Journal of Mathematical Inequalities
Volume 2, Number 3 (2008), 343–361

GENERAL THREE–POINT QUADRATURE FORMULAE WITH APPLICATIONS FOR α–L–HÖLDER TYPE FUNCTIONS

M. KLARIČIĆ BAKULA, J. PEČARIĆ, M. RIBIČIĆ PENAVA

(*communicated by N. Elezović*)

Abstract. In this paper we present two types of general three-point weighted quadrature formulae. The obtained formulae are used to establish several Ostrowski type inequalities for α-L-Hölder functions and some error estimates for three-point Gauss-Chebyshev quadratures.

1. Introduction

The most elementary quadrature rules in three nodes are:
Simpson's rule based on Simpson's formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{(b-a)^5}{2880} f^{(4)}(\xi), \tag{1.1}$$

the dual Simpson's rule based on the following three point formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] + \frac{7(b-a)^5}{23040} f^{(4)}(\xi), \tag{1.2}$$

and Maclaurin's rule based on Maclaurin's formula

$$\int_a^b f(t)\mathrm{d}t = \frac{b-a}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right] + \frac{7(b-a)^5}{51840} f^{(4)}(\xi), \tag{1.3}$$

where in all three formulae we take $\xi \in [a,b]$. These formulae are valid for any function f with continuous fourth derivative $f^{(4)}$ on $[a,b]$.

Let $f : [a,b] \to \mathbf{R}$ be differentiable on $[a,b]$ and $f' : [a,b] \to \mathbf{R}$ integrable on $[a,b]$. Then the Montgomery identity holds [3]

$$f(x) = \frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t + \int_a^b P(x,t) f'(t)\,\mathrm{d}t, \tag{1.4}$$

Mathematics subject classification (2000): 26D15, 65D32.
Key words and phrases: Three-point quadrature formulae, Montgomery identity.

where $P(x,t)$ is the Peano kernel defined by

$$P(x,t) = \begin{cases} \frac{t-a}{b-a}, & a \leqslant t \leqslant x \\ \frac{t-b}{b-a}, & x < t \leqslant b \end{cases} . \tag{1.5}$$

Now, let's suppose $w : [a,b] \to [0,\infty\rangle$ is some probability density function, that is an integrable function satisfying $\int_a^b w(t)\,\mathrm{d}t = 1$, and $W(t) = \int_a^t w(x)\,\mathrm{d}x$ for $t \in [a,b]$, $W(t) = 0$ for $t < a$ and $W(t) = 1$ for $t > b$. In [4] J. Pečarić proved a weighted generalization of the well known Montgomery identity :

$$f(x) = \int_a^b w(t) f(t)\,\mathrm{d}t + \int_a^b P_w(x,t) f'(t)\,\mathrm{d}t,$$

where the weighted Peano kernel is defined by

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x \\ W(t) - 1, & x < t \leqslant b \end{cases} .$$

In [2] G. A. Anastassiou proved the following equality:

$$g(y) - g(x) - \sum_{i=1}^{n} \frac{g^{(i)}(x)}{i!}(y-x)^i = \frac{1}{(n-1)!}\int_x^y \left(g^{(n)}(t) - g^{(n)}(x)\right)(y-t)^{n-1}\,\mathrm{d}t,$$

where $g : I \subseteq \mathbf{R} \to \mathbf{R}$ is such that for some $n \in \mathbf{N}$ the derivative $g^{(n)}$ exists for all $t \in [a,b] \subset I$ $(a < b)$ and where x, y belong to $[a,b]$.

These two results were used in the recent paper [1], where A. Aglić Aljinović and J. Pečarić introduced two new extensions of the weighted Montgomery identity.

In this paper we use those new weighted Montgomery identities to establish for each $x \in [a,(a+b)/2)$ two general three-point quadrature formulae of the type

$$\int_a^b w(t) f(t)\,\mathrm{d}t = A(x)\left[f(x) + f(a+b-x)\right] + (1 - 2A(x)) f\left(\frac{a+b}{2}\right) + R(f,w;x), \tag{1.6}$$

where $R(f,w;x)$ stands for the reminder. Considering the function $A : [a,(a+b)/2) \to \mathbf{R}$ as an arbitrary coefficient function we see that (1.6) defines a family of quadrature formulae which contains the formulae with same nodes as Simpson's formula, the dual Simpson's formula and Maclaurin's formula. The obtained three-point formulae are used to prove several Ostrowski type inequalities for α-L-Hölder functions. At the end of the paper we show how these results can be applied to obtain some error estimates for three-point Gauss-Chebyshev quadrature rules.

2. Variant I of general three-point quadrature formula

Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 2$. In [1] the following extension of the Montgomery identity has been proved:

$$\begin{aligned}\int_a^b w(t) f(t)\,\mathrm{d}t = f(x) &- \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \int_a^x W(t)(t-a)^i\,\mathrm{d}t \\ &+ \sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \int_x^b (1-W(t))(t-b)^i\,\mathrm{d}t \\ &+ \frac{1}{(n-2)!}\left\{\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(a)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right. \\ &\left.+ \int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(b)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\}, \end{aligned} \tag{2.1}$$

where x belongs to $[a,b]$ and $w : [a,b] \to [0,\infty\rangle$ is some probability density function. In this section we use (2.1) to study for each number $x \in \left[a, \frac{a+b}{2}\right)$ the general three-point quadrature formula of the type (1.6).

Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)}$ exists on $[a,b]$ for some $n \geqslant 2$. We introduce the following notation for each $x \in \left[a, \frac{a+b}{2}\right)$:

$$D(x) = A(x)\left[f(x) + f(a+b-x)\right] + (1-2A(x)) f\left(\frac{a+b}{2}\right).$$

Furthermore, we define

$$\begin{aligned} t_n(x) = A(x)&\left\{\sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!}\left[\int_x^b (1-W(t))(t-b)^i\,\mathrm{d}t + \int_{a+b-x}^b (1-W(t))(t-b)^i\,\mathrm{d}t\right]\right. \\ &\left.- \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!}\left[\int_a^x W(t)(t-a)^i\,\mathrm{d}t + \int_a^{a+b-x} W(t)(t-a)^i\,\mathrm{d}t\right]\right\} \\ &+ (1-2A(x))\left\{\sum_{i=0}^{n-1} \frac{f^{(i+1)}(b)}{i!} \int_{\frac{a+b}{2}}^b (1-W(t))(t-b)^i\,\mathrm{d}t\right. \\ &\left.- \sum_{i=0}^{n-1} \frac{f^{(i+1)}(a)}{i!} \int_a^{\frac{a+b}{2}} W(t)(t-a)^i\,\mathrm{d}t\right\} \end{aligned}$$

and

$$\begin{aligned} T_n(x) = A(x)&\left[T_n^a(x) + T_n^b(x) + T_n^a(a+b-x) + T_n^b(a+b-x)\right] \\ &+ (1-2A(x))\left[T_n^a\left(\frac{a+b}{2}\right) + T_n^b\left(\frac{a+b}{2}\right)\right], \end{aligned}$$

where

$$T_n^a(x) = \frac{1}{(n-2)!}\int_a^x W(t)\left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,$$
$$T_n^b(x) = \frac{1}{(n-2)!}\int_x^b (1-W(t))\left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t.$$

In the next theorem we establish the first variant of the generalized three-point quadrature formula based on the extended Montgomery identity which will play the key role in this section.

THEOREM 1. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $w : [a,b] \to [0,\infty\rangle$ be some probability density function. Let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geqslant 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following identity holds*

$$\int_a^b w(t) f(t)\,\mathrm{d}t = D(x) + t_n(x) + T_n(x). \tag{2.2}$$

Proof. We put $x \equiv x, x \equiv \frac{a+b}{2}$ and $x \equiv a+b-x$ in (2.1) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x), A(x)$ respectively and adding, we obtain (2.2). □

Before we give an estimation of the term

$$\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - t_n(x)\right|,$$

let us recall that a function $\varphi : [a,b] \to \mathbf{R}$ is said to be of α-L-Hölder type if $|\varphi(x) - \varphi(y)| \leqslant L|x-y|^{\alpha}$ for every $x, y \in [a,b]$, where $L > 0$ and $\alpha \in \langle 0, 1]$. We will also make use of the Beta function of Euler type which is for $x, y > 0$ defined by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t.$$

In what follows for $x \in \left[a, \frac{a+b}{2}\right)$ we denote

$$W(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ 1 - W(t), & x < t \leqslant b \end{cases}$$

$$U_n(x,t) = \begin{cases} (t-a)^{\alpha+n-1}, & a \leqslant t \leqslant x, \\ (b-t)^{\alpha+n-1}, & x < t \leqslant b \end{cases}$$

THEOREM 2. *Suppose that all the assumptions of Theorem 1 hold and additionally assume that* $f^{(n)} : [a,b] \to \mathbf{R}$ *is an* α*-*L*-Hölder type function. Then for each* $x \in \left[a, \frac{a+b}{2}\right)$ *the following inequalities hold*

$$\begin{aligned}
&\left| \int_a^b w(t) f(t)\, \mathrm{d}t - D(x) - t_n(x) \right| \\
&\quad \leqslant \frac{B(\alpha+1, n-1)\, L}{(n-2)!} \left\{ |A(x)| \int_a^b W(x,t)\, U_n(x,t)\, \mathrm{d}t \right. \\
&\qquad + |A(x)| \int_a^b W(a+b-x,t)\, U_n(a+b-x,t)\, \mathrm{d}t \\
&\qquad \left. + |1-2A(x)| \int_a^b W\left(\frac{a+b}{2}, t\right) U_n\left(\frac{a+b}{2}, t\right) \mathrm{d}t \right\} \\
&\quad \leqslant \frac{2B(\alpha+1, n-1)\, L}{(\alpha+n)(n-2)!} \left\{ |A(x)| \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n} \right] \right. \\
&\qquad \left. + |1-2A(x)| \left(\frac{b-a}{2} \right)^{\alpha+n} \right\}.
\end{aligned}$$

Proof. From (2.2) we have

$$\begin{aligned}
&\left| \int_a^b w(t) f(t)\, \mathrm{d}t - D(x) - t_n(x) \right| \\
&\quad = \left| A(x) \left[T_n^a(x) + T_n^b(x) + T_n^a(a+b-x) + T_n^b(a+b-x) \right] \right. \\
&\qquad \left. + (1-2A(x)) \left[T_n^a\left(\frac{a+b}{2}\right) + T_n^b\left(\frac{a+b}{2}\right) \right] \right| \\
&\quad \leqslant |A(x)| \left[|T_n^a(x)| + \left|T_n^b(x)\right| + |T_n^a(a+b-x)| + \left|T_n^b(a+b-x)\right| \right] \\
&\qquad + |1-2A(x)| \left[\left| T_n^a\left(\frac{a+b}{2}\right) \right| + \left| T_n^b\left(\frac{a+b}{2}\right) \right| \right]
\end{aligned} \tag{2.3}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (2.3) we obtain

