

## HARDY–HILBERT’S TYPE INEQUALITIES FOR $(p, q)$ -Hö $(0, \infty)$ FUNCTIONS

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*Abstract.* New inequalities concerning functions of the form  $f(xy)$  similar to Hardy-Hilbert’s integral inequality are presented. A new class of functions denoted by  $(p, q)^-$ -Hö $(I)$  is defined. Many other new inequalities are also given.

### 1. Introduction

A mapping  $f : I \subseteq [0, \infty) \rightarrow [0, \infty)$  (where  $I = (0, \infty), [0, \infty), (0, 1), [0, 1), (0, 1]$  or  $[0, 1]$ ) is called  $(p, q)$ -Hölder type on  $I$ , where  $p, q$  are fixed with the property  $p, q > 1, 1/p + 1/q = 1$ , if (see [1])

$$f(xy) \leq (f(x^p))^{1/p} (f(y^q))^{1/q} \quad \text{for all } x, y \in I.$$

The class of these functions is denoted by  $(p, q)$ -Hö $(I)$ . If the inequality is reversed, then we say that  $f \in (p, q)^-$ -Hö $(I)$ .

In [1], the authors proved that the following functions belong to  $(p, q)$ -Hö $(0, \infty)$

1.  $f(x) = e^{g(\ln x)}$ ,  $g$  is convex on  $(0, \infty)$ .
2.  $f$  is logarithmically convex and monotonic non-decreasing on  $(0, \infty)$ .
3.  $f$  is absolutely monotonic on  $(0, \infty)$ .
4.  $f$  is exponentially convex on  $(0, \infty)$ .

If  $f, g \geq 0$  are such that

$$0 < \int_0^\infty f^2(x) dx < \infty, \quad 0 < \int_0^\infty g^2(x) dx < \infty,$$

then the famous Hilbert’s integral inequality is given by

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \pi \left( \int_0^\infty f^2(x) dx \int_0^\infty g^2(x) dx \right)^{1/2}, \quad (1)$$

where the constant factor  $\pi$  is the best possible (see [3]). Inequality (1) has been generalized by Hardy-Riesz [2] as follows.

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If  $p > 1, \frac{1}{p} + \frac{1}{q} = 1,$

$$0 < \int_0^\infty f^p(x)dx < \infty, \quad 0 < \int_0^\infty g^q(x)dx < \infty,$$

then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy < \frac{\pi}{\sin(\pi/p)} \left( \int_0^\infty f^p(x)dx \right)^{1/p} \left( \int_0^\infty g^q(x)dx \right)^{1/q}, \quad (2)$$

where the constant factor  $\frac{\pi}{\sin(\pi/p)}$  is the best possible. We call (2) Hardy-Hilbert’s integral inequality, which is important in analysis and its applications (see [5]).

In the sequel the Beta function is denoted and defined by

$$\begin{aligned} B(p, q) &= \int_0^\infty \frac{x^{p-1}}{(1+x)^{p+q}} dx, \\ &= \int_0^1 x^{p-1}(1-x)^{q-1} dx, \quad p, q > 0. \end{aligned}$$

The following are examples for functions that belong to  $(p, q)^{-}\text{-H}\ddot{o}(0, \infty)$ . The proofs are similar to proofs in [1] and therefore are omitted (in [1] the proofs are related to convex and logarithmically convex functions).

PROPOSITION 1.1. *Let  $g : \mathbf{R} \rightarrow \mathbf{R}$  be a concave mapping and let  $f : (0, \infty) \rightarrow (0, \infty)$  given by  $f(x) = e^{g(\ln x)}$ . Then  $f \in (p, q)^{-}\text{-H}\ddot{o}(0, \infty)$ .*

PROPOSITION 1.2. *Suppose that  $f : (0, \infty) \rightarrow (0, \infty)$  logarithmically concave and monotonically non-increasing on  $(0, \infty)$ . Then  $f \in (p, q)^{-}\text{-H}\ddot{o}(0, \infty)$ .*

The following remarks are needed for our aim.

REMARK 1.3. The function  $f(t) = \frac{t}{1+t}$  is non-decreasing. Hence  $f^\lambda(t)$  is so for any  $\lambda > 0$ .

REMARK 1.4. Let  $\alpha, \beta > 0, 0 < t < 1$ . Then the function  $g(t) = \frac{t^\alpha}{(1-t)^\beta}$  is non-decreasing.

