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ON THE RASSIAS STABILITY OF A BI–JENSEN FUNCTIONAL EQUATION

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Abstract. In this paper, we investigate the stability of a bi-Jensen functional equation

$$4f\left(\frac{x+y}{2},\frac{z+w}{2}\right)=f(x,z)+f(x,w)+f(y,z)+f(y,w)$$

in the sense of Th. M. Rassias. Also, we establish the superstability of a bi-Jensen functional equation.

1. Introduction

In 1940, S.M. Ulam [14] raised a question concerning the stability of homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(\cdot,\cdot)$. Given $\varepsilon > 0$, does there exists a $\delta > 0$ such that if a mapping $h : G_1 \to G_2$ satisfies the inequality

$$d(h(xy),h(x)h(y)) < \delta$$

for all $x,y \in G_1$ then there is a homomorphism $H : G_1 \to G_2$ with

$$d(h(x),H(x)) < \varepsilon$$

for all $x \in G_1$? The case of approximately additive mappings was solved by D. H. Hyers [4] under the assumption that G_1 and G_2 are Banach spaces. In 1978, Th. M. Rassias [13] gave a generalization. Recently, P. Găvruta [2] obtained a further generalization of the Hyers-Ulam theorem following the spirit of the Th. M. Rassias Theorem for the stability of the linear mapping. Since then, a further generalization of the Hyers-Ulam theorem has been extensively investigated by a number of mathematicians[3,5,6,8-12].

Throughout this paper, let X be a normed space and Y a Banach space. A mapping $g : X \to Y$ is called a Jensen mapping if g satisfies the functional equation $2g(\frac{x+y}{2}) = g(x)+g(y)$. For a given mapping $f : X \times X \to Y$, we define

$$Jf(x,y,z,w) = 4f\left(\frac{x+y}{2},\frac{z+w}{2}\right) - f(x,z) - f(x,w) - f(y,z) - f(y,w)$$

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for all $x, y, z, w \in X$. A mapping $f : X \times X \to Y$ is called a bi-Jensen mapping if f satisfies the equation $Jf(x, y, z, w) = 0$ and the functional equation $Jf = 0$ is called a bi-Jensen functional equation.

When $X = Y = \mathbb{R}$, the function $f(x, y) := axy + bx + cy + d$ is a solution of the functional equation $Jf = 0$. Bae and Park [1] obtained the general solution and the generalized Hyers-Ulam stability of a bi-Jensen functional equation. Jun, Lee and Han [7] obtained another stability result of a bi-Jensen functional equation.

In this paper, we investigate the stability of a bi-Jensen functional equation in the sense of Th. M. Rassias. Also, we establish the superstability of a bi-Jensen functional equation.

2. Stability of a bi-Jensen functional equation

We need the following lemma to prove the main theorems.

LEMMA 1. *Let $f : X \times X \to Y$ be a bi-Jensen mapping. Then the following equalities hold;*

$$f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (\frac{1}{2^n} - \frac{1}{4^n})(f(2^n x, 0) + f(0, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0),$$

$$f(x,y) = \frac{1}{4^n} f(2^n x, 2^n y) + (2^n - 1)(f(\frac{x}{2^n}, 0) + f(0, \frac{y}{2^n})) - (2^{n+1} - 3 + \frac{1}{4^n})f(0,0)),$$

$$f(x,y) = 4^n f(\frac{x}{2^n}, \frac{y}{2^n}) + (2^n - 4^n)(f(\frac{x}{2^n}, 0) + f(0, \frac{y}{2^n})) + (2^n - 1)^2 f(0,0)),$$

$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^n}(1 - \frac{1}{2^n}) f(0, 2^n y) + (1 - \frac{1}{2^n})^2 f(0,0) \quad \textit{and}$$

$$f(x,y) = \frac{1}{2^n} f(2^n x, y) + \frac{1}{2^{n+1}}(1 - \frac{1}{2^n})(f(x, 2^n y) + f(-x, 2^n y)) + (1 - \frac{1}{2^n})^2 f(0,0)$$

for all $x, y \in X$ and $n \in \mathbb{N}$.

Proof. Since

$$f(x,y) - f(x,0) - f(0,y) + f(0,0) - \frac{f(2^n x, 2^n y) - f(2^n x, 0) - f(0, 2^n y) + f(0,0)}{4^n}$$
$$= \sum_{j=1}^{n} \frac{1}{4^j}(Jf(2^j x, 0, 2^j y, 0) - Jf(2^j x, 0, 0, 0) - Jf(0, 0, 2^j y, 0)) = 0,$$

$$f(x,y) - f(0,y) - \frac{1}{2^n}(f(2^n x, y) - f(0,y)) = \sum_{j=1}^{n} \frac{1}{2^{j+1}} Jf(2^j x, 0, y, y) = 0,$$

