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ON SOME HADAMARD–TYPE INEQUALITIES FOR h–CONVEX FUNCTIONS

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Abstract. In this paper, some inequalities Hadamard-type for h-convex functions are given. We also proved some Hadamard-type inequalities for products of two h-convex functions.

1. Introduction

Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a convex mapping and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as Hadamard's inequality for convex mapping. Note that some of the the classical inequalities for means can be derived from (1.1) for appropriate particular selections of the mapping f. Both inequalities hold in the reversed direction if f is concave.

DEFINITION 1. [5] We say that $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is a Godunova-Levin function or that f belongs to the class $Q(I)$ if f is nonnegative and for all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leqslant \frac{f(x)}{\alpha} + \frac{f(y)}{1-\alpha}.$$

The class $Q(I)$ was firstly described in [5] by Godunova and Levin. Some further properties of it are given in [3], [8] and [9]. Among others, it is noted that nonnegative monotone and nonnegative convex functions belong to this class of functions.

DEFINITION 2. [1] Let $s \in (0, 1]$ be a fixed real number. A function f : $[0, \infty) \to [0, \infty)$ is said to be s-convex (in the second sense), or that f belongs to the class K_s^2, if f

$$f(\alpha x + (1-\alpha)y) \leqslant \alpha^s f(x) + (1-\alpha)^s f(y)$$

for all $x, y \in [0, \infty)$ and $\alpha \in [0, 1]$.

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In 1978, Breckner introduced s-convex functions as a generalization of convex functions [1]. Also, in that one work Breckner proved the important fact that the set-valued map is s-convex only if the associated support function is s-convex function [2]. A number of properties and connections with s-convexity in the first sense are discussed in paper [6]. Of course, s-convexity means just convexity when $s = 1$.

DEFINITION 3. [3] We say that $f : I \to \mathbf{R}$ is a P- function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $\alpha \in [0, 1]$, we have

$$f(\alpha x + (1-\alpha)y) \leqslant f(x) + f(y).$$

DEFINITION 4. [11] Let $h : J \subseteq \mathbf{R} \to \mathbf{R}$ be a positive function. We say that $f : I \subseteq \mathbf{R} \to \mathbf{R}$ is h-convex function, or that f belongs to the class $SX(h, I)$, if f is nonnegative and for all $x, y \in I$ and $\alpha \in (0, 1)$ we have

$$f(\alpha x + (1-\alpha)y) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.2}$$

If inequality (1.2) is reversed, then f is said to be h-concave, i.e. $f \in SV(h, I)$.

Obviously, if $h(\alpha) = \alpha$, then all nonnegative convex functions belong to $SX(h, I)$ and all nonnegative concave functions belong to $SV(h, I)$; if $h(\alpha) = \frac{1}{\alpha}$, then $SX(h, I) = Q(I)$; if $h(\alpha) = 1$, then $SX(h, I) \supseteq P(I)$; and if $h(\alpha) = \alpha^s$, where $s \in (0, 1)$, then $SX(h, I) \supseteq K_s^2$.

In [4], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense.

THEOREM 1. [4] *Suppose that $f : [0, \infty) \to [0, \infty)$ is an s-convex function in the second sense, where $s \in (0, 1)$, and let $a, b \in [0, \infty)$, $a < b$. If $f \in L^1([a, b])$, then the following inequalities hold:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{s+1}. \tag{1.3}$$

The constant $k = \frac{1}{s+1}$ is the best possible in the second inequality in (1.3).

In [10], Pachpatte established two new Hadamard-type inequalities for products of convex functions. An analogous result for s-convex functions is due to Kirmaci at. al. [7].

THEOREM 2. [10] *Let $f, g : [a, b] \to [0, \infty)$ be convex functions on $[a, b] \subset \mathbf{R}$, $a < b$. Then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant \frac{1}{3}M(a, b) + \frac{1}{6}N(a, b) \tag{1.4}$$

and

$$2f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + \frac{1}{6}M(a, b) + \frac{1}{3}N(a, b), \tag{1.5}$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

THEOREM 3. [7] *Let $f,g:[a,b]\to\mathbf{R}$, $a,b\in[0,\infty)$, $a<b$, be functions such that g and fg are in $L_1([a,b])$. If f is convex and nonnegative on $[a,b]$, and if g is s-convex on $[a,b]$ for some fixed $s\in(0,1)$, then*

$$2^s f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant \frac{1}{(s+1)(s+2)}M(a,b)+\frac{1}{s+2}N(a,b). \tag{1.6}$$

In [3], Dragomir et. al. proved two inequalities of Hadamard type for classes of Godunova-Levin functions and P-functions.