LEMMA 1.5. *Let  $p_i > 0, \sum_{i=1}^n p_i = 1$ . Then*

$$1 + \prod_{i=1}^n x_i^{p_i} \leq \prod_{i=1}^n (1 + x_i)^{p_i}, \quad x_i \geq 0. \quad (3)$$

$$1 - \prod_{i=1}^n x_i^{p_i} \geq \prod_{i=1}^n (1 - x_i)^{p_i}, \quad 0 \leq x_i \leq 1. \quad (4)$$

*Proof.* The proof by induction. We first consider the case  $n = 2$ . Set

$$f(x_1) = (1 + x_1)^{p_1} (1 + x_2)^{p_2} - x_1^{p_1} x_2^{p_2} - 1, \quad (5)$$

we also assume that  $p_1 + p_2 = 1, x_1 \geq x_2, x_2$  is fixed, and  $x_1$  is variable. We have

$$f'(x_1) = p_1 \left( \frac{1+x_2}{1+x_1} \right)^{p_2} - p_1 \left( \frac{x_2}{x_1} \right)^{p_2} \geq 0.$$

Hence  $f$  is non-decreasing and attains its minimum when  $x_1 = x_2$ , which is 0. Therefore  $f(x_1) \geq 0$ .

Now we assume that the Theorem is true for  $n - 1$ . That is

$$1 + \prod_{i=1}^{n-1} x_i^{p_i} \leq \prod_{i=1}^{n-1} (1+x_i)^{p_i}, \quad \text{if } \sum_{i=1}^{n-1} p_i = 1.$$

Let  $\sum_{i=1}^n p_i = 1$ , then, as  $\sum_{i=1}^{n-1} \frac{p_i}{1-p_n} = 1$ , we have

$$\begin{aligned} 1 + \prod_{i=1}^n x_i^{p_i} &= 1 + x_n^{p_n} \prod_{i=1}^{n-1} x_i^{\frac{1-p_n}{1-p_n} p_i} \leq (1+x_n)^{p_n} \left( 1 + \prod_{i=1}^{n-1} x_i^{\frac{p_i}{1-p_n}} \right)^{1-p_n} \\ &\leq (1+x_n)^{p_n} \left( \prod_{i=1}^{n-1} (1+x_i)^{\frac{p_i}{1-p_n}} \right)^{1-p_n} \\ &= \prod_{i=1}^n (1+x_i)^{p_i}. \end{aligned}$$

The proof of the other part is similar. □

The first part of the following Lemma is quoted in [1], we prove the other new part.

LEMMA 1.6. *Let  $x_i > 0, p_i > 1, i = 1, \dots, n, \sum_{i=1}^n 1/p_i = 1$ . Then*

$$f \left( \prod_{i=1}^n x_i \right) \leq \prod_{i=1}^n (f(x_i^{p_i}))^{1/p_i} \quad \text{provided } f \in (p, q)\text{-Hö}(0, \infty). \tag{6}$$

$$f \left( \prod_{i=1}^n x_i \right) \geq \prod_{i=1}^n (f(x_i^{p_i}))^{1/p_i} \quad \text{provided } f \in (p, q)^-\text{-Hö}(0, \infty). \tag{7}$$

This Lemma appears in [1], and the second part is proved in a similar way and therefore will be omitted.

## 2. Main Results

In this chapter we prove seven theorems related to functions belonging to  $(p, q)$ -Hö $(I)$  and to functions belonging to  $(p, q)^-$ -Hö $(I)$ .

THEOREM 2.1. *Let  $f, F$  be non-negative and belong to  $(p, q)$ -Hö $(0, \infty)$ ,  $f'$  exists and strictly positive,  $f(0) = 0, f(\infty) = \infty, p > 1, q > 1, 1/p + 1/q = 1$ , let*

$\lambda > 0, \mu > 1, \max \left\{ \frac{p-1}{q}, \frac{q-1}{p} \right\} < \frac{\mu-1}{\lambda} < \min \left\{ \frac{p}{q}, \frac{q}{p} \right\}$ . Then

$$\int_0^\infty \int_0^\infty F(xy) \left( \frac{f(xy)}{1+f(xy)} \right)^\lambda dx dy \leq K \left( \int_0^\infty \frac{(f(x^p))^{(pq-2)\mu-\lambda-p/q} F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \right)^{1/p} \times \left( \int_0^\infty \frac{(f(y^q))^{(pq-2)\mu-\lambda-q/p} F(y^q)}{y^{q(q-1)/p} (f'(y^q))^{q/p}} dy \right)^{1/q} \tag{8}$$

where

$$K = B^{1/p}(\lambda p - \mu q + q, \lambda - \lambda p + \mu q - q) B^{1/q}(\lambda q - \mu p + p, \lambda - \lambda q + \mu p - p),$$

provided the integrals on the right do exist.