$$f(x,0) - f(0,0) - \frac{1}{2^n}(f(2^n x, 0) - f(0,0)) = \sum_{j=1}^{n} \frac{1}{2^{j+1}} Jf(2^j x, 0, 0, 0) = 0,$$

$$f(0,y) - f(0,0) - \frac{1}{2^n}(f(0, 2^n y) - f(0,0)) = \sum_{j=1}^{n} \frac{1}{2^{j+1}} Jf(0, 0, \frac{1}{2^j} y, 0) = 0 \quad \text{and}$$

$$f(0, 2^n y) = f(0, 2^n y) + \frac{1}{4} D(x, -x, 2^n y, 2^n y) = \frac{1}{2}[f(x, 2^n y) + f(-x, 2^n y)]$$

for all $x, y \in X$, we easily get

$$f(x,y) = \frac{1}{4^n}(f(2^n x, 2^n y) - f(2^n x, 0) - f(0, 2^n y) + f(0,0)) + \frac{1}{2^n}(f(2^n x, 0) - f(0,0)) + \frac{1}{2^n}(f(0, 2^n y) - f(0,0)) + f(0,0),$$

$$f(x,y) = \frac{1}{4^n}(f(2^n x, 2^n y) - f(2^n x, 0) - f(0, 2^n y) + f(0,0)) + 2^n(f(\frac{x}{2^n}, 0) - f(0,0)) + 2^n(f(0, \frac{y}{2^n}) - f(0,0)) + f(0,0),$$

$$f(x,y) = 4^n(f(\frac{x}{2^n}, \frac{y}{2^n}) - f(\frac{x}{2^n}, 0) - f(0, \frac{y}{2^n}) + f(0,0)) + 2^n(f(\frac{x}{2^n}, 0) - f(0,0)) + 2^n(f(0, \frac{y}{2^n}) - f(0,0)) + f(0,0),$$

for all $x, y \in X$. Using the above equalities, we obtain the desired results. □

Now we have the stability of a bi-Jensen mapping for the case $0 < p < 1$ in the following theorem.

THEOREM 2. *Let $0 < p < 1$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Jf(x,y,z,w)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p) \tag{1}$$

for all $x, y, z, w \in X$. Then there exists a unique bi-Jensen mapping $F : X \times X \to Y$ such that

$$\|f(x,y) - F(x,y)\| \leqslant \left(\frac{2^p}{|2(2-2^p)|} + \frac{2 \cdot 2^p}{|4-2^p|}\right)\varepsilon\,(\|x\|^p + \|y\|^p) \tag{2}$$

for all $x, y \in X$ with $F(0,0) = f(0,0)$. The mapping $F : X \times X \to Y$ is given by

$$F(x,y) := \lim_{j\to\infty}\frac{1}{4^j}f(2^j x, 2^j y) + \lim_{j\to\infty}\frac{1}{2^j}f(2^j x, 0) + \lim_{j\to\infty}\frac{1}{2^j}f(0, 2^j y) + f(0,0)$$

for all $x, y \in X$.

Proof. Since

$$\begin{aligned}\|\frac{1}{4^j}&(f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y) + f(0,0)) \\ &- \frac{1}{4^{j+1}}(f(2^{j+1}x, 2^{j+1}y) - f(2^{j+1}x, 0) - f(0, 2^{j+1}y) + f(0,0))\| \\ &= \frac{1}{4^{j+1}}\|Jf(2^{j+1}x, 0, 2^{j+1}y, 0) - Jf(2^{j+1}x, 0, 0, 0) - Jf(0, 0, 2^{j+1}y, 0)\| \\ &\leqslant 2\varepsilon \cdot (\frac{2^p}{4})^{j+1}(\|x\|^p + \|y\|^p),\end{aligned} \tag{3}$$

$$\begin{aligned}\|\frac{1}{2^j}&f(2^j x, 0) - f(0,0) - \frac{1}{2^{j+1}}(f(2^{j+1}x, 0) - f(0,0))\| \\ &= \frac{1}{2^{j+2}}\|Jf(2^{j+1}x, 0, 0, 0)\| \leqslant \frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\|x\|^p \quad \text{and}\end{aligned} \tag{4}$$

$$\begin{aligned}\|\frac{1}{2^j}f(0,2^jy)-f(0,0)-\frac{1}{2^{j+1}}(f(0,2^{j+1}y)-f(0,0))\|\\ =\frac{1}{2^{j+2}}\|Jf(0,0,2^{j+1}y,0)\|\leqslant\frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\|y\|^p\end{aligned}\tag{5}$$

for all $x,y\in X$ and $j\in\mathbb{N}$, we get

$$\begin{aligned}\|\frac{1}{4^l}(f(2^lx,2^ly)-f(2^lx,0)-f(0,2^ly)+f(0,0))\\ -\frac{1}{4^m}(f(2^mx,2^my)-f(2^mx,0)-f(0,2^my)+f(0,0))\|\\ \leqslant\sum_{j=l}^{m-1}(2\varepsilon\cdot(\frac{2^p}{4})^{j+1}(\|x\|^p+\|y\|^p),\end{aligned}\tag{6}$$