$$\begin{aligned}
&\left| \int_a^b w(t) f(t)\, \mathrm{d}t - D(x) - t_n(x) \right| \\
&\quad \leqslant \frac{|A(x)|}{(n-2)!} L \left\{ \int_a^x W(t) \left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\, \mathrm{d}s \right] \mathrm{d}t \right. \\
&\qquad + \int_a^{a+b-x} W(t) \left[\int_a^t (s-a)^\alpha (t-s)^{n-2}\, \mathrm{d}s \right] \mathrm{d}t \\
&\qquad + \int_x^b (1-W(t)) \left[\int_t^b (b-s)^\alpha (s-t)^{n-2}\, \mathrm{d}s \right] \mathrm{d}t
\end{aligned}$$

$$+ \int_{a+b-x}^{b} (1 - W(t)) \left[\int_{t}^{b} (b-s)^{\alpha} (s-t)^{n-2} \, \mathrm{d}s \right] \mathrm{d}t \Bigg\}$$

$$+ \frac{|1 - 2A(x)|}{(n-2)!} L \left\{ \int_{a}^{\frac{a+b}{2}} W(t) \left[\int_{a}^{t} (s-a)^{\alpha} (t-s)^{n-2} \, \mathrm{d}s \right] \mathrm{d}t \right.$$

$$\left. + \int_{\frac{a+b}{2}}^{b} (1 - W(t)) \left[\int_{t}^{b} (b-s)^{\alpha} (s-t)^{n-2} \, \mathrm{d}s \right] \mathrm{d}t \right\} \tag{2.4}$$

The first integral over $\mathrm{d}s$ in (2.4) can be written as

$$\begin{aligned}
\int_{a}^{t} (s-a)^{\alpha} (t-s)^{n-2} \, \mathrm{d}s &= (t-a)^{\alpha+n-2} \int_{a}^{t} \left(\frac{s-a}{t-a} \right)^{\alpha} \left(\frac{t-s}{t-a} \right)^{n-2} \mathrm{d}s \\
&= \left[u = \frac{s-a}{t-a} \right] \\
&= (t-a)^{\alpha+n-1} \int_{0}^{1} u^{\alpha} (1-u)^{n-2} \, \mathrm{d}u \\
&= (t-a)^{\alpha+n-1} B(\alpha+1, n-1).
\end{aligned}$$

Similarly can be done with other integrals in (2.4), hence we obtain

$$\begin{aligned}
&\left| \int_{a}^{b} w(t) f(t) \, \mathrm{d}t - D(x) - t_n(x) \right| \\
&\qquad \leqslant \frac{B(\alpha+1, n-1)}{(n-2)!} L \left\{ |A(x)| \int_{a}^{b} W(x,t) \, U_n(x,t) \, \mathrm{d}t \right. \\
&\qquad + |A(x)| \int_{a}^{b} W(a+b-x, t) \, U_n(a+b-x, t) \, \mathrm{d}t \\
&\qquad \left. + |1 - 2A(x)| \int_{a}^{b} W\left(\frac{a+b}{2}, t \right) U_n \left(\frac{a+b}{2}, t \right) \mathrm{d}t \right\}.
\end{aligned} \tag{2.5}$$

Since we have

$$0 \leqslant W(t) \leqslant 1, \quad t \in [a,b],$$

from (2.5) we obtain

$$\begin{aligned}
&\frac{B(\alpha+1, n-1)}{(n-2)!} L \left\{ |A(x)| \int_{a}^{b} W(x,t) \, U_n(x,t) \, \mathrm{d}t \right. \\
&\qquad + |A(x)| \int_{a}^{b} W(a+b-x, t) \, U_n(a+b-x, t) \, \mathrm{d}t \\
&\qquad \left. + |1-2A(x)| \int_{a}^{b} W\left(\frac{a+b}{2}, t \right) U_n \left(\frac{a+b}{2}, t \right) \mathrm{d}t \right\}
\end{aligned}$$

$$\leqslant \frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left\{|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right.$$
$$\left.+|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n}\right\},$$

which completes the proof. □

3. Variant I of non-weighted three-point quadrature formula and applications

Here we define

$$\widehat{t}_n(x) = A(x)\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{(x-a)^{i+2}+(b-x)^{i+2}}{i!\,(i+2)\,(b-a)}$$
$$+ (1-2A(x))\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{(b-a)^{i+1}}{i!\,(i+2)\,2^{i+2}},$$

and

$$\widehat{T}_n(x) = A(x)\left[\widehat{T}_n^a(x) + \widehat{T}_n^b(x) + \widehat{T}_n^a(a+b-x) + \widehat{T}_n^b(a+b-x)\right]$$
$$+ (1-2A(x))\left[\widehat{T}_n^a\left(\frac{a+b}{2}\right) + \widehat{T}_n^b\left(\frac{a+b}{2}\right)\right],$$

where

$$\widehat{T}_n^a(x) = \frac{1}{(n-2)!\,(b-a)}\int_a^x (t-a)\left[\int_a^t \left(f^{(n)}(a) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t,$$
$$\widehat{T}_n^b(x) = \frac{1}{(n-2)!\,(b-a)}\int_x^b (b-t)\left[\int_t^b \left(f^{(n)}(b) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t. \tag{3.1}$$

COROLLARY 1. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f: I\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D(x) + \widehat{t}_n(x) + \widehat{T}_n(x) \tag{3.2}$$

Proof. This is a special case of Theorem 1 for $w(t) = \frac{1}{b-a}$, $t\in[a,b]$. □

COROLLARY 2. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f: I\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)}:[a,b]\to\mathbf{R}$ is an α-L-Hölder* *type function for some* $n \geqslant 2$. *Then for each* $x \in \left[a, \frac{a+b}{2}\right)$ *the following inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D(x) - \widehat{t}_n(x) \right| \leqslant \frac{2B(\alpha+1, n-1)}{(b-a)(\alpha+n+1)(n-2)!} L \left\{ |A(x)| \left[(x-a)^{\alpha+n+1} + (b-x)^{\alpha+n+1} \right] + |1-2A(x)| \left(\frac{b-a}{2} \right)^{\alpha+n+1} \right\}.$$