*Proof.* By virtue of Remark 1.3,

$$\begin{aligned} & \int_0^\infty \int_0^\infty F(xy) \left( \frac{f(xy)}{1+f(xy)} \right)^\lambda dx dy \\ & \leq \int_0^\infty \int_0^\infty F^{\frac{1}{p}}(x^p) F^{\frac{1}{q}}(y^q) \left( \frac{f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)}{1+f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)} \right)^\lambda dx dy \\ & = \int_0^\infty \int_0^\infty \frac{F^{\frac{1}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q) f^{-\frac{\mu}{p}}(y^q) (f'(y^q) y^{q-1})^{\frac{1}{p}}}{f^{-\frac{\mu}{q}}(x^p) (f'(x^p) x^{p-1})^{\frac{1}{q}} \left( 1 + f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q) \right)^{\frac{\lambda}{p}}} \\ & \quad \times \frac{F^{\frac{1}{q}}(y^q) f^{\frac{\lambda}{p}}(x^p) f^{-\frac{\mu}{q}}(x^p) (f'(x^p) x^{p-1})^{\frac{1}{q}}}{f^{-\frac{\mu}{p}}(y^q) (f'(y^q) y^{q-1})^{\frac{1}{p}} \left( 1 + f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q) \right)^{\frac{\lambda}{q}}} dx dy \\ & \leq \left( \int_0^\infty \int_0^\infty \frac{F(x^p) f^{\frac{\lambda p}{q}}(y^q) f^{-\mu}(y^q) (f'(y^q) y^{q-1})}{f^{-\frac{\mu p}{q}}(x^p) (f'(x^p) x^{p-1})^{\frac{p}{q}} \left( 1 + f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q) \right)^\lambda dx dy \right)^{1/p} \\ & \quad \times \left( \int_0^\infty \int_0^\infty \frac{F(y^q) f^{\frac{\lambda q}{p}}(x^p) f^{-\mu}(x^p) (f'(x^p) x^{p-1})}{f^{-\frac{\mu q}{p}}(y^q) (f'(y^q) y^{q-1})^{\frac{q}{p}} \left( 1 + f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q) \right)^\lambda dx dy \right)^{1/q} \\ & = M^{1/p} N^{1/q}. \end{aligned}$$

To consider  $M$ , let  $f(y^q) = v^q$ , then

$$\begin{aligned} M & = \int_0^\infty \frac{(f(x^p))^{(pq-2)\mu-\lambda-q/p} F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \int_0^\infty \frac{(v f^{1/p}(x^p))^{\lambda p - \mu q + q - 1} f^{1/p}(x^p)}{(1 + v f^{1/p}(x^p))^\lambda} dv \\ & = \int_0^\infty \frac{(f(x^p))^{(pq-2)\mu-\lambda-q/p} F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \int_0^\infty \frac{z^{\lambda p - \mu q + q - 1}}{(1 + z)^\lambda} dz \\ & = B(\lambda p - \mu q + q, \lambda - \lambda p + \mu q - q) \int_0^\infty \frac{(f(x^p))^{(pq-2)\mu-\lambda-q/p} F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx. \end{aligned}$$

Similarly,

$$N = B(\lambda q - \mu p + p, \lambda - \lambda q + \mu p - p) \int_0^\infty \frac{(f(y^q))^{(pq-2)\mu - \lambda - p/q} F(y^q)}{y^{q(q-1)/p} (f'(y^q))^{q/p}} dy.$$

□

**THEOREM 2.2.** *Let  $f, F$  be non-negative and belong to  $(p, q)$ -Hö(0, ∞),  $f'$  exists and strictly positive,  $f(0) = 0, f(\infty) = \infty, p > 1, q > 1, 1/p + 1/q = 1$ . Let  $\lambda > 0, \mu > 1, \max\left\{\frac{p-1}{q}, \frac{q-1}{p}\right\} < \frac{\mu-1}{\lambda} < \min\left\{\frac{p}{q}, \frac{q}{p}\right\}$ . Then*

$$\int_0^\infty \int_0^\infty F(xy) \frac{f^\lambda(xy)}{1+f^\lambda(xy)} dx dy \leq K \left( \int_0^\infty \frac{F(x^p) (f(x^p))^{\mu(p-1) + \frac{\mu-1}{\lambda}(q-1)-1}}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \right)^{1/p} \times \left( \int_0^\infty \frac{F(y^q) (f(y^q))^{\mu(q-1) + \frac{\mu-1}{\lambda}(p-1)-1}}{y^{q(q-1)/p} (f'(y^q))^{q/p}} dy \right)^{1/q}, \tag{9}$$

where

$$K = \frac{1}{\lambda} B^{1/p} \left( p + \frac{q}{\lambda}(1-\mu), 1-p + \frac{q}{\lambda}(\mu-1) \right) B^{1/q} \left( q + \frac{p}{\lambda}(1-\mu), 1-q + \frac{p}{\lambda}(\mu-1) \right),$$

provided the integrals on the right do exist.