$$\|\frac{1}{2^l}f(2^lx,0)-f(0,0)-\frac{1}{2^m}(f(2^mx,0)-f(0,0))\|\leqslant\sum_{j=l}^{m-1}\frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\tag{7}$$

and

$$\|\frac{1}{2^l}f(0,2^ly)-f(0,0)-\frac{1}{2^m}(f(0,2^my)-f(0,0))\|\leqslant\sum_{j=l}^{m-1}\frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\|y\|^p\tag{8}$$

for given integers l,m $(0\leqslant l<m)$ and all $x,y\in X$. The sequences $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)+f(0,0))\}$, $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0)\}$ and $\{\frac{1}{2^j}(f(0,2^jy)-f(0,0)\}$ are Cauchy sequences for all $x,y\in X$. Since Y is complete, the sequences $\{\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)+f(0,0))\}$, $\{\frac{1}{2^j}(f(2^jx,0)-f(0,0)\}$ and $\{\frac{1}{2^j}(f(0,2^jy)-f(0,0)\}$ converge for all $x,y\in X$. Define $F_1,F_2,F_3:X\times X\to Y$ by

$$\begin{aligned}F_1(x,y)&:=\lim_{j\to\infty}\frac{1}{4^j}(f(2^jx,2^jy)-f(2^jx,0)-f(0,2^jy)),\\ F_2(x,y)&:=\lim_{j\to\infty}\frac{1}{2^j}f(2^jx,0),\\ F_3(x,y)&:=\lim_{j\to\infty}\frac{1}{2^j}f(0,2^jy)\end{aligned}$$

for all $x,y\in X$. Putting $l=0$ and taking $m\to\infty$ in (6), (7) and (8), one can obtain the inequalities

$$\|f(x,y)-f(x,0)-f(0,y)+f(0,0))-F_1(x,y)\|\leqslant\frac{2\cdot 2^p\varepsilon}{|4-2^p|}(\|x\|^p+\|y\|^p),\tag{9}$$

$$\|f(x,0)-f(0,0)-F_2(x,0)\|\leqslant\frac{2^p\varepsilon}{|2(2-2^p)|}\|x\|^p\quad\text{and}\tag{10}$$

$$\|f(0,y)-f(0,0)-F_3(0,y)\|\leqslant\frac{2^p\varepsilon}{|2(2-2^p)|}\|y\|^p\tag{11}$$

for all $x,y\in X$. Since

$$\lim_{n\to\infty}\frac{1}{4^n}f(2^nx,0)=\lim_{n\to\infty}\frac{1}{2^n}F_2(x,0)=0$$

and

$$\lim_{n\to\infty}\frac{1}{4^n}f(0,2^ny)=\lim_{n\to\infty}\frac{1}{2^n}F_3(0,y)=0,$$

we get

$$F_1(x,y):=\lim_{j\to\infty}\frac{1}{4^j}f(2^jx,2^jy)$$

for all $x,y\in X$. Since

$$\begin{aligned}JF_1(x,y,z,w)&=\lim_{n\to\infty}\frac{1}{4^n}Jf(2^nx,2^ny,2^nz,2^nw)=0,\\ JF_2(x,y,z,w)&=\lim_{n\to\infty}\frac{1}{2^n}Jf(2^nx,2^ny,0,0)=0,\\ JF_3(x,y,z,w)&=\lim_{n\to\infty}\frac{1}{2^n}Jf(0,0,2^nz,2^nw)=0\end{aligned}$$

for all $x,y,z,w\in X$ and all $n\in\mathbb{N}$, F is a bi-Jensen mapping satisfying (2), where $F:X\times X\to Y$ is given by

$$F(x,y)=F_1(x,y)+F_2(x,y)+F_3(x,y)+f(0,0)$$

for all $x,y\in X$. Now, let $F':X\times X\to Y$ be another bi-Jensen mapping satisfying (2) with $F(0,0)=F'(0,0)$. By Lemma 1, we have

$$\begin{aligned}&\|F(x,y)-F'(x,y)\|\\ &\quad\leqslant\|\frac{1}{4^n}(F-F')(2^nx,2^ny)+(\frac{1}{2^n}-\frac{1}{4^n})((F-F')(2^nx,0)+(F-F')(0,2^ny))\|\\ &\quad\leqslant\frac{1}{4^n}(\|(F-f)(2^nx,2^ny)\|+\|(f-F')(2^nx,2^ny)\|)+\frac{1}{2^n}\|(F-f)(2^nx,0)\|\\ &\qquad+\frac{1}{2^n}\|(f-F')(2^nx,0)\|+\frac{1}{2^n}\|(F-f)(0,2^ny)\|+\frac{1}{2^n}\|(f-F')(0,2^ny)\|\\ &\quad\leqslant(\frac{\varepsilon}{4^n}+\frac{\varepsilon}{2^n})(\frac{2^{(n+1)p}}{2-2^p}+\frac{4\cdot 2^{(n+1)p}}{4-2^p})(\|x\|^p+\|y\|^p)\end{aligned}$$

for all $n\in N$ and $x,y\in X$. As $n\to\infty$, we may conclude that $F(x,y)=F'(x,y)$ for all $x,y\in X$. Thus such a bi-Jensen mapping $F:X\times X\to Y$ is unique. □

Now we have the stability of a bi-Jensen mapping for the case $p>2$ in the following theorem.

