SHAFFER–FINK TYPE INEQUALITIES
FOR THE ELLIPTIC FUNCTION $sn(u|k)$

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Abstract. The inequalities of Shafer and Fink, namely,

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leq \sin^{-1}(x) \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}, \quad x \in [0, 1]$$

are generalized to similar inequalities for the elliptic function $sn(u|k)$.

1. Introduction

We start by presenting two very simple proofs of the Shafer and Fink inequalities. Consider the functions

$$f(\theta) = \pi \sin \theta - 2\theta - \theta \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

and

$$g(\theta) = 3 \sin \theta - 2\theta - \theta \cos \theta, \quad 0 \leq \theta \leq \frac{\pi}{2}$$

Each of these is concave over its interval and the following boundary conditions are satisfied $f(0) = f\left(\frac{\pi}{2}\right) = 0$ and $g(0) = g'(0) = 0$. Hence

$$g(\theta) \leq 0 \quad \text{and} \quad f(\theta) \geq 0, \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

That is

$$\frac{3 \sin \theta}{2 + \cos \theta} \leq \theta \leq \frac{\pi \sin \theta}{2 + \cos \theta}, \quad \theta \in \left[0, \frac{\pi}{2}\right]$$

or, on putting $x = \sin \theta$

$$\frac{3x}{2 + \sqrt{1 - x^2}} \leq \sin^{-1} x \leq \frac{\pi x}{2 + \sqrt{1 - x^2}}, \quad x \in [0, 1] \quad (1)$$

These are, respectively, the Shafer and Fink inequalities.

The origins of these are to be found in [1] and [2] and a large bibliography concerning them and their extensions appears in [3].


Key words and phrases: Shafer-Fink inequalities, Jacobi elliptic function.
Our purpose in this note is to generalize these results to the case in which \( \sin \theta \) is replaced by the Jacobi elliptic function \( \text{sn}(u|k) \). In short, it is our purpose to prove the following inequalities:

**THEOREM.** Let \( 0 < k < 1 \). Then if

\[
K(k) = \int_0^1 \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}
\]

we have

\[
\frac{3x}{2 + \sqrt{1 - x^2} \sqrt{1 - k^2 x^2}} \leq \text{sn}^{-1}(x|k) \leq \frac{2K(k)x}{2 + \sqrt{1 - x^2} \sqrt{1 - k^2 x^2}}, \quad x \in [0, 1] \quad (2)
\]

2. The Jacobi elliptic functions

A very succinct introduction to these functions, when the independent variable is real, can be found in [4]. And in [5] there is a comprehensive list of their properties. In this section we remind the reader of some of these facts.

**(a) Definitions.** With \( 0 < k < 1 \) the three Jacobi elliptic functions \( \text{sn}(u|k) \), \( \text{cn}(u|k) \) and \( \text{dn}(u|k) \) are usually defined by integrals such as, for example,

\[
u = \int_0^{\text{sn}(u|k)} \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}
\]

Let us follow [4] and write

\[x = \text{sn}(u|k), \quad y = \text{cn}(u|k) \quad \text{and} \quad z = \text{dn}(u|k)\]

Then, whenever the independent variable \( u \) is restricted to the real field an equivalent definition of these functions is

\[
\frac{dx}{du} = yz, \quad \frac{dy}{du} = -zx, \quad \frac{dz}{du} = -k^2 xy
\]

with

\[x(0) = 0, \quad y(0) = z(0) = 1\]

The parameter \( k \) is called the *elliptic modulus* and \( k' \), defined by

\[k' = \sqrt{1 - k^2}\]

is the *complementary elliptic modulus*.

Another constant involved in these matters is the following:

\[
K(k) = \int_0^1 \frac{dt}{\sqrt{1 - t^2} \sqrt{1 - k^2 t^2}}
\]

\( K \) is called the *complete elliptic integral of the first kind*. 
(b) **Properties.** We have, using the notations of \([4]\):

\[
x(0) = 0, \quad x(K) = 1
\]

\[
y(0) = 1, \quad y(K) = 0
\]

\[
z(0) = 1, \quad z(K) = k' = \sqrt{1 - k^2}
\]

\[
x^2 + y^2 = 1,
\]

\[
k^2x^2 + z^2 = 1,
\]

\[
k^2y^2 + k'^2 = z^2,
\]

\[
y^2 + k'^2x^2 = z^2
\]

Visualization of these functions may be helped by mentioning that \(x(u)\) increases from 0 to 1 and \(y(u)\) decreases from 1 to 0 in \([0, K]\), their graphs generally resembling those of \(\sin(u)\) and \(\cos(u)\). The function \(z\) decreases from 1 to \(k'\) in \([0, K]\) with a minimum at \(K\) when

\[
z = \sqrt{1 - k^2} > 0
\]

These properties have been proved in \([2]\), for example. The constant \(k\) will be fixed in \((0, 1)\) throughout but we note that as \(k \to 0\)

\[
x \to \sin, \quad y \to \cos, \quad z \to 1 \quad \text{and} \quad K \to \frac{\pi}{2}
\]

and, as \(k \to 1\)

\[
x \to \tanh, \quad y \to \sech, \quad z \to \sech \quad \text{and} \quad K \to \infty
\]

We shall not be concerned with behaviour outside \([0, K]\) but it may be mentioned here that \(4K\) is a period of \(x\) and \(y\) while \(2K\) is a period of \(z\).