Proof. This is a special case of Theorem 2 for $w(t) = \frac{1}{b-a}$, $t \in [a, b]$. □

The next step is setting

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2}, \; x \in \left[a, \tfrac{a+b}{2}\right).$$

This special choice of the function A enables us to establish our generalizations of the well known Simpson's formula $(x = a)$, dual Simpson's formula $(x = (3a+b)/4)$ and Maclaurin's formula $(x = (5a+b)/6)$. We will also show how to apply the results of Section 2 to obtain some error estimates for these quadrature rules if they involve α-L-Hölder type functions.

3.1. $x = a$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Simpson's formula states

$$\frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t = D(a) + \widehat{t}_n(a) + \widehat{T}_n(a),$$

where

$$D(a) = \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2} \right) + f(b) \right],$$

$$\widehat{t}_n(a) = \frac{1}{6} \sum_{i=0}^{n-1} \left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a) \right] \frac{(2^i+1)(b-a)^{i+1}}{2^i i! (i+2)}$$

and

$$\widehat{T}_n(a) = \frac{1}{6} \left[\widehat{T}_n^b(a) + 4\widehat{T}_n^a\left(\frac{a+b}{2} \right) + 4\widehat{T}_n^b\left(\frac{a+b}{2} \right) + \widehat{T}_n^a(b) \right],$$

where $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are as in (3.1).

COROLLARY 3. *Suppose that all the assumptions of Corollary 2 hold. Then the following inequality holds*

$$\left| \frac{1}{b-a} \int_a^b f(t)\,\mathrm{d}t - D(a) - \widehat{t}_n(a) \right| \leqslant \frac{B(\alpha+1, n-1)\left(2^{\alpha+n-1}+1\right)(b-a)^{\alpha+n}}{3 \cdot 2^{\alpha+n-1}(\alpha+n+1)(n-2)!} L.$$

Proof. This is a special case of Corollary 2 for $x = a$. □

For example, if in Corollary 3 we have $n = 2$ we obtain this estimation

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right] - \widehat{t}_2(a)\right| \leqslant \frac{\left(2^{\alpha+1}+1\right)(b-a)^{\alpha+2}L}{3\cdot 2^{\alpha+1}(\alpha+1)(\alpha+3)},$$

where

$$\widehat{t}_2(a) = \left(f'(b)-f'(a)\right)\frac{b-a}{6} - \left(f''(b)+f''(a)\right)\frac{(b-a)^2}{12}.$$

3.2. $x = \frac{3a+b}{4}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of the dual Simpson's formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D\left(\frac{3a+b}{4}\right) + \widehat{t}_n\left(\frac{3a+b}{4}\right) + \widehat{T}_n\left(\frac{3a+b}{4}\right)$$

where

$$D\left(\frac{3a+b}{4}\right) = \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right],$$

$$\widehat{t}_n\left(\frac{3a+b}{4}\right) = \frac{1}{3}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{\left[2\left(3^{i+2}+1\right)-2^{i+2}\right](b-a)^{i+1}}{4^{i+2}i!\,(i+2)}$$

and

$$\widehat{T}_n\left(\frac{3a+b}{4}\right) = \frac{1}{3}\left[2\widehat{T}_n^a\left(\frac{3a+b}{4}\right) + 2\widehat{T}_n^b\left(\frac{3a+b}{4}\right) - \widehat{T}_n^a\left(\frac{a+b}{2}\right) - \widehat{T}_n^b\left(\frac{a+b}{2}\right) + 2\widehat{T}_n^a\left(\frac{a+3b}{4}\right) + 2\widehat{T}_n^b\left(\frac{a+3b}{4}\right)\right],$$

where $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are as in (3.1).

COROLLARY 4. *Suppose that all the assumptions of Corollary 2 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{3a+b}{4}\right) - \widehat{t}_n\left(\frac{3a+b}{4}\right)\right| \leqslant \frac{B(\alpha+1,n-1)\left(3^{\alpha+n+1}+2^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{3\cdot 4^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 2 for $x = \frac{3a+b}{4}$. □

3.3. $x = \frac{5a+b}{6}$

Suppose that all the assumptions of Corollary 1 hold. Then our generalization of Maclaurin's formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D\left(\frac{5a+b}{6}\right) + \widehat{t}_n\left(\frac{5a+b}{6}\right) + \widehat{T}_n\left(\frac{5a+b}{6}\right)$$

where

$$D\left(\frac{5a+b}{6}\right) = \frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right],$$

$$\widehat{t}_n\left(\frac{5a+b}{6}\right) = \frac{3}{8}\sum_{i=0}^{n-1}\left[(-1)^i f^{(i+1)}(b) - f^{(i+1)}(a)\right]\frac{\left[2\cdot 3^{i+1}+5^{i+2}+1\right](b-a)^{i+1}}{6^{i+2}\cdot i!\,(i+2)}$$

and

$$\widehat{T}_n\left(\frac{5a+b}{6}\right) = \frac{1}{8}\left[3\widehat{T}_n^a\left(\frac{5a+b}{6}\right) + 3\widehat{T}_n^b\left(\frac{5a+b}{6}\right) + 2\widehat{T}_n^a\left(\frac{a+b}{2}\right) + 2\widehat{T}_n^b\left(\frac{a+b}{2}\right) + 3\widehat{T}_n^a\left(\frac{a+5b}{6}\right) + 3\widehat{T}_n^b\left(\frac{a+5b}{6}\right)\right],$$

where $\widehat{T}_n^a(x)$ and $\widehat{T}_n^b(x)$ are as in (3.1).

