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ON A VARIANT OF THE JENSEN–MERCER INEQUALITY FOR OPERATORS

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Abstract. Some refinements of the Jensen-Mercer inequality for operators are presented. Obtained results are used to refine some comparision inequalities between power and quasi-arithmetic means for operators.

1. Introduction

We assume that H is a Hilbert space, $\mathscr{B}(H)$ is the C^*-algebra of all bounded operators on H, $A_1,...,A_k \in \mathscr{B}(H)$ are selfadjoint operators with spectra contained in $[m,M]$ for some scalars $m<M$, 1_H is the identity operator in $\mathscr{B}(H)$, $w_1,\dots,w_k$ are positive real numbers and $W_k=\sum_{j=1}^k w_j$. We denote by $C([m,M])$ the set of all real valued continuous functions on an interval $[m,M]$. The following definition can be found in [1].

DEFINITION. A real valued continuous function f defined on an interval I is said to be *operator convex* if

$$f\left((1-\lambda)A+\lambda B\right)\leqslant(1-\lambda)f(A)+\lambda f(B) \tag{1.1}$$

for all $\lambda\in[0,1]$ and for all selfadjoint operators $A,B\in\mathscr{B}(H)$ whose spectra are contained in I. *A real valued continuous function* f is said to be *operator concave* if the reverse inequality (1.1) holds.

The following assertion is a special case of a general result proved in [3].

THEOREM A. *If $f\in C([m,M])$ is an operator convex function on $[m,M]$,*

$$f\left((m+M)1_H-\frac{1}{W_k}\sum_{j=1}^k w_jA_j\right)\leqslant\frac{1}{W_k}\sum_{j=1}^k w_jf\left((m+M)1_H-A_j\right)$$
$$\leqslant(f(m)+f(M))1_H-\frac{1}{W_k}\sum_{j=1}^k w_jf(A_j). \tag{1.2}$$

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If a function f is operator concave, then inequalities (1.2) *are reversed.*

REMARK 1. From a general result proved in [2] it follows that the Jensen-Mercer inequality for operators

$$f\left((m+M)1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j A_j\right) \leqslant (f(m)+f(M))1_H - \frac{1}{W_k}\sum_{j=1}^{k} w_j f(A_j) \quad (1.3)$$

holds, more generally, for all convex functions.

In this paper we give some refinements of the Jensen-Mercer inequality, and we present several applications of them. In Section 2 we prove refinements of (1.3), using an index set function. For related results in the real case see [4, p. 87]. In Section 3 we use these results to refine some inequalities among power and quasi-arithmetic means of Mercer's type for operators.

2. Main results

Let I be a finite nonempty set of positive integers. Let A_i $(i \in I)$ be selfadjoint operators in $\mathscr{B}(H)$ with spectra contained in $[m,M]$ for some scalars $m < M$, and let $w_i\ (i \in I)$ be positive real numbers. Observe that spectra of $\frac{1}{W_I}\sum_{i\in I} w_i A_i$ is also contained in $[m,M]$. If we define the index set function F as

$$F(I) = W_I\left[(f(m)+f(M))1_H - \frac{1}{W_I}\sum_{i\in I} w_i f(A_i) - f\left((m+M)1_H - \frac{1}{W_I}\sum_{i\in I} w_i A_i\right)\right],$$

then the following theorem is valid.

THEOREM 1. *Let I and J be finite nonempty sets of positive integers such that $I \cap J = \emptyset$. Let w_i $(i \in I \cup J)$ be positive real numbers and let A_i $(i \in I \cup J)$ be selfadjoint operators in $\mathscr{B}(H)$ with spectra contained in $[m,M]$. If $f \in C([m,M])$ is an operator convex function, then*

$$F(I \cup J) \geqslant F(I) + F(J). \quad (2.1)$$

If a function f is operator concave, then inequality (2.1) *is reversed.*

Proof. Suppose that f is operator convex. From Theorem A it follows that the following inequality holds for every selfadjoint operators $B_1, B_2 \in \mathscr{B}(H)$ with spectra contained in $[m,M]$, and for every $u_1, u_2 > 0$

$$(u_1+u_2)f\left((m+M)1_H - \frac{u_1B_1+u_2B_2}{u_1+u_2}\right) \leqslant u_1 f((m+M)1_H - B_1) + u_2 f((m+M)1_H - B_2). \quad (2.2)$$

