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COMPARISON OF OPERATOR MEAN GEODESICS

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Abstract. The space of positive invertible operators of a unital C*-algebra has a natural structure of reductive homogenious manifold with a Finsler metric. Then pairs of points A and B can be joined by a natural geodesic $A \sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ for $t \in [0,1]$, where is the geometric mean in the sense of Kubo and Ando. In this paper, we consider estimates of the upper bounds for the difference between the geodesic and extended interpolational paths by terms of the spectra of positive operators. As applications, we investigate some properties of the velocity vectors for interpolational paths. Also, we obtain estimates of the upper bounds for α-operator divergence as a noncommutative version of the α-divergence in the information geometry.

1. Introduction

Let $\mathscr{A}$ be a unital C*-algebra, $\mathscr{A}^+$ (resp. $\mathscr{A}^h$) be the set of all positive invertible (resp. selfadjoint) operators of $\mathscr{A}$. It is known that $\mathscr{A}^+$ is a real analytic open submanifold of $\mathscr{A}^h$ and its tangent space $(T\mathscr{A}^+)_A$ at any $A \in \mathscr{A}^+$ is naturally identified to $\mathscr{A}^h$, see for instance Corach, Porta and Recht [4, 5]. For each $A \in \mathscr{A}^+$, the norm $\|X\|_A = \|A^{-\frac{1}{2}}XA^{-\frac{1}{2}}\|$, $X \in (T\mathscr{A}^+)_A$ defined a Finsler structure on the tangent bundle $T\mathscr{A}^+$. For every $A, B \in \mathscr{A}^+$, there is a natural geodesic joining A and B:

$$\gamma_{A,B}(t) = A \sharp_t B \qquad \text{for } t \in [0,1],$$

where $A \sharp_t B$ is the geometric mean in the sense of Kubo-Ando theory [13]:

$$A \sharp_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}. \tag{1.1}$$

As usual, the length of a smooth curve γ in $\mathscr{A}^+$ is defined by

$$l(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t)} dt$$

and the geodesic distance between A and B in $\mathscr{A}^+$ is

$$d(A,B) = \inf\{l(\gamma) : \ \gamma \text{ joins } A \text{ and } B\}.$$

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Then it follows that

$$d(A,B) = \| \log(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})\|,$$

also see [2]. It is a general fact that $(\mathscr{A}^+, d)$ is a complete metric space.

Kamei and Fujii [9, 10] defined the relative operator entropy $S(A|B)$ which is a relative version of the operator entropy defined by Nakamura-Umegaki [14]:

$$S(A|B) = A^{\frac{1}{2}} \log \left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right) A^{\frac{1}{2}}. \tag{1.2}$$

Then the velocity vector $\dot{\gamma}_{A,B}(0)$ is exactly the relative operator entropy $S(A|B)$:

$$\dot{\gamma}_{A,B}(0) = \lim_{t\to 0} \frac{A \sharp_t B - A}{t} = S(A|B).$$

On the other hand, J. I. Fujii [7] showed that if the manifold $\mathscr{A}^+$ has a metric $L_a(X) = \|X\|$ (resp. $L_h(X) = \|A^{-1}XA^{-1}\|$) on the tangent space $T\mathscr{A}^+$, then the geodesic and the distance from A to B for $A, B \in \mathscr{A}^+$ are given by

the arithmetic mean $A \nabla_t B = (1-t)A + tB$ and $d_1(A,B) = \|B - A\|$

(resp.

the harmonic mean $A \,!_t\, B = \left((1-t)A^{-1} + tB^{-1}\right)^{-1}$ and $d_{-1}(A,B) = \|A^{-1} - B^{-1}\|$.

)

Here for $m_t = \sharp_t, \nabla_t$ and $!_t$, their paths have the following interpolationality [10]:

$$(A\ m_p\ B)\ m_t\ (A\ m_q\ B) = A\ m_{(1-t)p+tq}\ B$$

for $0 \leqslant p, q, t \leqslant 1$.

Typical interpolational paths from A to B which include the arithmetic, geometric and harmonic mean are as follows: For each $r \in [-1, 1]$

$$A\ m_{r,t}\ B = A^{\frac{1}{2}} \left(1 - t + t\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r\right)^{\frac{1}{r}} A^{\frac{1}{2}} \qquad \text{for } t \in [0,1]. \tag{1.3}$$

In particular, $A\ m_{1,t}\ B = A \nabla_t B$, $A\ m_{0,t}\ B = A \sharp_t B$ and $A\ m_{-1,t}\ B = A\ !_t\ B$.