COROLLARY 5. *Suppose that all the assumptions of Corollary 2 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{5a+b}{6}\right) - \widehat{t}_n\left(\frac{5a+b}{6}\right)\right| \leqslant \frac{B(\alpha+1,n-1)\left(5^{\alpha+n+1}+2\cdot 3^{\alpha+n}+1\right)(b-a)^{\alpha+n}}{8\cdot 6^{\alpha+n}(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 2 for $x = \frac{5a+b}{6}$. □

4. Variant II of general three-point formula

In the paper [1] another extension of the Montgomery identity has been proved: for $x \in [a,b]$ we have that

$$\int_a^b w(t)f(t)\,\mathrm{d}t = f(x) - \sum_{i=0}^{n-1}\frac{f^{(i+1)}(x)}{i!}\int_a^b P_w(x,t)(t-x)^i\,\mathrm{d}t + \frac{1}{(n-2)!}\int_a^b P_w(x,t)\left[\int_x^t \left(f^{(n)}(x) - f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t, \tag{4.1}$$

where f and w are as in Section 2 and

$$P_w(x,t) = \begin{cases} W(t), & a \leqslant t \leqslant x, \\ W(t) - 1, & x < t \leqslant b \end{cases} .$$

In this section we establish a general three-point quadrature formula based on the extended Montgomery identity (4.1). We denote

$$D(x) = A(x)\left[f(x) + f(a+b-x)\right] + (1 - 2A(x)) f\left(\frac{a+b}{2}\right),$$

$$\begin{aligned} r_n(x) = -A(x) \sum_{i=0}^{n-1} \Bigg[&\frac{f^{(i+1)}(x)}{i!} \int_a^b P_w(x,t)(t-x)^i \, \mathrm{d}t \\ &+ \frac{f^{(i+1)}(a+b-x)}{i!} \int_a^b P_w(a+b-x,t)(t-a-b+x)^i \, \mathrm{d}t \Bigg] \\ &- (1-2A(x)) \sum_{i=0}^{n-1} \frac{f^{(i+1)}\left(\frac{a+b}{2}\right)}{i!} \int_a^b P_w\left(\frac{a+b}{2}, t\right)\left(t - \frac{a+b}{2}\right)^i \mathrm{d}t \end{aligned}$$

and

$$\begin{aligned} R_n(x) = \frac{A(x)}{(n-2)!} \Bigg\{ &\int_a^b P_w(x,t) \left[\int_x^t \left(f^{(n)}(x) - f^{(n)}(s)\right)(t-s)^{n-2} \mathrm{d}s \right] \mathrm{d}t \\ &+ \int_a^b P_w(a+b-x,t) \left[\int_{a+b-x}^t \left(f^{(n)}(a+b-x) - f^{(n)}(s)\right)(t-s)^{n-2} \mathrm{d}s \right] \mathrm{d}t \Bigg\} \\ &+ \frac{1-2A(x)}{(n-2)!} \int_a^b P_w\left(\frac{a+b}{2}, t\right) \left[\int_{\frac{a+b}{2}}^t \left(f^{(n)}\left(\frac{a+b}{2}\right) - f^{(n)}(s)\right)(t-s)^{n-2} \mathrm{d}s \right] \mathrm{d}t. \end{aligned}$$

THEOREM 3. *Suppose that all the assumptions of Theorem 1 hold. Then for each* $x \in \left[a, \frac{a+b}{2}\right)$ *the following identity holds*

$$\int_a^b w(t) f(t)\, \mathrm{d}t = D(x) + r_n(x) + R_n(x). \tag{4.2}$$

Proof. We put $x \equiv x, x \equiv \frac{a+b}{2}$ and $x \equiv a+b-x$ in (4.1) to obtain three new formulae. After multiplying these three formulae by $A(x), 1-2A(x), A(x)$ respectively and adding, we get (4.2). □

THEOREM 4. *Suppose that all the assumptions of Theorem 2 hold. Then for each* $x \in \left[a, \frac{a+b}{2}\right)$ *the following inequalities hold*

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - r_n(x)\right| \\
&\leqslant \frac{B(\alpha+1,n-1)}{(n-2)!} L \left\{ |A(x)| \left[\int_a^b |P_w(x,t)|\,|x-t|^{\alpha+n-1}\,\mathrm{d}t \right.\right. \\
&\qquad \left. + \int_a^b |P_w(a+b-x,t)|\,|a+b-x-t|^{\alpha+n-1}\,\mathrm{d}t \right] \\
&\qquad \left. + |1-2A(x)| \int_a^b \left| P_w\left(\frac{a+b}{2},t\right)\right| \left|\frac{a+b}{2}-t\right|^{\alpha+n-1}\mathrm{d}t \right\} \\
&\leqslant \frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!} L \left\{ |A(x)| \left[(x-a)^{\alpha+n} + (b-x)^{\alpha+n}\right] \right. \\
&\qquad \left. + |1-2A(x)| \left(\frac{b-a}{2}\right)^{\alpha+n} \right\}.
\end{aligned}$$

Proof. From (4.2) we have that

$$\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - r_n(x)\right| = |R_n(x)|. \tag{4.3}$$