If we let

$$u_1 = W_I, u_2 = W_J, B_1 = \frac{1}{W_I}\sum_{i\in I} w_i A_i, B_2 = \frac{1}{W_J}\sum_{i\in J} w_i A_i$$

in (2.2), then we obtain

$$\begin{aligned} & W_{I\cup J} f\left((m+M)\,1_H - \frac{1}{W_{I\cup J}}\sum_{i\in I\cup J} w_i A_i\right) \\ & \leqslant W_I f\left((m+M)\,1_H - \frac{1}{W_I}\sum_{i\in I} w_i A_i\right) \\ & + W_J f\left((m+M)\,1_H - \frac{1}{W_J}\sum_{i\in J} w_i A_i\right). \end{aligned}$$

Multiplying the above inequality by (-1) and adding to the both sides the term

$$W_{I\cup J}\left[(f(m)+f(M))\,1_H - \frac{1}{W_{I\cup J}}\sum_{i\in I\cup J} w_i f(A_i)\right],$$

it follows that

$$\begin{aligned} & W_{I\cup J}\left[(f(m)+f(M))\,1_H - \frac{1}{W_{I\cup J}}\sum_{i\in I\cup J} w_i f(A_i)\right. \\ & \qquad \left. - f\left((m+M)\,1_H - \frac{1}{W_{I\cup J}}\sum_{i\in I\cup J} w_i A_i\right)\right] \\ & \geqslant W_I\left[(f(m)+f(M))\,1_H - \frac{1}{W_I}\sum_{i\in I} w_i f(A_i)\right. \\ & \qquad \left. - f\left((m+M)\,1_H - \frac{1}{W_I}\sum_{i\in I} w_i A_i\right)\right] \\ & + W_J\left[(f(m)+f(M))\,1_H - \frac{1}{W_J}\sum_{i\in J} w_i f(A_i)\right. \\ & \qquad \left. - f\left((m+M)\,1_H - \frac{1}{W_J}\sum_{i\in J} w_i A_i\right)\right]. \end{aligned}$$

Analogously, if f is operator concave, then reversed inequality (2.1) follows from reversed inequality (2.2). □

COROLLARY 1. *Let $I_1,\ldots,I_k$ be finite nonempty sets of positive integers such that $I_i \cap I_j = \emptyset$, for all $i \neq j \in \{1,\ldots,k\}$. Let w_i $(i \in \cup_{j=1}^k I_j)$ be positive real numbers and A_i $(i \in \cup_{j=1}^k I_j)$ selfadjoint operators in $\mathscr{B}(H)$ with spectra contained in $[m,M]$.*

If $f \in C([m,M])$ is an operator convex function, then

$$F\Big(\bigcup_{j=1}^{k} I_j\Big) \geqslant \sum_{j=1}^{k} F(I_j). \tag{2.3}$$

If a function f is operator concave, then the inequality (2.3) is reversed.

Proof. Directly from Theorem 1 by induction. □

The following corollaries give refinements of (1.3).

COROLLARY 2. *Let $I_k = \{1,\dots,k\}$ $(k = 1,\dots,n)$. Let $w_i\,(i \in I_n)$ be positive real numbers and $A_i\,(i \in I_n)$ selfadjoint operators in $\mathscr{B}(H)$ with spectra contained in $[m,M]$. If $f \in C([m,M])$ is an operator convex function, then*

$$F(I_n) \geqslant F(I_{n-1}) \geqslant \cdots \geqslant F(I_2) \geqslant F(I_1) \geqslant 0. \tag{2.4}$$

If a function f is operator concave, then inequalities (2.4) are reversed.

Proof. If we let $n = 1$ in Theorem A, then we have

$$f((m+M)1_H - A_1) \leqslant (f(m) + f(M))1_H - f(A_1).$$

Since $w_1 > 0$, it follows that $F(I_1) \geqslant 0$. Similarly, we may conclude that $F(\{k\}) \geqslant 0$ for all $k \in I_n$. Now, from Theorem 1 it follows that

$$F(I_k) = F(I_{k-1} \cup \{k\}) \geqslant F(I_{k-1}) + F(\{k\}) \geqslant F(I_{k-1})$$

for all $k \in \{2,\dots,n\}$. □

COROLLARY 3. *Let $I_k = \{1,\dots,k\}$ $(k = 1,\dots,n)$. Let $w_i\,(i \in I_n)$ be positive real numbers and $A_i\,(i \in I_n)$ selfadjoint operators in $\mathscr{B}(H)$ with spectra contained in $[m,M]$. If $f \in C([m,M])$ is an operator convex function, then*