In this paper, we investigate estimates of the upper bounds for the difference between the geodesic $A \sharp_t B$ and interpolational paths $A\ m_{r,t}\ B$ by terms of the spectra of positive operators: For each $s > 0$ and $t \in (0,1)$, there exists a suitable constant β such that

$$0 \leqslant A\ m_{s,t}\ B - A \sharp_t B \leqslant \beta A.$$

As applications, we investigate some properties of the velocity vectors for interpolational paths. Also, we obtain estimates of the upper bounds for α-operator divergence introduced by [6] as a noncommutative version of the α-divergence in the information geometry.

2. Interpolational paths

First of all, we recall an interpolational path for symmetric operator means. Following [10, 11], for a symmetric mean σ, a parametrized operator mean σ_t is called an interpolational path for σ if it satisfies

(1) $A\ \sigma_0\ B = A$, $A\ \sigma_{1/2}\ B = A\ \sigma\ B$ and $A\ \sigma_1\ B = B$

(2) $(A\ \sigma_p\ B)\ \sigma\ (A\ \sigma_q\ B) = A\ \sigma_{\frac{p+q}{2}}\ B$

(3) the map $t \to A\ \sigma_t\ B$ is norm continous for each A and B.

Typical interpolational means are so-called power means;

$$A\ m_r\ B = A^{\frac{1}{2}} \left(\frac{1 + (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r}{2} \right)^{\frac{1}{r}} A^{\frac{1}{2}} \qquad \text{for } r \in [-1,1]$$

and their interpolational paths from A to B via $A\ m_r\ B$ are given by (1.3).

Here we consider them in the general setting: For positive invertible operators A and B, an extended path $A\ m_{r,t}\ B$ is defined as

$$A\ m_{r,t}\ B = A^{\frac{1}{2}} \left(1 - t + t(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^r \right)^{\frac{1}{r}} A^{\frac{1}{2}} \qquad \text{for all real numbers } r \in \mathbb{R}$$

and $t \in [0,1]$. The representing function $f_{r,t}$ for $m_{r,t}$ is defined as

$$f_{r,t}(\xi) = 1\ m_{r,t}\ \xi = (1 - t + t\xi^r)^{\frac{1}{r}} \qquad \text{for } \xi > 0.$$

Notice that $A\ m_{r,t}\ B$ for $r \in \mathbb{R}$ is no longer a path of operator means for $r \notin [-1,1]$, but we list some properties of interpolational paths $m_{r,t}$ and the representing function $f_{r,t}$, also see [8].

Since every function $f_{r,t}(\xi)$ is strictly increasing and strictly convex (resp. strictly concave) for $r > 1$ (resp. $r < 1$), it follows that an extended path $A\ m_{r,t}\ B$ for each $t \in (0,1)$ is nondecreasing and norm continuous for $r \in \mathbb{R}$: For $r \leqslant s$

$$A\ m_{r,t}\ B \leqslant A\ m_{s,t}\ B.$$

Moreover, an extended path is also interpolational for all real numbers $r \in \mathbb{R}$. In particular, the transposition formula holds:

$$B\ m_{r,t}\ A = A\ m_{r,1-t}\ A. \tag{2.1}$$

For the sake of convenience, we prepare the following notation: For $k_2 > k_1 > 0$, $r \in \mathbb{R}$ and $t \in [0,1]$

$$a(r,t) = \frac{f_{r,t}(k_2) - f_{r,t}(k_1)}{k_2 - k_1} \quad \text{and} \quad b(r,t) = \frac{k_2 f_{r,t}(k_1) - k_1 f_{r,t}(k_2)}{k_2 - k_1}. \tag{2.2}$$

We investigate estimates of the upper bounds for the difference between extended interpolational paths:

LEMMA 2.1. *Let A and B be positive invertible operators such that $k_1A \leqslant B \leqslant k_2A$ for some scalars $0 < k_1 < k_2$. Then for $r \leqslant s$ and $t \in (0,1)$*

$$0 \leqslant A\, m_{s,t}\, B - A\, m_{r,t}\, B \leqslant \beta A \qquad \textit{for } r \leqslant 1 \tag{2.3}$$

and

$$0 \leqslant A\, m_{s,t}\, B - A\, m_{r,t}\, B \leqslant \beta' A \qquad \textit{for } r \geqslant 1 \tag{2.4}$$

hold for

$$\beta = \beta(r,s,t,k_1,k_2) = \max_{k_1 \leqslant \xi \leqslant k_2} \{f_{s,t}(\xi) - a(r,t)\xi - b(r,t)\},$$

and

$$\beta' = \beta'(r,s,t,k_1,k_2) = \max_{k_1 \leqslant \xi \leqslant k_2} \{a(s,t)\xi + b(s,t) - f_{r,t}(\xi)\},$$

where a,b are defined by (2.2).