Since $f^{(n)}$ is an α-L-Hölder type function, from (4.3) we obtain

$$\begin{aligned}
&\left|\int_a^b w(t) f(t)\,\mathrm{d}t - D(x) - r_n(x)\right| \\
&\leqslant \frac{L}{(n-2)!} \left\{ |A(x)| \left[\int_a^b |P_w(x,t)| \left| \int_x^t |(s-x)^\alpha (t-s)^{n-2}|\,\mathrm{d}s \right| \mathrm{d}t \right.\right. \\
&\quad \left. + \int_a^b |P_w(a+b-x,t)| \left| \int_{a+b-x}^t |(s-a-b+x)^\alpha (t-s)^{n-2}|\,\mathrm{d}s \right| \mathrm{d}t \right] \\
&\quad \left. + |1-2A(x)| \int_a^b \left| P_w\left(\frac{a+b}{2},t\right)\right| \left| \int_{\frac{a+b}{2}}^t \left| \left(s-\frac{a+b}{2}\right)^\alpha (t-s)^{n-2} \right| \mathrm{d}s \right| \mathrm{d}t \right\}.
\end{aligned} \tag{4.4}$$

From (4.4) similarly as in Theorem 2 we get

$$\begin{aligned}
&\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(x) - r_n(x)\right| \\
&\qquad \leqslant \frac{B(\alpha+1,n-1)}{(n-2)!} L \left\{ |A(x)| \left[\int_a^x W(t)(x-t)^{\alpha+n-1}\,\mathrm{d}t + \right.\right. \\
&\qquad\qquad + \int_a^{a+b-x} W(t)(a+b-x-t)^{\alpha+n-1}\,\mathrm{d}t
\end{aligned}$$

$$+\int_x^b (1-W(t))(t-x)^{\alpha+n-1}\,\mathrm{d}t+\int_{a+b-x}^b (1-W(t))(t-a-b+x)^{\alpha+n-1}\,\mathrm{d}t\Bigg]$$

$$+|1-2A(x)|\left[\int_a^{\frac{a+b}{2}} W(t)\left(\frac{a+b}{2}-t\right)^{\alpha+n-1}\mathrm{d}t\right.$$

$$\left.\left.+\int_{\frac{a+b}{2}}^b (1-W(t))\left(t-\frac{a+b}{2}\right)^{\alpha+n-1}\mathrm{d}t\right]\right\}$$

$$\leqslant \frac{2B(\alpha+1,n-1)}{(\alpha+n)(n-2)!}L\left\{|A(x)|\left[(x-a)^{\alpha+n}+(b-x)^{\alpha+n}\right]\right.$$

$$\left.+|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n}\right\}.$$

and this completes the proof. □

5. Variant II of non-weighted three-point quadrature formula and applications

Here we define

$$D(x)=A(x)[f(x)+f(a+b-x)]+(1-2A(x))f\left(\frac{a+b}{2}\right),$$

$$\widehat{r}_n(x)=A(x)\sum_{i=0}^{n-1}\left[f^{(i+1)}(x)+(-1)^{i+1}f^{(i+1)}(a+b-x)\right]\frac{(b-x)^{i+2}-(a-x)^{i+2}}{(i+2)!\,(b-a)}$$

$$+(1-2A(x))\sum_{i=0}^{n-1}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.$$

and

$$\widehat{R}_n(x)=\frac{1}{(n-2)!}\left\{A(x)\left[\int_a^b P(x,t)\left(\int_x^t\left(f^{(n)}(x)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right)\mathrm{d}t\right.\right.$$

$$\left.+\int_a^b P(a+b-x,t)\left(\int_{a+b-x}^t\left(f^{(n)}(a+b-x)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right)\mathrm{d}t\right]$$

$$\left.+(1-2A(x))\int_a^b P\left(\frac{a+b}{2},t\right)\left[\int_{\frac{a+b}{2}}^t\left(f^{(n)}\left(\frac{a+b}{2}\right)-f^{(n)}(s)\right)(t-s)^{n-2}\,\mathrm{d}s\right]\mathrm{d}t\right\},$$

where P is defined as in (1.5).

COROLLARY 6. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f:I\to\mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n\geqslant 2$. Then for each $x\in\left[a,\frac{a+b}{2}\right)$ the following identity holds*

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t=D(x)+\widehat{r}_n(x)+\widehat{R}_n(x).$$

Proof. This is a special case of Theorem 3 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

COROLLARY 7. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous and that $f^{(n)} : [a,b] \to \mathbf{R}$ is an α-L-Hölder type function for some $n \geqslant 2$. Then for each $x \in \left[a, \frac{a+b}{2}\right)$ the following inequality holds*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(x) - \widehat{r}_n(x)\right|$$
$$\leqslant \frac{2B(\alpha+1,n-1)L}{(b-a)(\alpha+n)(\alpha+n+1)(n-2)!}\left\{|A(x)|\left[(x-a)^{\alpha+n+1}+(b-x)^{\alpha+n+1}\right]\right.$$
$$\left.+|1-2A(x)|\left(\frac{b-a}{2}\right)^{\alpha+n+1}\right\}.$$

Proof. This is a special case of Theorem 4 for $w(t) = \frac{1}{b-a}$, $t \in [a,b]$. □

Now similarly as in Section 3 we set

$$A(x) = \frac{(b-a)^2}{6(a+b-2x)^2},\ x \in \left[a, \frac{a+b}{2}\right),$$

and proceed with some special choices of x.