*Proof.* Making use of Remark 1.3,

$$\begin{aligned} & \int_0^\infty \int_0^\infty F(xy) \frac{f^\lambda(xy)}{1+f^\lambda(xy)} dx dy \\ & \leq \int_0^\infty \int_0^\infty \frac{F^{\frac{1}{p}}(x^p) F^{\frac{1}{q}}(y^q) f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)}{1+f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)} dx dy \\ & = \int_0^\infty \int_0^\infty \frac{F^{\frac{1}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q) f^{-\frac{\mu}{p}}(y^q) (f'(y)y^{q-1})^{\frac{1}{p}}}{f^{-\frac{\mu}{q}}(x^p) (f'(x^p)x^{p-1})^{\frac{1}{q}} \left(1+f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)\right)^{\frac{1}{p}}} \\ & \quad \times \frac{F^{\frac{1}{q}}(y^q) f^{\frac{\lambda}{p}}(x^p) f^{-\frac{\mu}{q}}(x^p) (f'(x^p)x^{p-1})^{\frac{1}{q}}}{f^{-\frac{\mu}{p}}(y^q) (f'(y^q)y^{q-1})^{\frac{1}{p}} \left(1+f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)\right)^{\frac{1}{q}}} dx dy \\ & \leq \left( \int_0^\infty \int_0^\infty \frac{F(x^p) f^{\frac{\lambda p}{q}}(y^q) f^{-\mu}(y^q) (f'(y^q)y^{q-1})}{f^{-\frac{\mu p}{q}}(x^p) (f'(x^p)x^{p-1})^{\frac{p}{q}} \left(1+f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)\right)} dx dy \right)^{1/p} \\ & \quad \times \left( \int_0^\infty \int_0^\infty \frac{F(y^q) f^{\frac{\lambda q}{p}}(x^q) f^{-\mu}(x^p) (f'(x^p)x^{p-1})}{f^{-\frac{\mu q}{p}}(y^q) (f'(y^q)y^{q-1})^{\frac{q}{p}} \left(1+f^{\frac{\lambda}{p}}(x^p) f^{\frac{\lambda}{q}}(y^q)\right)} dx dy \right)^{1/q} \\ & = S^{1/p} T^{1/q}. \end{aligned}$$

In order to consider  $S$ , let  $f(y^q) = v^{q/\lambda}$ . Then

$$\begin{aligned}
 S &= \frac{1}{\lambda} \int_0^\infty \frac{F(x^p) (f(x^p))^{\mu(p-1) + \frac{\mu-1}{\lambda}(q-1) - 1}}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \int_0^\infty \frac{(vf^{1/p}(x^p))^{p + \frac{q}{\lambda}(1-\mu) - 1} f^{1/p}(x^p)}{(1 + vf^{1/p}(x^p))} dv \\
 &= \frac{1}{\lambda} B\left(p + \frac{q}{\lambda}(1 - \mu), 1 - p + \frac{q}{\lambda}(\mu - 1)\right) \int_0^\infty \frac{F(x^p) (f(x^p))^{\mu(p-1) + \frac{\mu-1}{\lambda}(q-1) - 1}}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx
 \end{aligned}$$

Similarly,

$$T = \frac{1}{\lambda} B\left(q + \frac{p}{\lambda}(1 - \mu), 1 - q + \frac{p}{\lambda}(\mu - 1)\right) \int_0^\infty \frac{F(y^q) (f(y^q))^{\mu(q-1) + \frac{\mu-1}{\lambda}(p-1) - 1}}{y^{q(q-1)/p} (f'(y^q))^{q/p}} dy$$

□

**THEOREM 2.3.** *Let  $f, F$  be non-negative and belongs to  $(p, q)$ -Hö $(0, \infty)$ ,  $f'$  exists and strictly positive,  $f(0) = 0, f(1) = 1, p > 1, q > 1, 1/p + 1/q = 1, \alpha > 0, 0 < \beta < 1$ . Then*

$$\begin{aligned}
 \int_0^1 \int_0^1 F(xy) \frac{f^\alpha(xy)}{(1 - f(xy))^\beta} dx dy &\leq K \left( \int_0^1 \frac{F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{p/q}} dx \right)^{1/p} \\
 &\times \left( \int_0^1 \frac{F(y^q)}{y^{q(q-1)/p} (f'(y^q))^{q/p}} dy \right)^{1/q}, \quad (10)
 \end{aligned}$$

where

$$K = B^{1/p} \left( \frac{\alpha p}{q} + q, 1 - \beta \right) B^{1/q} \left( \frac{\alpha q}{p} + p, 1 - \beta \right),$$

provided the integrals on the right do exist.

*Proof.* By virtue of Remark 1.4, we have

$$\begin{aligned}
 \int_0^1 \int_0^1 F(xy) \frac{f^\alpha(xy)}{(1 - f(xy))^\beta} dx dy &\leq \int_0^1 \int_0^1 \frac{F^{\frac{1}{p}}(x^p) F^{\frac{1}{q}}(y^q) f^{\frac{\alpha}{p}}(x^p) f^{\frac{\alpha}{q}}(y^q)}{\left(1 - f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)\right)^\beta} dx dy \\
 &= \int_0^\infty \int_0^\infty \frac{F^{\frac{1}{p}}(x^p) f^{\frac{\alpha}{q}}(y^q) (f'(y^q) y^{q-1})^{\frac{1}{p}}}{(f'(x^p) x^{p-1})^{\frac{1}{q}} \left(1 - f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)\right)^{\frac{\beta}{p}}} \\
 &\quad \times \frac{F^{\frac{1}{q}}(y^q) f^{\frac{\alpha}{p}}(x^p) (f'(x^p) x^{p-1})^{\frac{1}{q}}}{(f'(y^q) y^{q-1})^{\frac{1}{p}} \left(1 - f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)\right)^{\frac{\beta}{q}}} dx dy \\
 &\leq \left( \int_0^1 \int_0^1 \frac{F(x^p) f^{\frac{\alpha p}{q}}(y^q) (f'(y^q) y^{q-1})}{(f'(x^p) x^{p-1})^{\frac{p}{q}} \left(1 - f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)\right)^\beta} dx dy \right)^{1/p}
 \end{aligned}$$