THEOREM 3. *Let $p>2$ and $\varepsilon>0$. Let $f:X\times X\to Y$ be a mapping such that the inequality (1) holds for all $x,y,z,w\in X$. Then there exists a unique bi-Jensen mapping $F:X\times X\to Y$ satisfying the inequality (2) for all $x,y\in X$. The mapping $F:X\times X\to Y$ is given by*

$$\begin{aligned}F(x,y):=&\lim_{j\to\infty}4^j(f(\frac{x}{2^j},\frac{y}{2^j})-f(\frac{x}{2^j},0)-f(0,\frac{y}{2^j})+f(0,0))\\ &+\lim_{j\to\infty}2^j(f(\frac{x}{2^j},0)-f(0,0))+\lim_{j\to\infty}2^j(f(0,\frac{y}{2^j})-f(0,0))+f(0,0)\end{aligned}$$

for all $x,y\in X$.

Proof. Since the inequalities (3), (4) and (5) hold for all negative integer j, we can use the similar method in Theorem 1 to get the sequences $\{4^j(f(\frac{x}{2^j},\frac{y}{2^j})-f(\frac{x}{2^j},0)-f(0,\frac{y}{2^j})+f(0,0))\}$, $\{2^j(f(\frac{x}{2^j},0)-f(0,0))\}$ and $\{2^j(f(0,\frac{y}{2^j})-f(0,0))\}$ converge for all $x,y\in X$. Define $F_1,F_2,F_3 : X\times X\to Y$ by

$$F_1(x,y) := \lim_{j\to\infty} 4^j(f(\frac{x}{2^j},\frac{y}{2^j})-f(\frac{x}{2^j},0)-f(0,\frac{y}{2^j})+f(0,0)),$$
$$F_2(x,y) := \lim_{j\to\infty} 2^j(f(\frac{x}{2^j},0)-f(0,0)),$$
$$F_3(x,y) := \lim_{j\to\infty} 2^j(f(0,\frac{y}{2^j})-f(0,0))$$

for all $x,y\in X$. By the similar method in Theorem 1, we obtain the inequalities (9), (10) and (11). Since

$$\begin{aligned} JF_1(x,y,z,w) &= \lim_{n\to\infty} 4^n(Jf(\frac{x}{2^n},\frac{y}{2^n},\frac{z}{2^n},\frac{w}{2^n}) \\ &\qquad - Jf(\frac{x}{2^n},\frac{y}{2^n},0,0) - Jf(0,0,\frac{z}{2^n},\frac{w}{2^n})) = 0, \\ JF_2(x,y,z,w) &= \lim_{n\to\infty} 2^n Jf(\frac{x}{2^n},\frac{y}{2^n},0,0) = 0 \quad \text{and} \\ JF_3(x,y,z,w) &= \lim_{n\to\infty} 2^n(Jf(0,0,\frac{z}{2^n},\frac{w}{2^n}) = 0 \end{aligned}$$

for all $x,y,z,w\in X$, F is a bi-Jensen mapping satisfying (2), where $F : X\times X\to Y$ is given by

$$F(x,y) = F_1(x,y)+F_2(x,y)+F_3(x,y)+f(0,0)$$

for all $x,y\in X$. Now, let $F' : X\times X\to Y$ be another bi-Jensen mapping satisfying (2). By Lemma 1 and $F(0,0)=f(0,0)=F'(0,0)$, we have

$$\begin{aligned} &\|F(x,y)-F'(x,y)\| \\ &\leqslant \|4^n(F-F')(\frac{x}{2^n},\frac{y}{2^n}) + (2^n-4^n)((F-F')(\frac{x}{2^n},0)+(F-F')(0,\frac{y}{2^n}))\| \\ &\leqslant 4^n\|(F-f)(\frac{x}{2^n},\frac{y}{2^n})\| + 4^n\|(f-F')(\frac{x}{2^n},\frac{y}{2^n})\| + 4^n\|(F-f)(\frac{x}{2^n},0)\| \\ &\quad + 4^n\|(f-F')(\frac{x}{2^n},0)\| + 4^n\|(F-f)(0,\frac{y}{2^n})\| + 4^n\|(f-F')(0,\frac{y}{2^n})\| \\ &\leqslant \frac{4^n\varepsilon}{2^{(n-1)p}}(\frac{2}{2^p-2}+\frac{8}{2^p-4})(\|x\|^p+\|y\|^p) \end{aligned}$$

for all $n\in\mathbb{N}$ and $x,y\in X$. As $n\to\infty$, we may conclude that $F(x,y)=F'(x,y)$ for all $x,y\in X$. Thus such a bi-Jensen mapping $F : X\times X\to Y$ is unique. □

Now we have the stability of a bi-Jensen mapping for the case $1<p<2$ in the following theorem.