5.1. $x = a$

Suppose that all the assumptions of Corollary 6 hold. Then our second generalization of Simpson's formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D(a) + \widehat{r}_n(a) + \widehat{R}_n(a),$$

where

$$D(a) = \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right],$$

$$\widehat{r}_n(a) = \frac{1}{6}\sum_{i=0}^{n-1}\left[f^{(i+1)}(a) + (-1)^{i+1}f^{(i+1)}(b)\right]\frac{(b-a)^{i+1}}{(i+2)!}$$
$$+\frac{2}{3}\sum_{i=0}^{n-1}f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.$$

COROLLARY 8. *Suppose that all the assumptions of Corollary 7 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D(a) - \widehat{r}_n(a)\right| \leqslant \frac{B(\alpha+1,n-1)\left(2^{\alpha+n-1}+1\right)(b-a)^{\alpha+n}}{3\cdot 2^{\alpha+n-1}(\alpha+n)(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 7 for $x = a$. $\square$

For example, if in Corollary 8 we have $n = 2$ we obtain the following estimation

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \widehat{r}_2(a)\right| \leqslant \frac{\left(2^{\alpha+1}+1\right)(b-a)^{\alpha+2} L}{3\cdot 2^{\alpha+1}(\alpha+1)(\alpha+2)(\alpha+3)},$$

where

$$\widehat{r}_2(a) = \left(f'(a) - f'(b)\right)\frac{b-a}{12} + \left(f''(a) + f''\left(\frac{a+b}{2}\right) + f''(b)\right)\frac{(b-a)^2}{36}$$

5.2. $x = \frac{3a+b}{4}$

Suppose that all the assumptions of Corollary 6 hold. Then our second generalization of the dual Simpson's formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D\left(\frac{3a+b}{4}\right) + \widehat{r}_n\left(\frac{3a+b}{4}\right) + \widehat{R}_n\left(\frac{3a+b}{4}\right)$$

where

$$D\left(\frac{3a+b}{4}\right) = \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right],$$

$$\widehat{r}_n\left(\frac{3a+b}{4}\right) = \frac{2}{3}\sum_{i=0}^{n-1}\left[f^{(i+1)}\left(\frac{3a+b}{4}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+3b}{4}\right)\right] \cdot \frac{\left(3^{i+2} + (-1)^{i+1}\right)(b-a)^{i+1}}{4^{i+2}(i+2)!} - \frac{1}{3}\sum_{i=0}^{n-1} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1 + (-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.$$

COROLLARY 9. *Suppose that all the assumptions of Corollary 7 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{3a+b}{4}\right) - \widehat{r}_n\left(\frac{3a+b}{4}\right)\right| \leqslant \frac{B(\alpha+1, n-1)\left(3^{\alpha+n+1} + 2^{\alpha+n} + 1\right)(b-a)^{\alpha+n}}{3\cdot 4^{\alpha+n}(\alpha+n)(\alpha+n+1)(n-2)!} L.$$

Proof. This is a special case of Corollary 7 for $x = \frac{3a+b}{4}$. $\square$

5.3. $x = \frac{5a+b}{6}$

Suppose that all the assumptions of Corollary 6 hold. Then our second generalization of Maclaurin's formula states

$$\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t = D\left(\frac{5a+b}{6}\right) + \widehat{r}_n\left(\frac{5a+b}{6}\right) + \widehat{R}_n\left(\frac{5a+b}{6}\right)$$

where

$$D\left(\frac{5a+b}{6}\right) = \frac{1}{8}\left[3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right],$$

$$\widehat{r}_n\left(\frac{5a+b}{6}\right) = \frac{3}{8}\sum_{i=0}^{n-1}\left[f^{(i+1)}\left(\frac{5a+b}{6}\right) + (-1)^{i+1} f^{(i+1)}\left(\frac{a+5b}{6}\right)\right] \cdot \frac{\left(5^{i+2} + (-1)^{i+1}\right)(b-a)^{i+1}}{6^{i+2}(i+2)!} + \frac{1}{4}\sum_{i=0}^{n-1} f^{(i+1)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^{i+1}\right)(b-a)^{i+1}}{2^{i+2}(i+2)!}.$$

COROLLARY 10. *Suppose that all the assumptions of Corollary 7 hold. Then we have*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,\mathrm{d}t - D\left(\frac{5a+b}{6}\right) - \widehat{r}_n\left(\frac{5a+b}{6}\right)\right| \leqslant \frac{B(\alpha+1,n-1)\left(5^{\alpha+n+1} + 2\cdot 3^{\alpha+n} + 1\right)(b-a)^{\alpha+n}}{8\cdot 6^{\alpha+n}(\alpha+n)(\alpha+n+1)(n-2)!}L.$$

Proof. This is a special case of Corollary 7 for $x = \frac{5a+b}{6}$. □

6. Applications to the Gauss-Chebyshev quadratures

Gaussian quadrature rules are formulae of the following type

$$\int_a^b \varpi(t) f(t)\,\mathrm{d}t \approx \sum_{i=1}^{k} A_i f(x_i), \tag{6.1}$$

where $k \in \mathbf{N}$. Without loss of generality we may restrict ourselves to the special case $[a,b] = [-1,1]$. Further, if in (6.1) the function ϖ is defined by

$$\varpi(t) = \frac{1}{\sqrt{1-t^2}}, \quad t \in (-1,1)$$

we obtain **Gauss-Chebyshev quadrature rule** of the first kind. In this case

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t \approx \pi \sum_{i=1}^{k} A_i f(x_i), \tag{6.2}$$

where the weights A_i are defined by

$$A_i = \frac{1}{k}, \quad i = 1, \ldots, k$$

and x_i are zeros of the Chebyshev polynomials of the first kind defined by

$$C_k(x) = \cos(k \arccos(x)).$$

Each $C_k(x)$ has exactly k distinct zeros

$$x_i = \cos\left(\frac{(2i-1)\pi}{2k}\right)$$

all of which lie in the interval $(-1,1)$ (see for instance [5]). Error of the approximation formula (6.2) is

$$E_k(f) = \frac{\pi}{2^{2k-1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k = 3$ (6.2) reduces to