$$\begin{aligned} & \times \left( \int_0^1 \int_0^1 \frac{F(y^q) f^{\frac{\alpha q}{p}}(x^p) (f'(x^p) x^{p-1})}{(f'(y) y^{q-1})^{\frac{q}{p}} \left(1 - f^{\frac{1}{p}}(x^p) f^{\frac{1}{q}}(y^q)\right)^\beta} dx dy \right)^{1/q} \\ & = G^{1/p} H^{1/q}. \end{aligned}$$

As before, we consider first  $G$ . Let  $f(y^q) = v^q$ . Then

$$G = \int_0^1 \frac{F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{\frac{p}{q}}} dx \int_0^1 \frac{v^{\frac{\alpha p}{q} + q - 1}}{\left(1 - v f^{\frac{1}{p}}(x^p)\right)^\beta} dv.$$

As  $f(x^p) \leq 1$ , then

$$\int_0^1 \frac{v^{\frac{\alpha p}{q} + q - 1}}{\left(1 - v f^{\frac{1}{p}}(x^p)\right)^\beta} dv \leq \int_0^1 \frac{v^{\frac{\alpha p}{q} + q - 1}}{(1 - v)^\beta} dv = B\left(\frac{\alpha p}{q} + q, 1 - \beta\right).$$

Therefore, we obtain

$$G \leq B\left(\frac{\alpha p}{q} + q, 1 - \beta\right) \int_0^1 \frac{F(x^p)}{x^{p(p-1)/q} (f'(x^p))^{\frac{p}{q}}} dx.$$

Similarly,

$$H \leq B\left(\frac{\alpha q}{p} + p, 1 - \beta\right) \int_0^1 \frac{F(y^q)}{y^{q(q-1)/p} (f'(y^q))^{\frac{q}{p}}} dy.$$

□

**THEOREM 2.4.** Let  $f \in (p, q)$ -Hö(0, ∞),  $\lambda > 0$ ,  $p_i > 1$ ,  $\sum_{i=1}^n 1/p_i = 1$ . Then

$$\int_0^\infty \cdots \int_0^\infty \left( \frac{f\left(\prod_{i=1}^n x_i\right)}{1 + f\left(\prod_{i=1}^n x_i\right)} \right)^\lambda dx_1 \cdots dx_n \geq \prod_{i=1}^n \left( \int_0^\infty \frac{f^{\lambda/p_i}(x_i^{p_i})}{(1 + f(x_i^{p_i}))^{\lambda/p_i}} dx_i \right). \quad (11)$$

*Proof.* Making use of Remark 1.3 and Lemma 1.5,

$$\begin{aligned} & \int_0^\infty \cdots \int_0^\infty \left( \frac{f\left(\prod_{i=1}^n x_i\right)}{1 + f\left(\prod_{i=1}^n x_i\right)} \right)^\lambda dx_1 \cdots dx_n \\ & \geq \int_0^\infty \cdots \int_0^\infty \left( \frac{\prod_{i=1}^n f^{1/p_i}(x_i^{p_i})}{1 + \prod_{i=1}^n f^{1/p_i}(x_i^{p_i})} \right)^\lambda dx_1 \cdots dx_n \\ & = \int_0^\infty \cdots \int_0^\infty \left( \frac{\prod_{i=1}^n f^{\lambda/p_i}(x_i^{p_i})}{\left(1 + \prod_{i=1}^n f^{1/p_i}(x_i^{p_i})\right)^\lambda} \right) dx_1 \cdots dx_n \\ & \geq \int_0^\infty \cdots \int_0^\infty \left( \frac{\prod_{i=1}^n f^{\lambda/p_i}(x_i^{p_i})}{\prod_{i=1}^n (1 + f(x_i^{p_i}))^{\lambda/p_i}} \right) dx_1 \cdots dx_n \\ & = \prod_{i=1}^n \left( \int_0^\infty \frac{f^{\lambda/p_i}(x_i^{p_i})}{1 + f(x_i^{p_i})} dx_i \right)^{\lambda/p_i}. \end{aligned}$$

