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BOUNDS ON CERTAIN NONLINEAR DISCRETE INEQUALITIES

WEIHONG SHENG AND WEI NIAN LI

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Abstract. Our aim in this paper is to investigate certain nonlinear discrete inequalities which provide explicit bounds on unknown functions. The inequalities given here can be used as tools in applications in the theory of certain finite difference and sum–difference equations.

1. Introduction

It is well known that the discrete inequalities play a fundamental role in the development of the theory of finite difference equations. During the past few years, a number of discrete inequalities had been established by many scholars, which are motivated by certain applications. For example, we refer the reader to literatures [1–7] and the references therein. In many cases, however, when studying the behavior of solutions of certain classes of finite difference and sum–difference equations, the bounds provided by the earlier inequalities are inadequate in applications and we need some new and specific type of finite difference inequalities.

In this paper, we investigate certain new nonlinear discrete inequalities, which can be used as tools in applications in the theory of certain finite difference and sum–difference equations. Our paper gives, in some sense, an extension of a result of Pachpatte [2].

2. Main Results

In what follows, R denotes the set of real numbers and $R_+ = [0,\infty)$ is the given subset of R, and $N_0 = \{0,1,2,\cdots\}$ denotes the set of nonnegative integers. We use the usual conventions that empty sums and products are taken to be 0 and 1 respectively. Throughout this paper, all the functions which appear in the inequalities are assumed to be real-valued and all the sums involved exist on the respective domains of their definitions, p and q are constants, and $p \geqslant 1,\ 0 < q \leqslant p$.

We firstly introduce two lemmas, which are useful in our main results.

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LEMMA 1. (Bernoulli's inequality) [8]. *Let* $0 < \alpha \leqslant 1$ *and* $x > -1$. *Then* $(1+x)^{\alpha} \leqslant 1+\alpha x$.

LEMMA 2. [3]. *Assume that* $u(n)$, $a(n)$, $b(n)$ *are nonnegative functions defined for* $n \in N_0$, *and* $a(n)$ *is nonincreasing for* $n \in N_0$. *If*

$$u(n) \leqslant a(n) + \sum_{s=n+1}^{\infty} b(s)u(s), \quad n \in N_0,$$

then

$$u(n) \leqslant a(n) \prod_{s=n+1}^{\infty} \Big(1+b(s)\Big), \quad n \in N_0.$$

Next, we establish our main results.

THEOREM 3. *Assume that* $a(n) > 0$ *and* $x(n)$, $m(n)$, $f(n)$, $g(n)$ *are nonnegative functions defined for* $n \in N_0$. *Then the inequality*

$$x^p(n) \leqslant a(n) + m(n) \sum_{s=n+1}^{\infty} \Big[f(s)x^p(s) + g(s)x^q(s)\Big], \quad n \in N_0, \tag{$I1$}$$

implies

$$x(n) \leqslant a^{\frac{1}{p}}(n) + \frac{1}{p}a^{\frac{1}{p}-1}(n)m(n)h(n) \prod_{s=n+1}^{\infty} \Big(1+F(s)\Big), \; n \in N_0, \tag{1}$$

where

$$h(n) = \sum_{s=n+1}^{\infty} \Big[f(s)a(s) + g(s)a^{\frac{q}{p}}(s)\Big], \tag{2}$$

and

$$F(n) = m(n)\Big(f(n) + \frac{q}{p}a^{\frac{q}{p}-1}(n)g(n)\Big). \tag{3}$$

Proof. Define a function $u(n)$ by

$$u(n) = \sum_{s=n+1}^{\infty} \Big[f(s)x^p(s) + g(s)x^q(s)\Big], \quad n \in N_0. \tag{4}$$

Then $(I1)$ can be restated as

$$x^p(n) \leqslant a(n) + m(n)u(n) = a(n)\Big(1 + \frac{m(n)u(n)}{a(n)}\Big). \tag{5}$$

Using Lemma 1, from (5), we easily obtain

$$x(n) \leqslant a^{\frac{1}{p}}(n) + \frac{1}{p}a^{\frac{1}{p}-1}(n)m(n)u(n), \tag{6}$$

and

$$x^q(n) \leqslant a^{\frac{q}{p}}(n) + \frac{q}{p} a^{\frac{q}{p}-1}(n) m(n) u(n). \tag{7}$$