Proof. Suppose that $r \leqslant 1$. If we put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then we have $k_2I \geqslant C \geqslant k_1I > 0$. Since $f_{r,t}(\xi)$ is concave for $r \leqslant 1$, it follows from the definition of β that

$$\beta \geqslant f_{s,t}(\xi) - a(r,t)\xi - b(r,t) \geqslant f_{s,t}(\xi) - f_{r,t}(\xi) \qquad \text{for all } \xi \in [k_1,k_2],$$

and hence

$$\beta I \geqslant f_{s,t}(C) - f_{r,t}(C).$$

This fact implies

$$\beta A \geqslant A^{\frac{1}{2}} f_{s,t}(C) A^{\frac{1}{2}} - A^{\frac{1}{2}} f_{r,t}(C) A^{\frac{1}{2}} = A\, m_{s,t}\, B - A\, m_{r,t}\, B,$$

which gives the desired result (2.3). Conversely, if $r \geqslant 1$, then $f_{s,t}(\xi)$ is convex for $1 \leqslant r \leqslant s$ and the latter part (2.4) of Lemma 2.1 follows from the same way. □

REMARK 2.2. The constant $\beta = \beta(s,r,t,m,M)$ and $\beta' = \beta'(s,r,t,m,M)$ in Lemma 2.1 can be written explicitly as

$$\beta = \begin{cases} a(r,t)\left(\dfrac{1-t}{t}\right)^{\frac{1}{s}}\left(t^{\frac{1}{s-1}} a(r,t)^{\frac{s}{1-s}} - 1\right)^{\frac{s-1}{s}} - b(r,t) & \text{for } k_1 \leqslant \xi_0 \leqslant k_2 \\ f_{s,t}(k_1) - f_{r,t}(k_1) & \text{for } \xi_0 \leqslant k_1 \\ f_{s,t}(k_2) - f_{r,t}(k_2) & \text{for } k_2 \leqslant \xi_0 \end{cases}$$

where $\xi_0 = \left(\frac{1}{1-t}\left(\frac{a(r,t)}{t}\right)^{\frac{s}{1-s}} - \frac{t}{1-t}\right)^{-\frac{1}{s}}$ and

$$\beta' = \begin{cases} -a(s,t)\left(\dfrac{1-t}{t}\right)^{\frac{1}{r}}\left(t^{\frac{1}{r-1}} a(s,t)^{\frac{r}{1-r}} - 1\right)^{\frac{r-1}{r}} + b(s,t) & \text{for } k_1 \leqslant \xi_1 \leqslant k_2 \\ f_{s,t}(k_1) - f_{r,t}(k_1) & \text{for } \xi_1 \leqslant k_1 \\ f_{s,t}(k_2) - f_{r,t}(k_2) & \text{for } k_2 \leqslant \xi_1 \end{cases}$$

where $\xi_1 = \left(\frac{1}{1-t}\left(\frac{a(s,t)}{t}\right)^{\frac{r}{1-r}} - \frac{t}{1-t}\right)^{-\frac{1}{r}}$.

By lemma 2.1, we obtain estimates of the upper bounds for the difference between the geodesic $A \sharp_t B$ and extended interpolational paths:

THEOREM 2.3. *Let A and B be positive invertible operators such that $k_1 A \leqslant B \leqslant k_2 A$ for some scalars $0 < k_1 < k_2$. Then for each $t \in (0,1)$*

$$0 \leqslant A\, m_{s,t}\, B - A \sharp_t B \leqslant \beta(0,s,t,k_1,k_2)A \qquad \text{for} \quad s \geqslant 0, \tag{2.5}$$

and

$$0 \leqslant A \sharp_t B - A\, m_{r,t}\, B \leqslant \beta(r,0,t,k_1,k_2)A \qquad \text{for} \quad r \leqslant 0, \tag{2.6}$$

where β is defined by Remark 2.2.

As special cases of Theorem 2.3, we obtain an estimate of the upper bound for the difference between the geodesic $A \sharp_t B$ and the arithmetic interpolational paths $A \nabla_t B$, and also the same for the geodesic $A \sharp_t B$ and the harmonic interpolational paths $A\,!_t\,B$:

THEOREM 2.4. *Let A and B be positive invertible operators such that $k_1 A \leqslant B \leqslant k_2 A$ for some scalars $0 < k_1 < k_2$. Then for each $t \in (0,1)$*

$$0 \leqslant A \nabla_t B - A \sharp_t B \leqslant \max\{1 - t + tk_1 - k_1^t,\, 1 - t + tk_2 - k_2^t\}A \tag{2.7}$$

and

$$0 \leqslant A \sharp_t B - A\,!_t\,B \leqslant \beta(-1,0,t,k_1,k_2)A, \tag{2.8}$$

where $\beta(-1,0,t,k_1,k_2) =$

$$\begin{cases} \frac{1-t}{((1-t)k_1+t)((1-t)k_2+t)} \left(((1-t)k_1+t)((1-t)k_2+t)^{\frac{1}{1-t}} - k_1k_2 \right) & \\ \qquad \text{for} \quad k_1^{1-t} \leqslant ((1-t)k_1+t)((1-t)k_2+t) \leqslant k_2^{1-t}, & \\ k_2^t - \frac{k_2}{(1-t)k_2+t} & \\ \qquad \text{for} \quad k_2^{1-t} \leqslant ((1-t)k_1+t)((1-t)k_2+t), & \\ k_1^t - \frac{k_1}{(1-t)k_1+t} & \\ \qquad \text{for} \quad k_1^{1-t} \geqslant ((1-t)k_1+t)((1-t)k_2+t). & \end{cases}$$

Proof. If we put $s = 1$ in (2.5) of Theorem 2.3, then $f_{1,t}(\xi) = 1 - t + t\xi$ and $f_{0,t}(\xi) = \xi^t$. Since $a(0,t) = \frac{k_2^t - k_1^t}{k_2 - k_1}$, the condition $f'_{1,t}(M) \leqslant a(0,t) \leqslant f'_{1,t}(m)$ is equivalent to $a(0,t) = t$. Therefore we have

$$\beta = \begin{cases} 1 - t + tk_2 - k_2^t & \text{for} \quad \frac{k_2^t - k_1^t}{k_2 - k_1} \leqslant t, \\ 1 - t + tk_1 - k_1^t & \text{for} \quad \frac{k_2^t - k_1^t}{k_2 - k_1} \geqslant t. \end{cases}$$

Similarly, we have the latter part of this theorem by using (2.6) of Theorem 2.3. □

Next, we show estimates of the lower bounds of the ratio for extended interpolational paths:

LEMMA 2.5. *Let A and B be positive invertible operators such that $k_1 A \leqslant B \leqslant k_2 A$ for some scalars $0 < k_1 < k_2$. Then for $r \leqslant s$ and $t \in (0,1)$*

$$A\, m_{r,t}\, B \geqslant \alpha A\, m_{s,t}\, B \qquad \text{for} \quad r \leqslant 1 \tag{2.9}$$

and

$$A\, m_{r,t}\, B \geqslant \alpha' A\, m_{s,t}\, B \qquad \textit{for} \quad r \geqslant 1 \tag{2.10}$$

hold for

$$\alpha = \alpha(r,s,t,k_1,k_2) = \min_{k_1 \leqslant \xi \leqslant k_2} \{\frac{a(r,t)\xi + b(r,t)}{f_{s,t}(\xi)}\}$$

and

$$\alpha' = \alpha'(r,s,t,k_1,k_2) = \min_{k_1 \leqslant \xi \leqslant k_2} \{\frac{f_{r,t}(\xi)}{a(s,t)\xi + b(s,t)}\},$$

where a, b *are defined by* (2.2).

Proof. Suppose that $r \leqslant 1$. Since $f_{r,t}(\xi)$ is concave for $r \leqslant 1$, it follows that

$$\frac{f_{r,t}(\xi)}{f_{s,t}(\xi)} \geqslant \frac{a(r,t)\xi + b(r,t)}{f_{s,t}(\xi)} \geqslant \alpha$$

and hence $f_{r,t}(\xi) \geqslant \alpha f_{s,t}(\xi)$ on $[k_1,k_2]$. Therefore we have

$$A\, m_{r,t}\, B = A^{\frac{1}{2}} f_{r,t}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} \geqslant \alpha A^{\frac{1}{2}} f_{s,t}(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} = \alpha A\, m_{s,t}\, B.$$

Similarly, since $f_{s,t}(\xi)$ is convex for $1 \leqslant r \leqslant s$, the latter part follows from the same way. □

REMARK 2.6. The constant $\alpha = \alpha(r,s,t,k_1,k_2)$ and $\alpha' = \alpha'(r,s,t,k_1,k_2)$ in Lemma 2.5 can be written explicitly as follows: In the case of $s \geqslant 1$,

$$\alpha = \min\{\frac{f_{r,t}(k_1)}{f_{s,t}(k_1)}, \frac{f_{r,t}(k_2)}{f_{s,t}(k_2)}\}.$$