□

**THEOREM 2.5.** *Let  $f \in (p, q)$ -Hö(0, ∞),  $f(x_i^{p_i}) < 1$ ,  $p_i > 1$ ,  $\sum_{i=1}^n 1/p_i = 1$ ,  $\alpha > 0$ ,  $\beta > 0$ . Then*

$$\int_0^\infty \dots \int_0^\infty \left( \frac{f^\alpha(\prod_{i=1}^n x_i)}{(1 - f(\prod_{i=1}^n x_i))^\beta} \right) dx_1 \dots dx_n \leq \prod_{i=1}^n \left( \int_0^\infty \left( \frac{f^{\alpha/p_i}(x_i^{p_i})}{(1 - f(x_i^{p_i}))^{\beta/p_i}} dx_i \right) \right) \tag{12}$$

*Proof.* Making use of Remark 1.4 and Lemma 1.5,

$$\begin{aligned} & \int_0^\infty \dots \int_0^\infty \left( \frac{f^\alpha(\prod_{i=1}^n x_i)}{(1 - f(\prod_{i=1}^n x_i))^\beta} \right) dx_1 \dots dx_n \\ & \leq \int_0^\infty \dots \int_0^\infty \left( \frac{\prod_{i=1}^n f^{\alpha/p_i}(x_i^{p_i})}{(1 - \prod_{i=1}^n f^{1/p_i}(x_i^{p_i}))^\beta} \right) dx_1 \dots dx_n \\ & \leq \int_0^\infty \dots \int_0^\infty \left( \frac{\prod_{i=1}^n f^{\alpha/p_i}(x_i^{p_i})}{\prod_{i=1}^n (1 - f(x_i^{p_i}))^{\beta/p_i}} \right) dx_1 \dots dx_n \\ & = \prod_{i=1}^n \left( \int_0^\infty \frac{f^{\alpha/p_i}(x_i^{p_i})}{(1 - f(x_i^{p_i}))^{\beta/p_i}} dx_i \right). \end{aligned}$$

□

A mapping  $f : I \subseteq \mathbf{R} \rightarrow \mathbf{R}$  is said to be submultiplicative if  $f(xy) \leq f(x)f(y)$ . Concerning such maps we have the following results.

**THEOREM 2.6.** *Let  $f, F$  be non-negative submultiplicative maps,  $f'$  is strictly positive,  $f(0) = 0$ ,  $f(\infty) = \infty$ ,  $p_i > 1$ ,  $\sum_{i=1}^n 1/p_i = 1$ ,  $\lambda > 0$ ,  $\max\{\frac{\lambda}{p}, \frac{\lambda}{q}\} < \mu < \lambda$ . Then*

$$\begin{aligned} \int_0^\infty \int_0^\infty F(xy) \left( \frac{f(xy)}{1 + f(xy)} \right)^\lambda dx dy & \leq K \left( \int_0^\infty \frac{(f(x))^{p(2\mu - \lambda) + p - 1} F^p(x)}{(f'(x))^{p-1}} dx \right)^{1/p} \\ & \times \left( \int_0^\infty \frac{(f(y))^{q(2\mu - \lambda) + q - 1} F^q(y)}{(f'(y))^{q-1}} dy \right)^{1/q} \end{aligned}$$

where

$$K = B^{1/p}(p(\lambda - \mu), \lambda - p(\lambda - \mu)) B^{1/q}(q(\lambda - \mu), \lambda - q(\lambda - \mu)),$$

provided the integrals on the right do exist.

*Proof.* By Remark 1.3,

$$\begin{aligned} & \int_0^\infty \int_0^\infty F(xy) \left( \frac{f(xy)}{1 + f(xy)} \right)^\lambda dx dy \leq \int_0^\infty \int_0^\infty F(x)F(y) \left( \frac{f(x)f(y)}{1 + f(x)f(y)} \right)^\lambda dx dy \tag{13} \\ & = \int_0^\infty \int_0^\infty \frac{F(x)f^{\lambda - \mu - \frac{1}{p}}(y)(f'(y))^{\frac{1}{p}}}{f^{-\mu - \frac{1}{q}}(x)(f'(x))^{\frac{1}{q}}(1 + f(x)f(y))^{\frac{\lambda}{p}} f^{-\mu - \frac{1}{p}}(y)(f'(y))^{\frac{1}{p}}(1 + f(x)f(y))^{\frac{\lambda}{q}}} dx dy \end{aligned}$$



$$\begin{aligned} &\leq \left( \int_0^\infty \int_0^\infty \frac{F^p(x)f^{p(\lambda-\mu)-1}(y)f'(y)}{f^{-\mu p-\frac{p}{q}}(x)(f'(x))^{\frac{p}{q}}(1+f(x)f(y))^\lambda} dx dy \right)^{1/p} \\ &\times \left( \int_0^\infty \int_0^\infty \frac{F^q(y)f^{q(\lambda-\mu)-1}(x)f'(x)}{f^{-\mu q-\frac{q}{p}}(y)(f'(y))^{\frac{q}{p}}(1+f(x)f(y))^\lambda} dx dy \right)^{1/q} \\ &= E^{1/p} F^{1/q}. \end{aligned}$$