Combining (4), (5) and (7), we have

$$\begin{aligned} u(n) &\leqslant \sum_{s=n+1}^{\infty} \left[f(s)\Big(a(s) + m(s)u(s)\Big) + g(s)\Big(a^{\frac{q}{p}}(s) + \frac{q}{p} a^{\frac{q}{p}-1}(s) m(s) u(s)\Big)\right] \\ &= h(n) + \sum_{s=n+1}^{\infty} F(s) u(s), \quad n \in N_0, \end{aligned} \tag{8}$$

where $h(n)$ and $F(n)$ are defined by (2) and (3) respectively. Obviously, $h(n)$ is nonnegative and nonincreasing for $n \in N_0$. By Lemma 2, from (8), we have

$$u(n) \leqslant h(n) \prod_{s=n+1}^{\infty} \Big(1 + F(s)\Big). \tag{9}$$

It is easy to see that the desired inequality (1) follows from (6) and (9). This completes the proof of Theorem 3. □

THEOREM 4. *Assume that* $a(n) > 0$, $x(n)$, $m(n)$, $b(n)$ *are nonnegative functions for* $n \in N_0$, *and* $L : N_0 \times R_+ \to R_+$. *If*

$$0 \leqslant L(n,x) - L(n,y) \leqslant K(n,y)(x-y), \tag{10}$$

for $x \geqslant y \geqslant 0$, *where* $K : N_0 \times R_+ \to R_+$, *then the inequality*

$$x^p(n) \leqslant a(n) + m(n) \sum_{s=n+1}^{\infty} \Big[b(s) x^q(s) + L(s, x(s)) \Big], \quad n \in N_0, \tag{$I2$}$$

implies

$$x(n) \leqslant a^{\frac{1}{p}}(n) + \frac{1}{p} a^{\frac{1}{p}-1}(n) m(n) G(n) \prod_{s=n+1}^{\infty} \Big(1 + H(s)\Big), \ n \in N_0, \tag{11}$$

where

$$G(n) = \sum_{s=n+1}^{\infty} \left[b(s) a^{\frac{q}{p}}(s) + L\Big(s, a^{\frac{1}{p}}(s)\Big)\right], \tag{12}$$

and

$$H(n) = \left[\frac{q}{p} b(n) a^{\frac{q}{p}-1}(n) + K\Big(n, a^{\frac{1}{p}}(n)\Big) \frac{1}{p} a^{\frac{1}{p}-1}(n) \right] m(n). \tag{13}$$

Proof. Define a function $u(n)$ by

$$u(n) = \sum_{s=n+1}^{\infty} \Big[b(s) x^q(s) + L(s, x(s)) \Big]. \tag{14}$$

Then, as in the proof of Theorem 3, from $(I2)$, we obtain (5)–(7). Therefore, we have

$$\sum_{s=n+1}^{\infty} b(s)x^q(s) \leqslant \sum_{s=n+1}^{\infty} b(s)\left(a^{\frac{q}{p}}(s) + \frac{q}{p}a^{\frac{q}{p}-1}(s)m(s)u(s)\right) \tag{15}$$

and

$$\begin{aligned}\sum_{s=n+1}^{\infty} L(s,x(s)) &\leqslant \sum_{s=n+1}^{\infty}\left\{L\left(s,a^{\frac{1}{p}}(s)+\frac{1}{p}a^{\frac{1}{p}-1}(s)m(s)u(s)\right)\right.\\ &\qquad \left.-L\left(s,a^{\frac{1}{p}}(s)\right)+L\left(s,a^{\frac{1}{p}}(s)\right)\right\}\\ &\leqslant \sum_{s=n+1}^{\infty} L\left(s,a^{\frac{1}{p}}(s)\right)+\sum_{s=n+1}^{\infty} K\left(s,a^{\frac{1}{p}}(s)\right)\frac{1}{p}a^{\frac{1}{p}-1}(s)m(s)u(s).\end{aligned} \tag{16}$$

It follows from (14)–(16) that

$$\begin{aligned}u(n) &\leqslant \sum_{s=n+1}^{\infty}\left[b(s)a^{\frac{q}{p}}(s)+L\left(s,a^{\frac{1}{p}}(s)\right)\right]\\ &\quad+\sum_{s=n+1}^{\infty}\left[\frac{q}{p}b(s)a^{\frac{q}{p}-1}(s)m(s)+K\left(s,a^{\frac{1}{p}}(s)\right)\frac{1}{p}a^{\frac{1}{p}-1}(s)m(s)\right]u(s) \qquad (17)\\ &= G(n)+\sum_{s=n+1}^{\infty} H(s)u(s),\end{aligned}$$

where $G(n)$ and $H(n)$ are defined by (12) and (13) respectively.