In the case of $s \leqslant 1$,

$$\alpha = \begin{cases} \frac{a(r,t)\xi_0 + b(r,t)}{(1-t+t\xi_0^s)^{\frac{1}{s}}} & \text{for} \quad k_1 \leqslant \xi_0 \leqslant k_2, \\ \frac{f_{r,t}(k_2)}{f_{s,t}(k_2)} & \text{for} \quad k_2 \leqslant \xi_0, \\ \frac{f_{r,t}(k_1)}{f_{s,t}(k_1)} & \text{for} \quad k_1 \geqslant \xi_0, \end{cases}$$

where $\xi_0 = \left(\frac{1-t}{t}\frac{a(r,t)}{b(r,t)}\right)^{\frac{1}{s-1}}$ and

$$\alpha' = \begin{cases} \frac{(1-t+t\xi_1^r)^{\frac{1}{r}}}{a(s,t)\xi_1 + b(s,t)} & \text{for} \quad k_1 \leqslant \xi_1 \leqslant k_2, \\ \frac{f_{r,t}(k_2)}{f_{s,t}(k_2)} & \text{for} \quad k_2 \leqslant \xi_1, \\ \frac{f_{r,t}(k_1)}{f_{s,t}(k_1)} & \text{for} \quad k_1 \geqslant \xi_1, \end{cases}$$

where $\xi_1 = \left(\frac{1-t}{t}\frac{a(s,t)}{b(s,t)}\right)^{\frac{1}{r-1}}$.

As a special case of Lemma 2.5, we have the following theorem.

THEOREM 2.7. *Let A and B be positive invertible operators such that $k_1A \leqslant B \leqslant k_2A$ for some scalars $0 < k_1 < k_2$. Then for each $t \in (0,1)$*

$$A \sharp_t B \geqslant \min\{\frac{k_1^t}{1-t+tk_1}, \frac{k_2^t}{1-t+tk_2}\} A \nabla_t B \tag{2.11}$$

and

$$A\, !_t\, B \geqslant \alpha(-1,0,t,k_1,k_2)\, A \sharp_t B \tag{2.12}$$

holds for

$$\alpha(-1,0,t,k_1,k_2) = \begin{cases} \frac{(k_1k_2)^{1-t}}{((1-t)k_1+t)((1-t)k_2+t)} & \text{for} \quad k_1 \leqslant 1 \leqslant k_2 \\ \frac{k_2^{1-t}}{(1-t)k_2+t} & \text{for} \quad 1 \leqslant k_1, \\ \frac{k_1^{1-t}}{(1-t)k_1+t} & \text{for} \quad k_2 \leqslant 1. \end{cases}$$

3. Velocity vector of extended paths

Following [9, 10], for positve invertible operators A and B, the relative operator entropy $S(A|B)$ is defined by

$$S(A|B) = A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

which is a relative version of the operator entropy $-A\log A$ considered by Nakamura-Umegaki [14]. The relative operator entropy $S(A|B)$ is exactly the velocity vector $\dot{\gamma}_{A,B}(0)$ of the geodesic $A \sharp_t B$ at $t=0$:

$$S(A|B) = \lim_{t\to 0}\frac{A \sharp_t B - A \sharp_0 B}{t} = \dot{\gamma}_{A,B}(0).$$

In [12], Kamei analogously generalizes the relative operator entropy: For each $r \in \mathbb{R}$

$$S_r(A|B) = \lim_{t\to 0}\frac{A\, m_{r,t}\, B - A\, m_{r,0}\, B}{t},$$

which is considered as the right differential coefficient at $t=0$ of the extended path $A\, m_{r,t}\, B$. By the fact that

$$\lim_{t\to 0}\frac{(1-t+t\xi^r)^{\frac{1}{r}}-1}{t} = \frac{\xi^r-1}{r},$$

it follows that

$$S_r(A|B) = \frac{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^r A^{\frac{1}{2}} - A}{r} \qquad \text{for } r \in \mathbb{R}$$

and the representing function is

$$f_r(\xi) = \frac{\xi^r - 1}{r}.$$

In particular,

$$S_1(A|B) = \lim_{t\to 0} \frac{A \nabla_t B - A}{t} = B - A,$$
$$S_0(A|B) = S(A|B),$$
$$S_{-1}(A|B) = \lim_{t\to 0} \frac{A\, !_t\, B - A}{t} = A - AB^{-1}A.$$

For the sake of convenience, we prepare the following notation:

$$a(r) = \frac{f_r(k_2) - f_r(k_1)}{k_2 - k_1} \qquad \text{and} \qquad b(r) = \frac{k_2 f_r(k_1) - k_1 f_r(k_2)}{k_2 - k_1} \tag{3.1}$$

for $0 < k_1 < k_2$ and $r \in \mathbb{R}$.