$$\begin{aligned} F(I_n) \geqslant (w_i + w_j)\Bigg[(f(m) + f(M))1_H - \frac{w_i f(A_i) + w_j f(A_j)}{w_i + w_j} \\ - f\left((m+M)1_H - \frac{w_i A_i + w_j A_j}{w_i + w_j}\right)\Bigg], \text{ for all } 1 \leqslant i < j \leqslant n \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} F(I_n) \geqslant w_i\,[(f(m) + f(M))1_H - f(A_i) \\ - f((m+M)1_H - A_i)], \text{ for all } 1 \leqslant i \leqslant n. \end{aligned} \tag{2.6}$$

If a function f is operator concave, then inequalities (2.5) and (2.6) are reversed.

Proof. Similarly as $F(I_n) \geqslant F(I_2)$ in Corollary 2, we may conclude that $F(I_n) \geqslant F(\{i,j\})$ for all $i \neq j \in \{1,\dots,n\}$ and analogously, that $F(I_n) \geqslant F(\{i\})$ for all $i \in \{1,\dots,n\}$. □

3. Applications

Let $M_n^{[r]}$ be the (weighted) power mean of order r of selfadjoint operators $A_i \in \mathscr{B}(H)$ with spectra contained in $[m,M]$, for some scalars $0<m<M$, formed with positive weights w_i $(i=1,\dots,n)$, i.e.,

$$M_n^{[r]} = \begin{cases} \left[\frac{1}{W_n}\sum_{i=1}^n w_i A_i^r\right]^{\frac{1}{r}}, & r \neq 0, \\ \exp\left(\frac{1}{W_n}\sum_{i=1}^k w_i \ln(A_i)\right), & r = 0. \end{cases}$$

If we define

$$\widetilde{M}_n^{[r]} := \begin{cases} \left((m^r + M^r)\,1_H - \frac{1}{W_n}\sum_{i=1}^n w_i A_i^r\right)^{\frac{1}{r}}, & r \neq 0, \\ \exp\left((\ln mM)\,1_H - \frac{1}{W_n}\sum_{i=1}^k w_i \ln(A_i)\right), & r = 0, \end{cases}$$

then we have the following results.

THEOREM 2.

$$\begin{aligned} W_n\left(\ln \widetilde{M}_n^{[0]} - \ln \widetilde{M}_n^{[1]}\right) &\leqslant W_{n-1}\left(\ln \widetilde{M}_{n-1}^{[0]} - \ln \widetilde{M}_{n-1}^{[1]}\right) \\ &\leqslant \cdots \leqslant W_1\left(\ln \widetilde{M}_1^{[0]} - \ln \widetilde{M}_1^{[1]}\right) \leqslant 0. \end{aligned} \tag{3.1}$$

Proof. Applying Corollary 2 to the operator concave function $f(x) = \ln x$ we obtain (3.1), since in this case

$$\begin{aligned} F(I_k) = W_k \Bigg[(\ln mM)\,1_H - \frac{1}{W_k}\sum_{i=1}^k w_i \ln(A_i) \\ - \ln\left((m+M)\,1_H - \frac{1}{W_k}\sum_{i=1}^k w_i A_i\right)\Bigg] \\ = W_k\left(\ln \widetilde{M}_k^{[0]} - \ln \widetilde{M}_k^{[1]}\right). \end{aligned}$$

□

COROLLARY 4.

$$\begin{aligned} W_n\left(\ln \widetilde{M}_n^{[-1]} - \ln \widetilde{M}_n^{[0]}\right) &\leqslant W_{n-1}\left(\ln \widetilde{M}_{n-1}^{[-1]} - \ln \widetilde{M}_{n-1}^{[0]}\right) \\ &\leqslant \cdots \leqslant W_1\left(\ln \widetilde{M}_1^{[-1]} - \ln \widetilde{M}_1^{[0]}\right) \leqslant 0. \end{aligned}$$

Proof. Directly from Theorem 2 by the substitutions $m \to m^{-1}$, $M \to M^{-1}$, $A_i \to A_i^{-1}$. □

THEOREM 3. *If* $r \leqslant -1$ *or* $\frac{1}{2} \leqslant r \leqslant 1$, *then*

$$W_n\left(\widetilde{M}_n^{[1]}-\widetilde{M}_n^{[r]}\right)W_{n-1}\left(\widetilde{M}_{n-1}^{[1]}-\widetilde{M}_{n-1}^{[r]}\right) \geqslant \cdots \geqslant W_1\left(\widetilde{M}_1^{[1]}-\widetilde{M}_1^{[r]}\right) \geqslant 0. \tag{3.2}$$