It is obvious that $G(n)$ is nonnegative and nonincreasing for $n \in N_0$. Using Lemma 2, from (17), we have

$$u(n) \leqslant G(n)\prod_{s=n+1}^{\infty}\left(1+H(s)\right). \tag{18}$$

Combining (6) and (18), we obtain the desired inequality (11). The proof is complete. $\square$

THEOREM 5. *Let $a(n)$, $x(n)$, $b(n)$, $L(n,s)$ and $K(n,s)$ be the same as in Theorem 4. If the condition (10) holds, then the inequality*

$$x^p(n) \leqslant a(n) + \sum_{s=n+1}^{\infty} b(s)x^p(s) + \sum_{s=n+1}^{\infty} L(s,x^q(s)), \quad n \in N_0, \tag{$I3$}$$

implies

$$x(n) \leqslant B^{\frac{1}{p}}(n)\left(a^{\frac{1}{p}}(n)+\frac{1}{p}a^{\frac{1}{p}-1}(n)J(n)M(n)\right), \ n \in N_0 \tag{19}$$

where

$$B(n) = \prod_{s=n+1}^{\infty}\left(1+b(s)\right), \tag{20}$$

$$J(n) = \sum_{s=n+1}^{\infty} L\left(s, B^{\frac{q}{p}}(s)a^{\frac{q}{p}}(s)\right), \tag{21}$$

and

$$M(n) = \prod_{s=n+1}^{\infty} \left(1 + K\left(s, B^{\frac{q}{p}}(s)a^{\frac{q}{p}}(s)\right)B^{\frac{q}{p}}(s)\frac{q}{p}a^{\frac{q}{p}-1}(s)\right). \tag{22}$$

Proof. Define a function $u(n)$ by

$$u(n) = a(n) + v(n), \tag{23}$$

where

$$v(n) = \sum_{s=n+1}^{\infty} L(s, x^q(s)). \tag{24}$$

Then $(I3)$ can be restated as

$$x^p(n) \leqslant u(n) + \sum_{s=n+1}^{\infty} b(s)x^p(s). \tag{25}$$

We easily see that $u(n)$ is a nonnegative and nonincreasing function for $n \in N_0$. Therefore, by Lemma 2, it follows from (25) that

$$x^p(n) \leqslant B(n)u(n),$$

i.e.,

$$x^p(n) \leqslant B(n)(a(n) + v(n)) = B(n)a(n)\left(1 + \frac{v(n)}{a(n)}\right), \tag{26}$$

where $B(n)$ is defined by (20). Using Lemma 1, from (26) we have

$$x(n) \leqslant B^{\frac{1}{p}}(n)\left(a^{\frac{1}{p}}(n) + \frac{1}{p}a^{\frac{1}{p}-1}(n)v(n)\right), \tag{27}$$

and

$$x^q(n) \leqslant B^{\frac{q}{p}}(n)\left(a^{\frac{q}{p}}(n) + \frac{q}{p}a^{\frac{q}{p}-1}(n)v(n)\right). \tag{28}$$

Combining (24) and (28), and noting the hypotheses (10), we obtain

$$\begin{aligned} v(n) \;&\leqslant\; \sum_{s=n+1}^{\infty} \left\{ L\left(s, B^{\frac{q}{p}}(s)\left(a^{\frac{q}{p}}(s) + \frac{q}{p}a^{\frac{q}{p}-1}(s)v(s)\right)\right) \right. \\ &\qquad \left. -L\left(s, B^{\frac{q}{p}}(s)a^{\frac{q}{p}}(s)\right) + L\left(s, B^{\frac{q}{p}}(s)a^{\frac{q}{p}}(s)\right)\right\} \\ &\leqslant\; J(n) + \sum_{s=n+1}^{\infty} K\left(s, B^{\frac{q}{p}}(s)a^{\frac{q}{p}}(s)\right)B^{\frac{q}{p}}(s)\frac{q}{p}a^{\frac{q}{p}-1}(s)v(s), \end{aligned} \tag{29}$$

where $J(n)$ is defined by (21).

It is obvious that $J(n)$ is nonnegative and nonincreasing for $n \in N_0$. Using Lemma 2, from (29), we have

$$v(n) \leqslant J(n)M(n), \tag{30}$$

where $M(n)$ is defined by (22).