Now, we consider

$$\begin{aligned} E &= \int_0^\infty \frac{(f(x))^{p(2\mu-\lambda)+p} F^p(x)}{(f'(x))^{p-1}} dx \int_0^\infty \frac{(f(x)f(y))^{p(\lambda-\mu)-1} f'(y)}{(1+f(x)f(y))^\lambda} dy \\ &= B(p(\lambda-\mu), \lambda-p(\lambda-\mu)) \int_0^\infty \frac{(f(x))^{p(2\mu-\lambda)+p-1} F^p(x)}{(f'(x))^{p-1}} dx \end{aligned}$$

Similarly,

$$F = B(q(\lambda-\mu), \lambda-q(\lambda-\mu)) \int_0^\infty \frac{(f(y))^{q(2\mu-\lambda)+q-1} F^q(y)}{(f'(y))^{q-1}} dy.$$

□

**THEOREM 2.7.** *Let  $f, F$  be non-negative, submultiplicative,  $f(0) = 0, f(\infty) = \infty, p > 1, 1/p + 1/q = 1, \lambda > 0, \max\{\frac{1}{p}, \frac{1}{q}\} < \mu < 1$ . Then*

$$\begin{aligned} \int_0^\infty \int_0^\infty F(xy) \frac{f^\lambda(xy)}{1+f^\lambda(xy)} dx dy &\leq K \left( \int_0^\infty \frac{(f(x))^{p[\lambda(2\mu-1)+1]-1} F^p(x)}{(f'(x))^{p-1}} dx \right)^{1/p} \\ &\times \left( \int_0^\infty \frac{(f(y))^{q[\lambda(2\mu-1)+1]-1} F^q(y)}{(f'(y))^{q-1}} dy \right)^{1/q}, \end{aligned} \tag{14}$$

where

$$K = \frac{1}{\lambda} B^{1/p}(p(1-\mu), 1-p(1-\mu)) B^{1/q}(q(1-\mu), 1-q(1-\mu)),$$

provided the integrals on the right do exist.

*Proof.* By Remark 1.3,

$$\begin{aligned} &\int_0^\infty \int_0^\infty F(xy) \frac{f^\lambda(xy)}{1+f^\lambda(xy)} dx dy \\ &\leq \int_0^\infty \int_0^\infty F(x)F(y) \frac{f^\lambda(x)f^\lambda(y)}{1+f^\lambda(x)f^\lambda(y)} dx dy \\ &= \int_0^\infty \int_0^\infty \frac{F(x)f^{\lambda-\mu\lambda-\frac{\lambda}{p}}(y)(\lambda f^{\lambda-1}(y)f'(y))^{\frac{1}{p}}}{f^{-\mu\lambda-\frac{\lambda}{q}}(x)(\lambda f^{\lambda-1}(x)f'(x))^{\frac{1}{q}}(1+f^\lambda(x)f^\lambda(y))^{\frac{1}{p}}} \\ &\times \frac{F(y)f^{\lambda-\mu\lambda-\frac{\lambda}{q}}(x)(\lambda f^{\lambda-1}(x))^{\frac{1}{q}}}{f^{-\mu\lambda-\frac{\lambda}{p}}(y)(\lambda f^{\lambda-1}(y)f'(y))^{\frac{1}{p}}(1+f^\lambda(x)f^\lambda(y))^{\frac{1}{q}}} dx dy \end{aligned}$$

$$\begin{aligned} &\leq \left( \int_0^\infty \int_0^\infty \frac{F^p(x) f^{p\lambda(1-\mu)-\lambda}(y) (\lambda f^{\lambda-1}(y) f'(y))}{f^{-\mu\lambda p - \frac{\lambda p}{q}}(x) (\lambda f^{\lambda-1}(x) f'(x))^{\frac{p}{q}} (1 + f^\lambda(x) f^\lambda(y))} dx dy \right)^{1/p} \\ &\quad \times \left( \int_0^\infty \int_0^\infty \frac{F^q(y) f^{q\lambda(1-\mu)-\lambda}(x) (\lambda f^{\lambda-1}(x) f'(x))}{f^{-\mu\lambda q - \frac{\lambda q}{p}}(y) (\lambda f^{\lambda-1}(y) f'(y))^{\frac{q}{p}} (1 + f^\lambda(x) f^\lambda(y))} dx dy \right)^{1/q} \\ &= C^{1/p} D^{1/q}. \end{aligned}$$

To consider  $C$ , we have

$$\begin{aligned} C &= \frac{1}{\lambda} \int_0^\infty \frac{F^p(x) (f(x))^{p[\lambda(2\mu-1)+1]-1}}{(f'(x))^{p/q}} dx \\ &\quad \times \int_0^\infty \frac{(f^\lambda(x) f^\lambda(y))^{p(1-\mu)-1} (\lambda f^{\lambda-1}(y) f'(y)) f^\lambda(x)}{(1 + f^\lambda(x) f^\lambda(y))} dy \\ &= \frac{1}{\lambda} B(p(1-\mu), 1-p(1-\mu)) \int_0^\infty \frac{(f(x))^{p[\lambda(2\mu-1)+1]-1} F^p(x)}{(f'(x))^{p-1}} dx. \end{aligned}$$