THEOREM 4. *Let* $1 < p < 2$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping satisfying the inequality (1). Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *satisfying the inequality (2) for all* $x, y \in X$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty} \frac{1}{4^j}(f(2^j x, 2^j y) - f(2^j x, 0) - f(0, 2^j y)) + f(0,0) + \lim_{j\to\infty} 2^j (f(\frac{x}{2^j}, 0) - f(0,0)) + \lim_{j\to\infty} 2^j (f(0, \frac{y}{2^j}) - f(0,0)) + f(0,0)$$

for all $x, y \in X$.

Proof. Let F_1 be as in Theorem 2 and let F_2 and F_3 be as in Theorem 3. Since

$$JF_1(x,y,z,w) = \lim_{n\to\infty} \frac{1}{4^n}(Jf(2^n x, 2^n y, 2^n z, 2^n w) - Jf(2^n x, 2^n y, 0, 0) - Jf(0,0,2^n z, 2^n w)) = 0,$$

F is a bi-Jensen mapping satisfying (11), where F is defined by

$$F(x,y) = F_1(x,y) + F_2(x,y) + F_3(x,y) + f(0,0)$$

for all $x, y \in X$. Now, let $F' : X \times X \to Y$ be another bi-Jensen mapping satisfying (2). By Lemma 1 and $F(0,0) = f(0,0) = F'(0,0)$, we have

$$\begin{aligned} &\|F(x,y) - F'(x,y)\| \\ &\leqslant \|\frac{1}{4^n}(F - F')(2^n x, 2^n y) + (2^n - 1)((F - F')(\frac{x}{2^n}, 0) + (F - F')(0, \frac{y}{2^n}))\| \\ &\leqslant \frac{1}{4^n}\|(F - f)(2^n x, 2^n y)\| + \frac{1}{4^n}\|(f - F')(2^n x, 2^n y)\| + 2^n\|(F - f)(\frac{x}{2^n}, 0)\| \\ &\quad + 2^n\|(f - F')(\frac{x}{2^n}, 0)\| + 2^n\|(F - f)(0, \frac{y}{2^n})\| + 2^n\|(f - F')(0, \frac{y}{2^n})\| \\ &\leqslant (\frac{2^{np}}{4^n} + \frac{2^n}{2^{np}})(\frac{2^p\varepsilon}{2^p - 2} + \frac{4\cdot 2^p\varepsilon}{4 - 2^p})(\|x\|^p + \|y\|^p) \end{aligned}$$

for all $n \in \mathbb{N}$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique.

Now we have another stability of a bi-Jensen mapping different from Theorem 2 for the case $0 < p < 1$.

THEOREM 5. *Let* $0 < p < 1, 0 < \delta$ *and* $\varepsilon > 0$. *Let* $f : X \times X \to Y$ *be a mapping satisfying the inequality (1). Then there exists a unique bi-Jensen mapping* $F : X \times X \to Y$ *such that*

$$\|f(x,y) - F(x,y)\| \leqslant \frac{2^p\varepsilon}{2(2-2^p)}\|x\|^p + (\frac{2^p\varepsilon}{2(2-2^p)} + \varepsilon)\|y\|^p + \delta \qquad (12)$$

for all $x, y \in X$ *with* $F(0,0) = f(0,0)$. *The mapping* $F : X \times X \to Y$ *is given by*

$$F(x,y) := \lim_{j\to\infty} \frac{1}{2^j}(f(2^j x, y) + f(0, 2^j y)) + f(0,0)$$

for all $x, y \in X$.

Proof. Since

$$\|\frac{1}{2^j}f(2^jx,y)-f(0,y)-\frac{1}{2^{j+1}}(f(2^{j+1}x,y)-f(0,y))\|$$
$$=\frac{1}{2^{j+2}}\|Jf(2^{j+1}x,0,y,y)\|\leqslant\frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\|x\|^p+\frac{\varepsilon}{2^{j+1}}\|y\|^p+\frac{1}{2^{j+2}}\delta$$

for all $x,y\in X$, we get

$$\|\frac{1}{2^l}f(2^lx,y)-f(0,y)-\frac{1}{2^m}(f(2^mx,y)-f(0,y))\|$$
$$\leqslant\sum_{j=l}^{m-1}(\frac{\varepsilon}{2}(\frac{2^p}{2})^{j+1}\|x\|^p+\frac{\varepsilon}{2^{j+1}}\|y\|^p+\frac{1}{2^{j+2}}\delta)\qquad(13)$$