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t = \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right] + \frac{\pi}{23040} f^{(6)}(\xi),$$

where $\xi \in (-1,1)$. If in (6.1) the function ϖ is defined by

$$\varpi(t) = \sqrt{1-t^2}, \ \ t \in [-1,1]$$

we obtain **Gauss-Chebyshev quadrature rule** of the second kind

$$\int_{-1}^{1} \sqrt{1-t^2} f(t)\,\mathrm{d}t \approx \frac{\pi}{2} \sum_{i=1}^{k} A_i f(x_i), \tag{6.3}$$

where the weights A_i are given by

$$A_i = \frac{2}{k+1} \sin^2 \frac{i\pi}{k+1}, \quad i = 1, \ldots, k$$

and x_i are zeros of the Chebyshev polynomials of the second kind defined by

$$U_k(x) = \frac{\sin[(k+1)\arccos(x)]}{\sin[\arccos(x)]}.$$

$U_k(x)$ has exactly k distinct zeros

$$x_i = \cos\left(\frac{i\pi}{k+1}\right)$$

all of which lie in the interval $(-1,1)$. Error of the approximation formula (6.3) is

$$E_k(f) = \frac{\pi}{2^{2k+1}(2k)!} f^{(2k)}(\xi), \quad \xi \in (-1,1).$$

In case $k=3$ (6.3) reduces to

$$\int_{-1}^{1}\sqrt{1-t^2}f(t)\,\mathrm{d}t=\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]+\frac{\pi}{92160}f^{(6)}(\xi),$$

where $\xi\in(-1,1)$.

Next we show how to apply the results of Section 2 to obtain some error estimates for the Gauss-Chebyshev quadrature rules involving α-L-Hölder type functions.

THEOREM 5. *Let I be an open interval in $\mathbf{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbf{R}$ be such that for some $n\geqslant 2$ the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}$ is an α-L-Hölder function. Then*

$$\left|\int_{-1}^{1}\frac{1}{\pi\sqrt{1-t^2}}f(t)\,\mathrm{d}t-\frac{1}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right)+f(0)+f\left(\frac{\sqrt{3}}{2}\right)\right]-t_n\left(-\frac{\sqrt{3}}{2}\right)\right|$$
$$\leqslant\frac{2B(\alpha+1,n-1)}{3(\alpha+n)(n-2)!}L\left[\left(1-\frac{\sqrt{3}}{2}\right)^{\alpha+n}+\left(1+\frac{\sqrt{3}}{2}\right)^{\alpha+n}+1\right],$$

where t_n is defined as in Section 2 and $W(t)=\frac{1}{\pi}\left(\arcsin t+\frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 2 for $[a,b]=[-1,1]$, $x=-\sqrt{3}/2$, $A\left(-\sqrt{3}/2\right)=1/3$ and

$$w(t)=\frac{1}{\pi\sqrt{1-t^2}},\quad t\in(-1,1).$$

□

THEOREM 6. *Let I be an open interval in $\mathbf{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbf{R}$ be such that for some $n\geqslant 2$ the derivative $f^{(n-1)}$ is absolutely continuous and $f^{(n)}$ is an α-L-Hölder function. Then*

$$\left|\int_{-1}^{1}\frac{2}{\pi}\sqrt{1-t^2}f(t)\,\mathrm{d}t-\frac{1}{4}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]-t_n\left(-\frac{\sqrt{2}}{2}\right)\right|$$
$$\leqslant\frac{B(\alpha+1,n-1)}{2(\alpha+n)(n-2)!}L\left[\left(1-\frac{\sqrt{2}}{2}\right)^{\alpha+n}+\left(1+\frac{\sqrt{2}}{2}\right)^{\alpha+n}+2\right],$$

where t_n is defined as in Section 2 and $W(t)=\frac{1}{\pi}\left(t\sqrt{1-t^2}+\arcsin t+\frac{\pi}{2}\right)$.

Proof. This is a special case of Theorem 2 for $[a,b]=[-1,1]$, $x=-\sqrt{2}/2$, $A\left(-\sqrt{2}/2\right)=1/4$ and

$$w(t)=\frac{2}{\pi}\sqrt{1-t^2},\quad t\in[-1,1].$$

□

REFERENCES

[1] A. AGLIĆ ALJINOVIĆ, J. PEČARIĆ, *Extensions of Montgomery identity with applications for* α*-L-Hölder type functions*, J. Concr. Appl. Math., **5** (1) (2007), 9–24.
[2] G. A. ANASTASSIOU, *Ostrowski type inequalities*, Proc. Amer. Math. Soc., **123** (1995), 3775–3781.
[3] D. S. MITRINOVIĆ, J. E. PEČARIĆ, A. M. FINK, *Inequalities for functions and their Integrals and Derivatives*, Kluwer Academic Publishers, Dordrecht (1994).
[4] J. PEČARIĆ, *On the Čebyšev inequality*, Bul. Inst. Politehn. Temisioara, **25** (39) (1980), 10–11.
[5] A. RALSTON, P. RABINOWITZ, *A First Course in Numerical Analysis*, Dover Publications, Inc., Mineola, New York (2001).

(Received May 5, 2008)

M. Klaričić Bakula
Department of Mathematics
Faculty of Natural Sciences, Mathematics and Education
University of Split
Teslina 12
21000 Split
Croatia
e-mail: milica@pmfst.hr

J. Pečarić
Faculty of Textile Technology
University of Zagreb
Prilaz Baruna Filipovića 30
10000 Zagreb
Croatia
e-mail: pecaric@hazu.hr

M. Ribičić Penava
Department of Mathematics
University of Osijek
Trg Ljudevita Gaja 6
31000 Osijek
Croatia
e-mail: mihaela@mathos.hr