Since $f_r(\xi)$ is monotone increasing on $r \in \mathbb{R}$, the velocity vectors $S_r(A|B)$ is monotone increasing on $r \in \mathbb{R}$:

$$r \leqslant s \qquad \text{implies} \qquad S_r(A|B) \leqslant S_s(A|B).$$

The left differentiable coefficient of $A\, m_{r,t}\, B$ at $t = 1$ is $-S_r(B|A)$:

$$\lim_{t\to 1} \frac{A\, m_{r,t}\, B - A\, m_{r,1}\, B}{t-1} = -S_r(B|A).$$

If $B \geqslant A$, then the velocity vectors of exteded paths at $t = 0, 1$ are positive:

$$S_r(A|B) \geqslant 0 \qquad \text{and} \qquad -S_r(B|A) \geqslant 0.$$

We investigate estimetaes of the upper bounds for the difference between velocity vectors of extended interpolational paths.

LEMMA 3.1. *Let A and B be positive invertible operators such that $k_1 A \leqslant B \leqslant k_2 A$ for some scalars $0 < k_1 < k_2$. Then for $r \leqslant s$*

$$S_s(A|B) - S_r(A|B) \leqslant \gamma A \qquad \text{for} \quad r \leqslant s \leqslant 1, \tag{3.2}$$

$$S_s(A|B) - S_r(A|B) \leqslant \max\{\frac{k_1^s - 1}{s} - \frac{k_1^r - 1}{r}, \frac{k_2^s - 1}{s} - \frac{k_2^r - 1}{r}\}A \qquad \text{for} \quad r \leqslant 1 \leqslant s \tag{3.3}$$

and

$$S_s(A|B) - S_r(A|B) \leqslant \gamma' A \qquad \text{for} \quad 1 \leqslant r \leqslant s \tag{3.4}$$

hold for

$$\gamma = \gamma(r,s,k_1,k_2) = \max_{k_1 \leqslant \xi \leqslant k_2} \{f_s(\xi) - a(r)\xi - b(r)\}$$

and

$$\gamma' = \gamma'(r,s,k_1,k_2) = \max_{k_1 \leqslant \xi \leqslant k_2} \{a(s)\xi + b(s) - f_r(\xi)\},$$

where a, b are defined by (3.1).

Proof. Suppose that $r \leqslant 1$. If we put $C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, then we have $0 < k_1 I \leqslant C \leqslant k_2 I$. Since $f_r(\xi)$ is concave for $r \leqslant 1$, it follows that

$$f_s(\xi) - f_r(\xi) \leqslant f_s(\xi) - a(r)\xi - b(r) \leqslant \gamma$$

and hence we have the desired result (3.2) and (3.3). The remainder part (3.4) follow in the same way. □

REMARK 3.2. The constant $\gamma = \gamma(r,s,k_1,k_2)$ and $\gamma' = \gamma'(r,s,k_1,k_2)$ in Lemma 3.1 can be written explicitly as

$$\gamma = \begin{cases} \frac{1-s}{s} a(r)^{\frac{s}{s-1}} - b(r) - \frac{1}{s} & \text{for} \quad k_1 \leqslant a(r)^{\frac{1}{s-1}} \leqslant k_2, \\ \frac{k_2^s - 1}{s} - \frac{k_2^r - 1}{r} & \text{for} \quad k_2 \leqslant a(r)^{\frac{1}{s-1}}, \\ \frac{k_1^s - 1}{s} - \frac{k_1^r - 1}{r} & \text{for} \quad k_1 \geqslant a(r)^{\frac{1}{s-1}}, \end{cases}$$

and

$$\gamma' = \begin{cases} \frac{r-1}{r} a(s)^{\frac{r}{r-1}} + b(s) + \frac{1}{r} & \text{for} \quad k_1 \leqslant a(s)^{\frac{1}{r-1}} \leqslant k_2, \\ \frac{k_2^s - 1}{s} - \frac{k_2^r - 1}{r} & \text{for} \quad k_2 \leqslant a(s)^{\frac{1}{r-1}}, \\ \frac{k_1^s - 1}{s} - \frac{k_1^r - 1}{r} & \text{for} \quad k_1 \geqslant a(s)^{\frac{1}{r-1}}. \end{cases}$$

By Lemma 3.1, we obtain estimates of the upper bound for the difference between the velocity vectors $S(A|B)$ and $S_r(A|B)$ of the extended interpolational paths $A\ m_{r,t}\ B$ at $t = 0$:

THEOREM 3.3. *Let A and B be positive invertible operators such that $k_1 A \leqslant B \leqslant k_2 A$ for some scalars $0 < k_1 < k_2$. Then*

$$S_s(A|B) - S(A|B) \leqslant \gamma A \qquad \textit{for} \quad 0 \leqslant s \leqslant 1 \tag{3.5}$$

and

$$S_s(A|B) - S(A|B) \leqslant \max\{\frac{k_1^s - 1}{s} - \log k_1, \frac{k_2^s - 1}{s} - \log k_2\}A \qquad \textit{for} \quad 1 \leqslant s, \tag{3.6}$$