for given integers l,m $(0\leqslant l<m)$ and all $x,y\in X$. By $0<p<1$, the sequence $\{\frac{1}{2^j}(f(2^jx,y)-f(0,y)\}$ is a Cauchy sequence for all $x,y\in X$. Since Y is complete, the sequence $\{\frac{1}{2^j}(f(2^jx,y)-f(0,y)\}$ converges for all $x,y\in X$. Define $F_4:X\times X\to Y$ by

$$F_4(x,y):=\lim_{j\to\infty}\frac{1}{2^j}f(2^jx,y)$$

for all $x,y\in X$. Putting $l=0$ and taking $m\to\infty$ in (13), one can obtain the inequality

$$\|f(x,y)-f(0,y)-F_4(x,y)\|\leqslant\frac{2^p\varepsilon}{2(2-2^p)}\|x\|^p+\varepsilon\|y\|^p+\frac{1}{2}\delta$$

for all $x,y\in X$. We can define F_3 as in the proof of Theorem 2 and the equality

$$\|f(0,y)-f(0,0)-F_3(0,y)\|\leqslant\frac{2^p\varepsilon}{|2(2-2^p)|}\|y\|^p+\frac{1}{2}\delta$$

holds for all $x,y\in X$. Since

$$JF_4(x,y,z,w)=\lim_{n\to\infty}\frac{1}{2^n}Jf(2^nx,2^ny,z,w)=0$$

for all $x,y,z,w\in X$, F is a bi-Jensen mapping satisfying (12), where $F:X\times X\to Y$ is given by

$$F(x,y)=F_3(x,y)+F_4(x,y)+f(0,0)$$

for all $x,y\in X$. Now, let $F':X\times X\to Y$ be another bi-Jensen mapping satisfying (12) with $F(0,0)=F'(0,0)$. By Lemma 1, we have

$$\|F(x,y)-F'(x,y)\|$$
$$\leqslant\frac{1}{2^n}\|F(2^nx,y)-F'(2^nx,y)+\frac{1}{2^n}(1-\frac{1}{2^n})F(0,2^ny)-F'(0,2^ny)\|$$
$$\leqslant\frac{1}{2^n}(\|(F(2^nx,y)-f(2^nx,y)\|+\|(2^nx,y)-F'(2^nx,y)\|$$
$$+\|F(0,2^ny)-f(0,2^ny)\|+\|f(0,2^ny)-F'(0,2^ny))\|)$$
$$\leqslant(\frac{2^p}{2})^n(\frac{2^p\varepsilon}{2-2^p}\|x\|^p+(\frac{2\cdot2^p\varepsilon}{2-2^p}+4\varepsilon)\|y\|^p)+\frac{4}{2^n}\delta$$

for all $n \in \mathbb{N}$ and $x, y \in X$. As $n \to \infty$, we may conclude that $F(x,y) = F'(x,y)$ for all $x, y \in X$. Thus such a bi-Jensen mapping $F : X \times X \to Y$ is unique. $\square$

3. Superstability of a bi-Jensen functional equation

We need the following lemma to prove the main theorem.

LEMMA 6. *If F satisfies the equality*

$$JF(x,y,z,w) = 0$$

for all $x, y, z, w \in X \backslash \{0\}$, then F satisfies the equality

$$JF(x,y,z,w) = 0$$

for all $x, y, z, w \in X$.

Proof. Since

$$\begin{aligned}
JF(x,y,z,0) &= JF(x,y,\frac{z}{2},\frac{z}{2}) + \frac{1}{4}[JF(x,x,\frac{3z}{2},-\frac{z}{2}) - JF(x,x,\frac{3z}{2},\frac{z}{2}) \\
&\quad - JF(x,x,\frac{z}{2},-\frac{z}{2}) + JF(y,y,\frac{3z}{2},-\frac{z}{2}) - JF(y,y,\frac{3z}{2},\frac{z}{2}) \\
&\quad - JF(y,y,\frac{z}{2},-\frac{z}{2})] = 0, \\
JF(x,0,z,w) &= JF(\frac{x}{2},\frac{x}{2},z,w) + \frac{1}{4}[JF(\frac{3x}{2},-\frac{x}{2},z,z) - JF(\frac{3x}{2},\frac{x}{2},z,z) \\
&\quad - JF(\frac{x}{2},-\frac{x}{2},z,z) + JF(\frac{3x}{2},-\frac{x}{2},w,w) - JF(\frac{3x}{2},\frac{x}{2},w,w) \\
&\quad - JF(\frac{x}{2},-\frac{x}{2},w,w)] = 0, \\
JF(x,y,0,w) &= JF(x,y,w,0) = 0, \\
JF(0,y,z,w) &= JF(y,0,z,w) = 0, \\
JF(x,0,0,0) &= \frac{1}{2}[JF(\frac{3x}{2},-\frac{x}{2},z,-z,) - JF(\frac{3x}{2},\frac{x}{2},z,-z,) \\
&\quad - JF(\frac{x}{2},-\frac{x}{2},z,-z,) + JF(\frac{x}{2},\frac{x}{2},z,-z,)] = 0, \\
JF(0,y,0,0) &= JF(y,0,0,0) = 0, \\
JF(0,0,z,0) &= \frac{1}{2}[JF(x,-x,\frac{3z}{2},-\frac{z}{2}) - JF(x,-x,\frac{3z}{2},\frac{z}{2}) \\
&\quad - JF(x,-x,\frac{z}{2},-\frac{z}{2}) + JF(x,-x,\frac{z}{2},\frac{z}{2})] = 0, \\
JF(0,0,0,w) &= JF(0,0,w,0) = 0,
\end{aligned}$$