THEOREM 4. [3] *Let $f\in Q(I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{4}{b-a}\int_a^b f(x)dx. \tag{1.7}$$

THEOREM 5. [3] *Let $f\in P(I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$f\left(\frac{a+b}{2}\right)\leqslant\frac{2}{b-a}\int_a^b f(x)dx\leqslant 2\left[f(a)+f(b)\right]. \tag{1.8}$$

For several recent results concerning Hadamard's inequality we refer the interested reader to ([3], [4], [7] and [10]).

The main purpose of this paper is to establish new inequalities like those given in the above theorems, but now for the class h-convex functions.

2. Main Results

In the sequel of the paper, I and J are intervals in $\mathbf{R}$, $[0,1]\subseteq J$ and functions h and f are real nonnegative functions defined on J and I, respectively.

THEOREM 6. *Let $f\in SX(h,I)$, $a,b\in I$, with $a<b$ and $f\in L_1([a,b])$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}f\left(\frac{a+b}{2}\right)\leqslant\frac{1}{b-a}\int_a^b f(x)dx\leqslant\left[f(a)+f(b)\right]\int_0^1 h(\alpha)d\alpha. \tag{2.1}$$

Proof. According to (1.2) with $x=ta+(1-t)b$, $y=(1-t)a+tb$ and $\alpha=\frac{1}{2}$ we find that

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant h\left(\tfrac{1}{2}\right)f(ta+(1-t)b)+h\left(\tfrac{1}{2}\right)f((1-t)a+tb) \\ &\leqslant h\left(\tfrac{1}{2}\right)\left[f(ta+(1-t)b)+f((1-t)a+tb)\right]. \end{aligned}$$

Thus, by integrating, we obtain

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leqslant h\left(\tfrac{1}{2}\right)\left[\int_0^1 f(ta+(1-t)b)dt+\int_0^1 f((1-t)a+tb)dt\right] \\
&\leqslant h\left(\tfrac{1}{2}\right)\frac{2}{b-a}\int_a^b f(x)dx, \qquad (2.2)
\end{aligned}
$$

and the first inequality is proved. The proof of the second inequality follows by using (1.2) with $x=a$ and $y=b$ and integrating with respect to α over $[0,1]$. That is,

$$
\frac{1}{b-a}\int_a^b f(x)dx \leqslant [f(a)+f(b)]\int_0^1 h(\alpha)d\alpha. \qquad (2.3)
$$

We obtain inequalities (2.1) from (2.2) and (2.3).

The proof is complete. □

REMARK 1. In Theorem 6, if we choose $h(\alpha)=\alpha$, $h(\alpha)=1$ and $h(\alpha)=\alpha^s$, where $s\in(0,1)$, then (2.1) reduce to (1.1), (1.3) and (1.8), respectively.

THEOREM 7. *Let $f\in SX(h_1,I)$, $g\in SX(h_2,I)$, $a,b\in I$, $a<b$, be functions such that $fg\in L_1([a,b])$, and $h_1h_2\in L_1([0,1])$, then*

$$
\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant M(a,b)\int_0^1 h_1(t)h_2(t)dt+N(a,b)\int_0^1 h_1(t)h_2(1-t)dt \qquad (2.4)
$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.

Proof. Since $f\in SX(h_1,I)$ and $g\in SX(h_2,I)$, we have

$$
\begin{aligned}
f(ta+(1-t)b) &\leqslant h_1(t)f(a)+h_1(1-t)f(b) \\
&\leqslant g(ta+(1-t)b)\leqslant h_2(t)g(a)+h_2(1-t)g(b),
\end{aligned}
$$

for all $t\in[0,1]$. Since f and g are nonnegative, so

$$
\begin{aligned}
f(ta+(1-t)b)g(ta+(1-t)b) \leqslant\ & h_1(t)h_2(t)f(a)g(a)+h_1(t)h_2(1-t)f(a)g(b) \\
&+h_2(t)h_1(1-t)f(b)g(a) \\
&+h_1(1-t)h_2(1-t)f(b)g(b).
\end{aligned}
$$