### 3. The proofs

In the proofs of the Lemma and of the Theorem which follow it is convenient to write

\[
Q \equiv [z^2 + k^2y^2] - 4k^2x^2
\]

We have:

**Lemma.**

*If \(Q > 0\), then \(\frac{x}{yz} \geq u\), \(u \in [0, K]\)*

*Proof.* Consider

\[
w(u) = x - uyz
\]

Then

\[
w'(u) = yz - yz + u[xz^2 + k^2xy^2]
\]

\[
= ux[z^2 + k^2y^2]
\]

\[
w''(u) = x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2] + ux[-k^22zxy - k^22yxz]
\]

\[
= x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2] - 4k^2x^2
\]

\[
= x[z^2 + k^2y^2] + uyzQ
\]
Since \( Q > 0 \) then \( w''(u) > 0 \). So \( w(u) \) is convex and, since \( w(0) = w'(0) = 0 \) we have 
\[ w(u) \geq 0, \quad u \in [0, K] \]
and so
\[ \frac{x}{yz} \geq u, \quad u \in [0, K) \]
as was to be proved.

**Proof of the Theorem (left side).** Consider
\[ f(u) = 3 \text{sn}(u|k) - 2u - u \text{cn}(u|k) \text{dn}(u|k) \]
or equivalently,
\[ f(u) = 3x - 2u - uyz \]
Differentiating with respect to \( u \), we get
\[ f'(u) = 2yz - 2ux[z^2 + k^2y^2] \]
Then
\[
\begin{align*}
f''(u) &= -2xz^2 - 2k^2xy^2 + x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2] \\
&\quad + ux[-k^22zxy - k^22yuz] \\
&= -x[z^2 + k^2y^2] + uyz[z^2 + k^2y^2 - 4k^2x^2] \\
&= -x[z^2 + k^2y^2] + uyzQ
\end{align*}
\]
where, again,
\[ Q \equiv [z^2 + k^2y^2] - 4k^2x^2 \]
When \( u = 0 \), \( Q = 1 + k^2 > 0 \) and when \( u = K \), \( Q = 1 - 5k^2 \), so that, depending on \( k \), \( Q \) may take both signs.
When \( Q < 0 \) we see that \( f''(u) \) is negative. And when \( Q > 0 \) we find that \( f''(u) \) is negative again, by virtue of the Lemma and the fact that
\[ z^2 + k^2y^2 > Q = [z^2 + k^2y^2] - 4k^2x^2 \]
So \( f(u) \) is concave. And since \( f(0) = f'(0) = 0 \) then 
\[ f(u) < 0, \quad u \in [0, K] \]
and so
\[ u > \frac{3 \text{sn}(u|k)}{2 + \text{cn}(u|k) \text{dn}(u|k)}, \quad u \in [0, K]. \]
Putting \( x = \text{sn}(u) \) this reads
\[ \text{sn}^{-1}(x|k) > \frac{3x}{2 + \sqrt{1 - x^2} \sqrt{1 - k^2x^2}}, \quad x \in [0, 1] \]
which is the left inequality of (2).

Proof of the Theorem (right side). The proof of this is very similar. We consider
\[ g(u) = 2K \text{sn}(u|k) - 2u - u \text{cn}(u|k) \text{dn}(u|k) \]
or equivalently,
\[ g(u) = 2Kx - 2u - uy \]

Differentiating with respect to \( u \), we get
\[ g'(u) = (2K - 1)yz - 2 + ux[z^2 + k^2y^2] \]
and
\[ g''(u) = -2(K - 1)x[z^2 + k^2y^2] + uyQ \]

Just as previously we see that if \( Q < 0 \) then \( g''(u) < 0 \). And if \( Q > 0 \), then \( g''(u) < 0 \) by virtue of the fact that
\[ z^2 + k^2y^2 > Q = [z^2 + k^2y^2] - 4k^2x^2 \]

and the Lemma, which, in this case, gives
\[ \frac{x}{yz} > \frac{u}{2K - 2} \]
since \( 2K - 2 \geq \pi - 2 > 1 \).

(Note that since \( K(0) = \pi/2 \) and \( K = K(k) \) increases with \( k \) (see [2]), this inequality persists for \( k \in (0, 1) \)). So \( g(u) \) is concave. And since \( g(0) = g(K) = 0 \) we have
\[ g(u) \geq 0, \quad u \in [0, K] \]

Hence
\[ u < \frac{2K \text{sn}(u|k)}{2 + \text{cn}(u|k) \text{dn}(u|k)}, \quad u \in [0, K]. \]

Putting \( x = \text{sn}(u|k) \) this reads
\[ \text{sn}^{-1}(x|k) < \frac{2Kx}{2 + \sqrt{1 - x^2} \sqrt{1 - k^2x^2}}, \quad x \in [0, 1] \]

and this is the right side of (2).

So the proof of the Theorem is complete.

A final note. If we let \( k \to 0 \) in (2) we recover (1) and if we let \( k \to 1 \) (2) becomes
\[ \tanh^{-1}x > \frac{3x}{3 - x^2}, \quad x \in [0, 1]. \]
REFERENCES


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