$$\begin{aligned}
&JF(x,0,z,0) = JF(x,0,\frac{z}{2},\frac{z}{2}) + \frac{1}{2}JF(x,x,z,0) + \frac{1}{2}JF(0,0,z,0) = 0,\\
&JF(x,0,0,z) = JF(0,x,z,0) = JF(0,x,0,z) = JF(x,0,z,0) = 0,\\
&JF(0,0,z,w) = JF(x,-x,z,w) - \frac{1}{2}JF(x,-x,z,z) - \frac{1}{2}JF(x,-x,w,w) = 0,\\
&JF(x,y,0,0) = JF(x,y,z,-z) - \frac{1}{2}JF(x,x,z,-z) - \frac{1}{2}JF(y,y,z,-z) = 0,\\
&JF(0,0,0,0) = 0
\end{aligned}$$

for all $x,y,z,w \in X\backslash\{0\}$, we get the desired result. □

Now we have the superstability of a bi-Jensen mapping for the case $p < 0$ in the following theorem.

THEOREM 7. *Let $p < 0$ and $\varepsilon > 0$. Let $f : X \times X \to Y$ be a mapping such that*

$$\|Jf(x,y,z,w)\| \leqslant \varepsilon(\|x\|^p + \|y\|^p + \|z\|^p + \|w\|^p)$$

for all $x,y,z,w \in X\backslash\{0\}$. Then $f : X \times X \to Y$ is a bi-Jensen mapping.

Proof. Using the inequalities

$$\begin{aligned}
&\|\frac{1}{2^j}f(2^jx,y) - f(0,y) - \frac{1}{2^{j+1}}(f(2^{j+1}x,y) - f(0,y))\|\\
&\quad = \frac{1}{2^{j+3}}\|Jf(3\cdot 2^jx,-2^jx,y,y) - Jf(3\cdot 2^jx,2^jx,y,y) - Jf(2^jx,-2^jx,y,y)\|\\
&\quad \leqslant \frac{(3^p+2)2^{jp}}{2^{j+2}}\varepsilon\|x\|^p + \frac{3\varepsilon}{2^{j+2}}\|y\|^p
\end{aligned}$$

and

$$\begin{aligned}
&\|\frac{1}{2^j}f(0,2^jy) - f(0,0) - \frac{1}{2^{j+1}}(f(0,2^{j+1}y) - f(0,0))\|\\
&\qquad = \frac{1}{2^{j+3}}\|Jf(x,-x,3\cdot 2^jy,-2^jy) - Jf(x,-x,3\cdot 2^jy,2^jy)\\
&\qquad\quad - Jf(x,-x,2^jy,-2^jy) + Jf(x,-x,2^jy,2^jy)\|\\
&\qquad \leqslant \frac{\varepsilon}{2^j}\|x\|^p + \frac{(3^p+3)2^{jp}}{2^{j+2}}\varepsilon\|y\|^p
\end{aligned}$$

for all $x,y \in X\backslash\{0\}$ and the similar method in Theorem 2, we can define $F_1', F_2' : X\backslash\{0\} \times X\backslash\{0\} \to Y$ by

$$F_1'(x,y) := \lim_{j\to\infty}\frac{1}{2^j}[f(2^jx,y) - f(0,y)] = \lim_{j\to\infty}\frac{1}{2^j}f(2^jx,y)$$

and

$$F_2'(x,y) := \lim_{j\to\infty}\frac{1}{2^j}[f(0,2^jy) - f(0,0)] = \lim_{j\to\infty}\frac{1}{2^j}f(0,2^jy)$$

for all $x, y \in X\backslash\{0\}$ and obtain the inequalities

$$\|f(x,y) - f(0,y) - F_1'(x,y)\| \leqslant \frac{3^p+2}{2(2-2^p)}\varepsilon\|x\|^p + \frac{3}{2}\varepsilon\|y\|^p$$