where

$$\gamma = \begin{cases} \frac{1-s}{s}\left(\frac{\log k_2 - \log k_1}{k_2 - k_1}\right)^{\frac{s}{s-1}} - \frac{k_2 \log k_1 - k_1 \log k_2}{k_2 - k_1} - \frac{1}{s} & \textit{for} \quad k_1 \leqslant \left(\frac{\log k_2 - \log k_1}{k_2 - k_1}\right)^{\frac{1}{s-1}} \leqslant k_2, \\ \frac{k_2^s - 1}{s} - \log k_2 & \textit{for} \quad k_2 \leqslant \left(\frac{\log k_2 - \log k_1}{k_2 - k_1}\right)^{\frac{1}{s-1}}, \\ \frac{k_1^s - 1}{s} - \log k_1 & \textit{for} \quad k_1 \geqslant \left(\frac{\log k_2 - \log k_1}{k_2 - k_1}\right)^{\frac{1}{s-1}}. \end{cases}$$

4. α-operator divergence

The concept of α-divergence plays an important role in the information geometry. Following [1], α-divergence is defined as follows: For positive valued measurable functions p and q, and $\alpha \in \mathbb{R}$

$$\begin{aligned} D_\alpha(p\|q) &\equiv \frac{4}{1-\alpha^2}\int\{\frac{1-\alpha}{2}p+\frac{1+\alpha}{2}q-p^{\frac{1-\alpha}{2}}q^{\frac{1+\alpha}{2}}\}dx, \quad (\alpha \neq \pm 1), \\ D_{-1}(p\|q) &= D_1(q\|p) \equiv \int\{q-p+p\log\frac{p}{q}\}dx. \end{aligned} \tag{4.1}$$

If we put $t = \frac{1+\alpha}{2}$ in (4.1), then

$$D_\alpha(p\|q) \equiv \frac{1}{t(1-t)}\int\{(1-t)p+tq-p^{1-t}q^t\}dx, \quad (t \neq 0,1).$$

From the viewpoint of this, J. I. Fujii [6] defined the following operator version of α-divergence in the differential geometry: For positive invertible operators A and B,

$$D_\alpha(A,B) \equiv \frac{1}{\alpha(1-\alpha)}\left(A \nabla_\alpha B - A \sharp_\alpha B\right) \quad (0<\alpha<1).$$

In particular,

$$\begin{aligned} D_1(A,B) &\equiv \lim_{\alpha\uparrow 1} D_\alpha(A,B) = \lim_{\alpha\uparrow 1}\left(\frac{A-B}{\alpha} - \frac{B\sharp_{1-\alpha}A - B}{\alpha(1-\alpha)}\right) \\ &= A - B - S(B|A) \\ D_0(A,B) &\equiv \lim_{\alpha\downarrow 0} D_\alpha(A,B) = \lim_{\alpha\downarrow 0}\left(\frac{B-A}{1-\alpha} - \frac{A\sharp_\alpha B - A}{\alpha(1-\alpha)}\right) \\ &= B - A - S(A|B). \end{aligned}$$

By definition, α-operator divergence is considered as the difference between the arithmetic and the geometric interpolational paths. In particular, for the case of $\alpha = 1/2$, it follows that α-operator divergence coinsides with by four times the difference of the geometric mean and the arithmetic mean. For the case of density operators, it coinsides with a relative entropy introduced by Beravkin and Staszewski [3] in C*-algebra setting.

Also we have the following different interpretation of α-operator divergence:

THEOREM 4.1. *The α-operator divergence is the difference between two velocity vectors $S_1(A|B)$ and $S_\alpha(A|B)$: For each $\alpha \in (0,1)$*

$$\begin{aligned} D_\alpha(A,B) &= \frac{1}{1-\alpha}\left(S_1(A|B) - S_\alpha(A|B)\right) \\ &= \frac{1}{\alpha}\left(S_1(B|A) - S_{1-\alpha}(B|A)\right). \end{aligned}$$

We investigate estimates of the upper bounds for α-operator divergence:

THEOREM 4.2. *Let A and B be positive invertible operators such that $k_1A \leqslant B \leqslant k_2A$ for some scalars $0 < k_1 < k_2$. Then α-operator divergence is positive and for every operator mean ρ and $\alpha \in (0,1)$*

$$(\beta A)\ \rho\ (\overline{\beta}B) \geqslant D_\alpha(A,B) \geqslant 0$$

holds for

$$\beta = \max\{\frac{1-\alpha+\alpha k_1 - k_1^\alpha}{\alpha(1-\alpha)}, \frac{1-\alpha+\alpha k_2 - k_2^\alpha}{\alpha(1-\alpha)}\},$$
$$\overline{\beta} = \max\{\frac{\alpha+(1-\alpha)k_2^{-1} - k_2^{\alpha-1}}{\alpha(1-\alpha)}, \frac{\alpha+(1-\alpha)k_1^{-1} - k_1^{\alpha-1}}{\alpha(1-\alpha)}\}.$$