Therefore, the desired inequality (19) follows from (27) and (30). This completes the proof of Theorem 5. □

REMARK 6. Noting that p and q are constants, and $p \geqslant 1,\ 0 < q \leqslant p$, we can obtain many peculiar discrete inequalities by using our main results. For example, let $p = 1,\ q = \frac{1}{2}$, and $p = q = 2$, respectively, from Theorem 3, we obtain the following corollaries.

COROLLARY 7. *Assume that $a(n) > 0$ and $x(n)$, $m(n)$, $f(n)$, $g(n)$ are nonnegative functions defined for $n \in N_0$. Then the inequality*

$$x(n) \leqslant a(n) + m(n) \sum_{s=n+1}^{\infty} \left[f(s)x(s) + g(s)x^{\frac{1}{2}}(s) \right], \quad n \in N_0, \tag{$I4$}$$

implies

$$x(n) \leqslant a(n) + m(n)\overline{h}(n) \prod_{s=n+1}^{\infty} \left(1 + \overline{F}(s)\right), \ n \in N_0, \tag{31}$$

where

$$\overline{h}(n) = \sum_{s=n+1}^{\infty} \left[f(s)a(s) + g(s)a^{\frac{1}{2}}(s) \right], \tag{32}$$

and

$$\overline{F}(n) = m(n)\left(f(n) + \frac{1}{2}a^{-\frac{1}{2}}(n)g(n) \right). \tag{33}$$

COROLLARY 8. *Assume that $a(n) > 0$ and $x(n)$, $m(n)$, $f(n)$ are nonnegative functions defined for $n \in N_0$. Then the inequality*

$$x^2(n) \leqslant a(n) + m(n) \sum_{s=n+1}^{\infty} f(s)x^2(s), \quad n \in N_0, \tag{$I5$}$$

implies

$$x(n) \leqslant a^{\frac{1}{2}}(n) + \frac{1}{2}a^{-\frac{1}{2}}(n)m(n) \sum_{s=n+1}^{\infty} [f(s)a(s)] \prod_{s=n+1}^{\infty} \left(1 + m(s)f(s)\right), \ n \in N_0. \tag{34}$$

3. An Application

In this section, using Theorem 3, we obtain the bound on the solution of a nonlinear sum–difference equation.

EXAMPLE. Consider the following sum–difference equation

$$x^p(n) = a(n) + b(n)\sum_{s=n+1}^{\infty} J(s,x(s)),\ n \in N_0, \tag{35}$$

where $p \geqslant 1$ is a constant, $a,b : N_0 \to R$, $|a(n)| > 0$, and $J : N_0 \times R \to R$.
Assume that

$$|J(n,x(n))| \leqslant f(n)|x(n)|^p + g(n)|x(n)|^q,\ n \in N_0, \tag{36}$$

where $f,g : N_0 \to R_+$, and $0 < q \leqslant p$ is a constant. If $x(n)$ is a solution of the problem (35), then

$$|x(n)| \leqslant |a(n)|^{\frac{1}{p}} + \frac{1}{p}|a(n)|^{\frac{1}{p}-1}|b(n)|\widetilde{h}(n)\prod_{s=n+1}^{\infty}\left(1+\widetilde{F}(s)\right),\ n \in N_0, \tag{37}$$

where

$$\widetilde{h}(n) = \sum_{s=n+1}^{\infty}\left[f(s)|a(s)| + g(s)|a(s)|^{\frac{q}{p}}\right],\quad \widetilde{F}(n) = |b(n)|\left[f(n) + \frac{q}{p}|a(n)|^{\frac{q}{p}-1}g(n)\right]. \tag{38}$$

In fact, let $x(n)$ be a solution of the problem (35). Then we have

$$|x(n)|^p \leqslant |a(n)| + |b(n)|\sum_{s=n+1}^{\infty}|J(s,x(s))|,\ n \in N_0. \tag{39}$$

Noting the assumption (35), we easily obtain

$$|x(n)|^p \leqslant |a(n)| + |b(n)|\sum_{s=n+1}^{\infty}\left(f(s)|x(s)|^p + g(s)|x(s)|^q\right). \tag{40}$$

Now a suitable application of Theorem 3 to (40) immediately yields (37).

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Weihong Sheng
Department of Mathematics
Binzhou University
Shandong 256603
P. R. China
e-mail: wh-sheng@163.com

Wei Nian Li
Department of Mathematics
Binzhou University
Shandong 256603
P. R. China
e-mail: wnli@263.net