and

$$\|f(0,y) - f(0,0) - F_2'(x,y)\| \leqslant 2\varepsilon\|x\|^p + \frac{3^p+3}{2(2-2^p)}\varepsilon\|y\|^p$$

for all $x, y \in X\backslash\{0\}$. Since

$$\lim_{j\to\infty}\frac{1}{2^j}f(0,y) = 0$$

and

$$\lim_{j\to\infty}\frac{1}{2^j}f(2^jx,0) = \lim_{j\to\infty}\frac{1}{2^j}\left[\frac{1}{4}Jf(2^jx,2^jx,y,-y) + \frac{1}{2}f(2^jx,y) + \frac{1}{2}f(2^jx,-y)\right]$$

for all $x, y \in X\backslash\{0\}$, we can define $F_1, F_2 : X \times X \to Y$ by

$$F_1(x,y) := \lim_{j\to\infty}\frac{1}{2^j}f(2^jx,y) \quad \text{and}$$
$$F_2(x,y) := \lim_{j\to\infty}\frac{1}{2^j}f(0,2^jy)$$

for all $x, y \in X$. Since

$$JF_1(x,y,z,w) = \lim_{n\to\infty}\frac{1}{2^n}Jf(2^nx,2^ny,z,w) = 0,$$
$$JF_2(x,y,z,w) = \lim_{n\to\infty}\frac{1}{2^n}\left[Jf(x,-x,2^nz,2^nw) - \frac{1}{2}Jf(x,-x,2^nz,2^nz) - \frac{1}{2}Jf(x,-x,2^nw,2^nw)\right] = 0$$

for all $x, y, z, w \in X\backslash\{0\}$, F_1 and F_2 are bi-Jensen mappings by Lemma 6. Hence F is a bi-Jensen mapping satisfying

$$\|f(x,y) - F(x,y)\| \leqslant \left(\frac{3^p+2}{2(2-2^p)} + 2\right)\varepsilon\|x\|^p + \left(\frac{3}{2} + \frac{3^p+3}{2(2-2^p)}\right)\varepsilon\|y\|^p \qquad (14)$$

for all $x, y \in X\backslash\{0\}$, where $F : X \times X \to Y$ is given by

$$F(x,y) = F_1(x,y) + F_2(x,y) + f(0,0)$$

for all $x, y \in X$. From (14), we get

$$\begin{aligned}\|f(x,y) - F(x,y)\| = \frac{1}{4}\|&Jf((k+2)x,-kx,(k+2)y,-ky) + (f-F)(-kx,-ky)\\ &+(f-F)((k+2)x,-ky) + (f-F)((k+2)x,(k+2)y)\\ &+(f-F)(-kx,(k+2)y) - JF((k+2)x,-kx,(k+2)y,-ky)\|\end{aligned}$$

$$\leqslant ((k+2)^p+k^p)\left[\left(\frac{3^p+2}{4(2-2^p)}+\frac{5}{4}\right)\varepsilon\|x\|^p+\left(1+\frac{3^p+3}{4(2-2^p)}\right)\varepsilon\|y\|^p\right]$$

for all $x,y\in X\backslash\{0\}$ and $k\in\mathbb{N}$. Similarly we get

$$\|f(x,0)-F(x,0)\|\leqslant ((k+2)^p+k^p)\left(\frac{3^p+2}{4(2-2^p)}+\frac{5}{4}\right)\varepsilon\|x\|^p+k^p\left(2+\frac{3^p+3}{2(2-2^p)}\right)\varepsilon\|y\|^p,$$
$$\|f(0,y)-F(0,y)\|\leqslant k^p\left(\frac{3^p+2}{2(2-2^p)}+\frac{5}{2}\right)\varepsilon\|x\|^p+((k+2)^p+k^p)\left(1+\frac{3^p+3}{4(2-2^p)}\right)\varepsilon\|y\|^p,$$
$$\|f(0,0)-F(0,0)\|\leqslant k^p\left[\left(\frac{3^p+2}{2(2-2^p)}+\frac{5}{2}\right)\varepsilon\|x\|^p+\left(2+\frac{3^p+3}{2(2-2^p)}\right)\varepsilon\|y\|^p\right]$$

for all $x,y\in X\backslash\{0\}$ and $k\in\mathbb{N}$. As $k\to\infty$ in the above inequalities, we have $F(x,y)=f(x,y)$ for all $x,y\in X$. □

We can prove the following theorem by the similar method used in Theorem 7.

THEOREM 8. *Let $p,q<0$ and $\varepsilon>0$. Let $f:X\times X\to Y$ be a mapping such that*

$$\|Jf(x,y,z,w)\|\leqslant\varepsilon(\|x\|^p+\|y\|^p)(\|z\|^q+\|w\|^q)$$

for all $x,y,z,w\in X\backslash\{0\}$. Then $f:X\times X\to Y$ is a bi-Jensen mapping.

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