Proof. Since $A\ \nabla_\alpha\ B \geqslant A\ \sharp_\alpha\ B$ ($0 \leqslant \alpha \leqslant 1$), it follows that α-operator divergence is positive, that is, $D_\alpha(A,B) \geqslant 0$. On the other hand, it follows from Theorem 2.4 that $\beta A \geqslant D_\alpha(A,B) \geqslant 0$. Since $A\,\nabla_\alpha\,B - A\,\sharp_\alpha\,B = B\,\nabla_{1-\alpha}\,A - B\,\sharp_{1-\alpha}\,A$ by (2.1), we applied B,A and $1-\alpha$ in Theorem 2.4 to obtain the constant $\overline{\beta} = \beta(0,1,1-\alpha,k_2^{-1},k_1^{-1})$ such that $\overline{\beta}B \geqslant D_\alpha(A,B) \geqslant 0$ because $k_2^{-1}B \leqslant A \leqslant k_1^{-1}B$. Therefore we have for every operator mean ρ

$$(\beta A)\ \rho\ (\overline{\beta}B) \geqslant D_\alpha(A,B)\ \rho\ D_\alpha(A,B) = D_\alpha(A,B) \geqslant 0.$$

□

If we put $\alpha \to 0,1$ in Theorem 4.2, then we have the following corollary:

COROLLARY 4.3. *Let A and B be positive invertible operators such that $k_1A \leqslant B \leqslant k_2A$ for some scalars $0 < k_1 < k_2$. Then for every operator mean ρ*

$$(\beta A)\ \rho\ (\overline{\beta}B) \geqslant D_0(A,B) = S_1(A|B) - S_0(A|B),$$

holds for $\beta = \max\{k_1 - 1 - \log k_1, k_2 - 1 - \log k_2\}$ *and* $\overline{\beta} = \max\{1 - k_2^{-1} - k_2^{-1}\log k_2, 1 - k_1^{-1} - k_1^{-1}\log k_1\}$ *and*

$$(\beta A)\ \rho\ (\overline{\beta}B) \geqslant D_1(A,B) = S_1(B|A) - S_0(B|A)$$

holds for $\beta = \max\{1 - k_2^{-1} - k_2^{-1}\log k_2, 1 - k_1^{-1} + k_1^{-1}\log k_1\}$ *and* $\overline{\beta} = \max\{k_1 - 1 + \log k_1, k_2 - 1 - \log k_2\}$.

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REFERENCES

[1] S. AMARI, *Differential Geometrical Methods in Statistics*, Springer Lecture Notes in Statistics, **28** (1985).

[2] E. ANDRUCHOW, G. CORACH, D. STOJANOFF, *Geometrical significance of the Lowner-Heinz inequality*, Proc. Amer. Math. Soc., **128** (1999), 1031–1037.

[3] V. P. BELAVKIN, P. STASZEWSKI, *C^*-algebraic generalization of relative entropy and entropy*, Ann. Inst. H. Poincaré Sect., **A.37** (1982), 51–58.

[4] G. CORACH, H. PORTA, L. RECHT, *Geodesics and operator means in the space of positive operators*, Int. J. Math., **4** (1993), 193–202.

[5] G. CORACH, H. PORTA, L. RECHT, *Convexity of the geodesic distance on spaces of positive operators*, Illinois J. Math., **38** (1994), 87–94.

[6] J. I. FUJII, *On the relative operator entropy (in Japanese)*, RIMS Kokyuroku **903** (1995), 49–56.

[7] J. I. FUJII, *The Corach-Porta-Recht theorem and operator means*, preprint.

[8] J. I. FUJII, M. FUJII, Y. SEO, *Bounds for interpolational path of positive operators*, Recent Topics on Operator Inequalities, RIMS Kokyuroku, **1359** (2004), 46-57.

[9] J. I. FUJII, E. KAMEI, *Relative operator entropy in noncommutative information theory*, Math. Japon., **34** (1989), 341–348.

[10] J. I. FUJII, E. KAMEI, *Uhlmann's interpolational method for operator means*, Math. Japon., **34** (1989), 541–547.

[11] T. FURUTA, J. MIĆIĆ, J. E. PEČARIĆ, Y. SEO, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[12] E. KAMEI, *Paths of operators parametrized by operator means*, Math. Japon., **39** (1994), 395–400.

[13] F. KUBO, T. ANDO, *Means of positive linear operators*, Math. Ann., **246** (1980), 205–224.

[14] M. NAKAMURA, H. UMEGAKI, *A note on the entropy for operator algebras*, Proc. Japan. Acad., **37** (1961), 149–151.

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