Integrating both sides of the above inequality over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt \\
&\qquad \leqslant f(a)g(a)\int_0^1 h_1(t)h_2(t)dt + f(a)g(b)\int_0^1 h_1(t)h_2(1-t)dt \\
&\qquad\quad + f(b)g(a)\int_0^1 h_2(t)h_1(1-t)dt + f(b)g(b)\int_0^1 h_1(1-t)h_2(1-t)dt \\
&\qquad = [f(a)g(a)+f(b)g(b)]\int_0^1 h_1(t)h_2(t)dt \\
&\qquad\quad + [f(a)g(b)+f(b)g(a)]\int_0^1 h_1(t)h_2(1-t)dt.
\end{aligned}$$

Then

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leqslant M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt,$$

where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$. □

REMARK 2. In Theorem 7, if we take $h_1(t)=h_2(t)=t$, then inequality (2.4) reduces to inequality (1.4).

THEOREM 8. *Let $f\in SX(h_1,I)$, $g\in SX(h_2,I)$, $a,b\in I$, $a<b$, be functions such that $fg\in L_1([a,b])$, and $h_1h_2\in L_1([0,1])$, then*

$$\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)} f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)g(x)dx \\
&\qquad \leqslant M(a,b)\int_0^1 h_1(t)h_2(t)dt + N(a,b)\int_0^1 h_1(t)h_2(1-t)dt. \qquad (2.5)
\end{aligned}$$

Proof. We can write $\dfrac{a+b}{2}=\dfrac{at+(1-t)b}{2}+\dfrac{(1-t)a+tb}{2}$, so

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\
&\quad = f\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right) g\left(\frac{at+(1-t)b}{2}+\frac{(1-t)a+tb}{2}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant h_1\left(\tfrac{1}{2}\right)\left[f\left(at+(1-t)b\right)+f\left((1-t)a+tb\right)\right]\times \\
&\qquad\times h_2\left(\tfrac{1}{2}\right)\left[g\left(at+(1-t)b\right)+g\left((1-t)a+tb\right)\right] \\
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\times \\
&\qquad\times\left\{f\left(at+(1-t)b\right)g\left(at+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right. \\
&\quad\left.+f\left(at+(1-t)b\right)g\left((1-t)a+tb\right)+f\left((1-t)a+tb\right)g\left(at+(1-t)b\right)\right\} \\
&\leqslant h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\times \\
&\qquad\times\left\{f\left(at+(1-t)b\right)g\left(at+(1-t)b\right)+f\left((1-t)a+tb\right)g\left((1-t)a+tb\right)\right\} \\
&\quad+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\left\{\left[h_1(t)f(a)+h_1(1-t)f(b)\right]\left[h_2(1-t)g(a)+h_2(1-t)g(b)\right]\right. \\
&\quad\left.+\left[h_1(1-t)f(a)+h_1(t)f(b)\right]\left[h_2(t)g(a)+h_2(1-t)g(b)\right]\right\}.
\end{aligned}
$$

Thus we get

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&= h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\left\{f\left(ta+(1-t)b\right)g(ta+(1-t)b)+f\left((1-t)a+tb\right)g((1-t)a+tb)\right\} \\
&\quad+h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\left\{\left(h_1(t)h_2(1-t)+h_1(1-t)h_2(t)\right)M(a,b)\right. \\
&\qquad\qquad\left.+\left(h_1(t)h_2(t)+h_1(1-t)h_2(1-t)\right)N(a,b)\right\}.
\end{aligned}
$$

Integrating over $[0,1]$, we obtain

$$
\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}{b-a}\int_a^b f(x)dx \\
&\qquad\leqslant 2h_1\left(\tfrac{1}{2}\right)h_2\left(\tfrac{1}{2}\right)\left\{M(a,b)\int_0^1 h_1(t)h_2(1-t)dt+N(a,b)\int_0^1 h_1(t)h_2(t)dt\right\}.
\end{aligned}
$$

□

REMARK 3. If in Theorem 8 we take $h_1(t)=h_2(t)=t$ and $h_1(t)=t,\ h_2(t)=t^s$, where $s\in(0,1)$, then (2.5) reduce to (1.5) and (1.6), respectively.

REMARK 4. If in Theorem 8 we take $f:[a,b]\to\mathbf{R}$ defined as $f(x)=1$ for all $x\in[a,b]$, we obtain the following inequality for h-convex functions:

$$
\begin{aligned}
&\frac{1}{2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}g\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b g(x)dx \\
&\qquad\leqslant\left[g(a)+g(b)\right]\int_0^1 h_1(t)\left[h_2(t)+h_2(1-t)\right]dt.
\end{aligned}
$$

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