2.2.2) holds and

$$C_1(st)\text{-}\lim_n \sum_{k\in E} |\ a_{nk}-\alpha_k\ |=0 \tag{2.3.2}$$

for every $E\subseteq\mathbb{N}$ with $\delta(E)=0$.

Proof. Necessity. Let (2.3.1) hold. Since $\beta(x)\leqslant L(x)$ and $\alpha(-x)\leqslant -l(x)$, (2.2.2) follows from Theorem 2.2. Now let us show the necessity of (2.3.2). For any

$E \subseteq \mathbb{N}$ with $\delta(E) = 0$, let us define a matrix $B = (b_{nk})$ as follows

$$b_{nk} = \begin{cases} a_{nk} - \alpha_k & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

Then, it is clear that B satisfies the conditions of Lemma 2.2 and hence there exists a $y \in l_\infty$ such that $||y|| \leqslant 1$ and

$$C_1(st)\text{-}\limsup_n \sum_k b_{nk} y_k = C_1(st)\text{-}\limsup_n \sum_k |b_{nk}|.$$

Let us define the sequence $y = (y_k)$ by

$$y_k = \begin{cases} 1 & \text{for } k \in E, \\ 0 & \text{for } k \notin E. \end{cases}$$

Using the fact that $C_1(st)\text{-}\lim y = \beta(y) = \alpha(y) = 0$ and (2.3.1), we get

$$C_1(st)\text{-}\limsup_n \sum_{k \in E} |a_{nk} - \alpha_k| \leqslant \frac{\lambda + \kappa}{2} \beta(y) + \frac{\lambda - \kappa}{2} \alpha(-y) = 0,$$

and hence we get (2.3.2).

Sufficiency. Let (2.2.2) and (2.3.2) hold and $x \in l_\infty$. As in Theorem 2.1, we can write

$$\sum_k (a_{nk} - \alpha_k) x_k = \sum_{k \in E} (a_{nk} - \alpha_k) x_k + \sum_{k \notin E} (a_{nk} - \alpha_k)^+ x_k - \sum_{k \notin E} (a_{nk} - \alpha_k)^- x_k.$$

Using Lemma 2.3 and $C_1(st)$-conservativeness of A, we have

$$C_1(st)\text{-}\limsup_n \sum_k (a_{nk} - \alpha_k) x_k \leqslant \frac{\lambda + \kappa}{2} \beta(x) + \frac{\lambda - \kappa}{2} \alpha(-x) + \lambda \epsilon.$$

But ϵ was arbitrary, so (2.3.1) holds.

This completes the proof of theorem. □

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