If $r \geqslant 1$, *then inequalities* (3.2) *are reversed.*

Proof. Suppose that $r \leqslant -1$. Applying Corollary 2 to the operator convex function $f(x) = x^{\frac{1}{r}}$, and replacing m, M, and A_i with m^r, M^r, and A_i^r respectively, we obtain (3.2), since in this case

$$\begin{aligned} F(I_k) &= W_k\left[(m+M)\,1_H - \frac{1}{W_k}\sum_{i=1}^{k} w_i A_i \right. \\ &\quad \left. - \left((m^r+M^r)\,1_H - \frac{1}{W_k}\sum_{i=1}^{k} w_i A_i^r\right)^{\frac{1}{r}}\right] \\ &= W_k\left(\widetilde{M}_k^{[1]}-\widetilde{M}_k^{[r]}\right). \end{aligned}$$

If $\frac{1}{2} \leqslant r \leqslant 1$, then the function $f(x) = x^{\frac{1}{r}}$ is operator convex, so (3.2) also hold. If $r \geqslant 1$, then the function $f(x) = x^{\frac{1}{r}}$ is operator concave, so (3.2) are reversed. □

THEOREM 4. *Let* $r,s \in \mathbf{R}$, $r \leqslant s$.

(i) *If* $0 < r$ *and* $s \leqslant 2r$, *or* $0 < s \leqslant -r$, *then*

$$W_n\left(\left(\widetilde{M}_n^{[s]}\right)^s - \left(\widetilde{M}_n^{[r]}\right)^s\right) \geqslant W_{n-1}\left(\left(\widetilde{M}_{n-1}^{[s]}\right)^s - \left(\widetilde{M}_{n-1}^{[r]}\right)^s\right) \geqslant \cdots \geqslant W_1\left(\left(\widetilde{M}_1^{[s]}\right)^s - \left(\widetilde{M}_1^{[r]}\right)^s\right) \geqslant 0. \tag{3.3}$$

(ii) *If* $s < 0$, *then inequalities* (3.3) *are reversed.*

(iii) *If* $s < 0$ *and* $2s \leqslant r$, *or* $0 < -r \leqslant s$, *then*

$$W_n\left(\left(\widetilde{M}_n^{[r]}\right)^r - \left(\widetilde{M}_n^{[s]}\right)^r\right) \geqslant W_{n-1}\left(\left(\widetilde{M}_{n-1}^{[r]}\right)^r - \left(\widetilde{M}_{n-1}^{[s]}\right)^r\right) \geqslant \cdots \geqslant W_1\left(\left(\widetilde{M}_1^{[r]}\right)^r - \left(\widetilde{M}_1^{[s]}\right)^r\right) \geqslant 0. \tag{3.4}$$

(iv) *If* $0 < r$, *then inequalities* (3.4) *are reversed.*

Proof. (i) Suppose that $0 < r \leqslant s \leqslant 2r$. Applying Corollary 2 to the operator convex function $f(x) = x^{\frac{s}{r}}$, and replacing m, M, and A_i with m^r, M^r, and A_i^r

respectively, we obtain (3.3), since in this case

$$F(I_k) = W_k \left[(m^s + M^s) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i A_i^s - \left((m^r + M^r) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i A_i^r \right)^{\frac{s}{r}} \right]$$
$$= W_k \left(\left(\widetilde{M}_k^{[s]} \right)^s - \left(\widetilde{M}_k^{[r]} \right)^s \right).$$

If $r \leqslant s$ and $0 < s \leqslant -r$, then the function $f(x) = x^{\frac{s}{r}}$ is operator convex, so (3.3) also hold.

(ii) If $r \leqslant s < 0$, then the function $f(x) = x^{\frac{s}{r}}$ is operator concave, so (3.3) are reversed.

(iii) Suppose that $2s \leqslant r \leqslant s < 0$. Applying Corollary 2 to the operator convex function $f(x) = x^{\frac{r}{s}}$, and replacing m, M, and A_i with m^s, M^s, and A_i^s respectively, we obtain (3.3), since in this case

$$F(I_k) = W_k \left[(m^r + M^r) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i A_i^r - \left((m^s + M^s) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i A_i^s \right)^{\frac{r}{s}} \right]$$
$$= W_k \left(\left(\widetilde{M}_k^{[r]} \right)^r - \left(\widetilde{M}_k^{[s]} \right)^r \right).$$

If $r \leqslant s$ and $0 < -r \leqslant s$, then the function $f(x) = x^{\frac{r}{s}}$ is operator convex, so (3.4) also hold.