Similarly,

$$D = \frac{1}{\lambda} B(q(1-\mu), 1-q(1-\mu)) \int_0^\infty \frac{(f(y))^{q[\lambda(2\mu-1)+1]-1} F^q(y)}{(f'(y))^{q-1}} dy.$$

□

### 3. Applications.

**3.1.** Theorems 2.1, 2.2, 2.3 and 2.5 are satisfied by the following functions (see [1]).

1.  $f(xy) = \sinh(|xy|)$ ,  $x, y \in \mathbf{R}$ .
2.  $f(xy) = \cosh(|xy|)$ ,  $x, y \in \mathbf{R}$ .
3.  $f(xy) = \ln\left(\frac{1}{1-|xy|}\right)$ ,  $x, y \in (-1, 1)$ .
4.  $f(xy) = \ln\left(\frac{1+|xy|}{1-|xy|}\right)$ ,  $x, y \in (-1, 1)$ .
5.  $f(xy) = \sin^{-1}(|xy|)$ ,  $x, y \in (-1, 1)$ .

The above examples of functions that belong to  $(p, q)$ -Hö( $I$ ) are in addition to those functions mentioned in the beginning of the introduction and are also quoted from [1].

**3.2.** Let  $f$  be non-negative belong to  $(p, q)$ -Hö( $0, 1$ ),  $f'(x)$  exist and positive, on the interval  $0 < x < 1$ ,  $f(x^p) < 1$ ,  $p > 1$ ,  $f(0) = 0$ ,  $f(1) = 1$ . Let  $g(x) = \frac{f(x)}{1-f(x)}$ ,  $\frac{1}{p} + \frac{1}{q} = 1$ . Then

$$\int_0^1 \int_0^1 g\left(x^{\frac{1}{p}} y^{\frac{1}{q}}\right) f'(x) f'(y) dx dy \leq \left(\frac{\pi}{\sin(\pi/p)}\right)^2. \tag{15}$$

*Proof.* We claim that  $g \in (p, q)$ -Hö $(0, 1)$ . In fact via Lemma 1.5, we have

$$\begin{aligned} g(xy) &= \frac{f(xy)}{1-f(xy)} \leq \frac{f^{\frac{1}{p}}(x^p)f^{\frac{1}{q}}(y^q)}{1-f^{\frac{1}{p}}(x^p)f^{\frac{1}{q}}(y^q)} \leq \frac{f^{\frac{1}{p}}(x^p)}{1-f^{\frac{1}{p}}(x^p)} \frac{f^{\frac{1}{q}}(y^q)}{1-f^{\frac{1}{q}}(y^q)} \\ &\leq g^{\frac{1}{p}}(x^p)g^{\frac{1}{q}}(y^q). \end{aligned}$$

Therefore

$$\begin{aligned} \int_0^1 \int_0^1 g\left(x^{\frac{1}{p}}y^{\frac{1}{q}}\right)f'(x)f'(y)dxdy &\leq \int_0^1 g^{1/p}(x)f'(x)dx \int_0^1 g^{1/q}(y)f'(y)dy \\ &= \int_0^1 \frac{f^{1/p}(x)f'(x)}{(1-f(x))^{1/p}}dx \int_0^1 \frac{f^{1/q}(y)f'(y)}{(1-f(y))^{1/q}}dy \\ &= B^2\left(\frac{1}{p}, \frac{1}{q}\right) = \left(\frac{\pi}{\sin \pi/p}\right)^2. \end{aligned}$$

□

**3.3.** Let  $f : (0, 1) \rightarrow \mathbf{R}$  defined by

$$f(x) = \frac{x^\alpha}{(1-x)^\beta}, \quad \alpha > -1, 0 < \beta < 1.$$

Then

$$\int_0^1 \int_0^1 f(xy)dxdy \leq B^2(\alpha + 1, \beta - 1). \tag{16}$$

*Proof.*  $f$  is submultiplicative, as for  $0 \leq x, y \leq 1$ ,  $2xy \leq 2\sqrt{xy} \leq x + y$ , which implies  $(1-x)(1-y) \leq 1-xy$ , hence

$$\frac{1}{(1-xy)^\beta} \leq \frac{1}{(1-x)^\beta(1-y)^\beta}.$$

Therefore

$$f(xy) = \frac{(xy)^\alpha}{1-(xy)^\beta} \leq \frac{x^\alpha}{(1-x)^\beta} \frac{y^\alpha}{(1-y)^\beta} = f(x)f(y).$$

$$\begin{aligned} \int_0^1 \int_0^1 f(xy)dxdy &\leq \int_0^1 f(x)dx \int_0^1 f(y)dy = \left(\int_0^1 f(x)dx\right)^2 \\ &= \left(\int_0^1 \frac{x^\alpha}{(1-x)^\beta}dx\right)^2 = B^2(\alpha + 1, \beta - 1). \end{aligned}$$

□

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