(iv) If $0 < r \leqslant s$, then the function $f(x) = x^{\frac{r}{s}}$ is operator concave, so (3.4) are reversed. □

Let $\varphi \in C([m,M])$ be strictly monotonic function on an interval $[m,M]$. Let $M_\varphi^{[n]}$ be the quasi-arithmetic mean of selfadjoint operators $A_i \in \mathscr{B}(H)$ with spectra contained in $[m,M]$, for some scalars $0 < m < M$, formed with positive weights w_i $(i = 1,\ldots,n)$, i.e.,

$$M_\varphi^{[n]} = \varphi^{-1} \left(\frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(A_i) \right)$$

If we define

$$\widetilde{M}_\varphi^{[n]} := \varphi^{-1} \left((\varphi(m) + \varphi(M)) 1_H - \frac{1}{W_n} \sum_{i=1}^{n} w_i \varphi(A_i) \right).$$

then we have the following results.

THEOREM 5. *Let* $\varphi, \psi \in C([m,M])$ *be strictly monotonic functions. If* $\psi \circ \varphi^{-1}$ *is operator convex, then*

$$W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right) \geqslant W_{n-1}\left(\psi\left(\widetilde{M}_{\psi}^{[n-1]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[n-1]}\right)\right) \geqslant \cdots \geqslant W_1\left(\psi\left(\widetilde{M}_{\psi}^{[1]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[1]}\right)\right) \geqslant 0. \tag{3.5}$$

If $\psi \circ \varphi^{-1}$ *is operator concave, then inequalities* (3.5) *are reversed.*

Proof. Applying Corollary 2 to the operator convex function $f = \psi \circ \varphi^{-1}$, and replacing m, M, and A_i with $\varphi(m)$, $\varphi(M)$, and $\varphi(A_i)$ respectively, we obtain (3.5), since in this case

$$\begin{aligned} F(I_k) = W_k \Bigg(& (\psi(m) + \psi(M)) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i \psi(A_i) \\ & - \left(\psi \circ \varphi^{-1}\right)\left((\varphi(m) + \varphi(M)) 1_H - \frac{1}{W_k} \sum_{i=1}^{k} w_i \varphi(A_i)\right)\Bigg) \\ = {} & W_k\left(\psi\left(\widetilde{M}_{\psi}^{[k]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[k]}\right)\right). \end{aligned}$$

□

REMARK 2. Theorems 2, 3 and 4 follow from Theorem 5 by choosing adequate functions φ and ψ, and appropriate substitutions.

THEOREM 6. *Let* $\varphi, \psi \in C([m,M])$ *be strictly monotonic functions. If* $\psi \circ \varphi^{-1}$ *is operator convex, then*

$$\begin{aligned} & W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right) \\ & \geqslant (w_i + w_j)\left[(\psi(m) + \psi(M)) 1_H - \frac{w_i \psi(A_i) + w_j \psi(A_j)}{w_i + w_j}\right. \\ & \left. - \left(\psi \circ \varphi^{-1}\right)\left((\varphi(m) + \varphi(M)) 1_H - \frac{w_i \varphi(A_i) + w_j \varphi(A_j)}{w_i + w_j}\right)\right], \text{ for all } 1 \leqslant i < j \leqslant n \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} & W_n\left(\psi\left(\widetilde{M}_{\psi}^{[n]}\right) - \psi\left(\widetilde{M}_{\varphi}^{[n]}\right)\right) \\ & \geqslant w_i\left[(\psi(m) + \psi(M)) 1_H - \psi(A_i)\right. \\ & \left. - \left(\psi \circ \varphi^{-1}\right)\left((\varphi(a) + \varphi(M)) 1_H - \varphi(A_i)\right)\right], \text{ for all } 1 \leqslant i \leqslant n \end{aligned} \tag{3.7}$$

If $\psi \circ \varphi^{-1}$ *is operator concave, then inequalities* (3.6) *and* (3.7) *are reversed.*

REMARK 3. From Theorem 6, analogous assertions for the power means $\widetilde{M}_n^{[r]}$ follow by choosing adequate functions φ and ψ, and appropriate substitutions.

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