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INEQUALITIES FOR THE RATIOS OF CERTAIN BIVARIATE MEANS

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Dedicated to
Professor I. Gy. Maurer
on occasion of his eightieth birthday

(*communicated by J. Pečarić*)

Abstract. Inequalities connecting ratios of bivariate homogeneous means whose variables satisfy certain monotonicity conditions are obtained. Derived results include the Stolarsky, Gini, Schwab-Borchardt, and lemniscatic means.

1. Introduction

In recent years a problem of comparison of ratios of certain bivariate homogeneous means has attracted attention of researchers (see, e.g., [17], [6]).

In order to formulate this problem let us introduce a notation which will be used throughout the sequel. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ stand for vectors whose components are positive numbers. To this end we will always assume that a and b satisfy the monotonicity conditions

$$\frac{a_1}{a_2} \geqslant \frac{b_1}{b_2} \geqslant 1. \tag{1.1}$$

Further, let Φ and Ψ be bivariate means. We will always assume that Φ and Ψ are homogeneous of degree 1 (or simply homogeneous) in their variables. The central problem discussed in this paper is formulated as follows. Assume that the variables a_i and b_i $(i = 1, 2)$ satisfy monotonicity conditions (1.1). For what means Φ and Ψ does the following inequality

$$\frac{\Phi(a)}{\Phi(b)} \leqslant \frac{\Psi(a)}{\Psi(b)} \tag{1.2}$$

hold true? In [6] the authors have proven that the inequality (1.2) is valid for power means of certain order, logarithmic, identric and the Heronian mean of order ω. For the definition of the latter mean see [7] and formula (2.6).

In this paper we shall obtain inequalities of the form (1.2) for the Stolarsky, Gini, Schwab-Borchardt, and the lemniscatic means. Definitions and basic properties of these

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means are presented in Section 2. The main results are derived in Section 3. We close this paper with a result which deals with the relationship of the Ky Fan inequality and the inequality (1.2).

2. Definitions and Basic Properties of Certain Bivariate Means

We begin with the definition of the Stolarsky means which have been introduced in [18] and studied extensively by numerous researchers (see, e.g., [4], [8], [10], [11], [15]). For $x > 0$, $y > 0$ and $p, q \in \mathbf{R}$, they are denoted by $D_{p,q}(x,y)$, and defined for $x \neq y$ as

$$D_{p,q}(x,y) = \begin{cases} \left[\frac{q(x^p - y^p)}{p(x^q - y^q)}\right]^{\frac{1}{p-q}}, & pq(p-q) \neq 0 \\ \exp\left(-\frac{1}{p} + \frac{x^p \ln x - y^p \ln y}{x^p - y^p}\right), & p = q \neq 0 \\ \left[\frac{x^p - y^p}{p(\ln x - \ln y)}\right]^{\frac{1}{p}}, & p \neq 0,\ q = 0 \\ \sqrt{xy}, & p = q = 0. \end{cases} \tag{2.1}$$

Also, $D_{p,q}(x,x) = x$.

Stolarsky means are sometimes called the extended means or the difference means (see [8], [10], [15]).

A second family of bivariate means employed in this paper was introduced by C. Gini [5]. Throughout the sequel they will be denoted by $S_{p,q}(x,y)$. Following [5]

$$S_{p,q}(x,y) = \begin{cases} \left[\frac{x^p + y^p}{x^q + y^q}\right]^{\frac{1}{p-q}}, & p \neq q \\ \exp\left(\frac{x^p \ln x + y^p \ln y}{x^p + y^p}\right), & p = q \neq 0 \\ \sqrt{xy}, & p = q = 0. \end{cases} \tag{2.2}$$

Gini means are also called the sum means (see, e.g., [10]).

For the reader's convenience we recall basic properties of these two families of means. Properties (P1)–(P3) follow directly from (2.1) and (2.2). Properties (P4)–(P6) are established in [8], [18] and [11]. For the sake of presentation, let $\Phi_{p,q}$ stand either for the Stolarsky or Gini mean of order (p,q). We have

(P1) $\Phi_{p,q}(\cdot,\cdot) = \Phi_{q,p}(\cdot,\cdot)$.

(P2) $\Phi_{\cdot,\cdot}(x,y) = \Phi_{\cdot,\cdot}(y,x)$.

(P3) $\Phi_{p,q}(x,y)$ is homogeneous of degree 1 in its variables, i.e.,

$$\Phi_{p,q}(\lambda x, \lambda y) = \lambda \Phi_{p,q}(x,y), \quad \lambda > 0.$$

(P4) $\Phi_{p,q}(\cdot,\cdot)$ increases with increase in either p or q.

(P5)

$$\ln D_{p,q}(x,y) = \frac{1}{q-p}\int_p^q \ln I_t(x,y)\,dt \quad (p \neq q)$$

where

$$I_p(x,y) = D_{p,p}(x,y) \tag{2.3}$$

is the identric mean of order p. Similarly

(P6)

$$\ln S_{p,q}(x,y) = \frac{1}{q-p}\int_p^q \ln J_t(x,y)\,dt \quad (p \neq q)$$

where

$$J_p(x,y) = S_{p,p}(x,y). \tag{2.4}$$

Other means used in this paper include the power mean A_p of order $p \in \mathbf{R}$. Recall that

$$A_p(x,y) = \begin{cases} \left(\frac{x^p+y^p}{2}\right)^{1/p}, & p \neq q \\ \sqrt{xy}, & p = 0. \end{cases} \tag{2.5}$$

The Heronian mean $\mathscr{H}_\omega$ of order $\omega \geqslant 0$ is defined as

$$\mathscr{H}_\omega(x,y) = \frac{x+y+\omega\sqrt{xy}}{2+\omega} \tag{2.6}$$

(see [7]). Also we will deal with the harmonic, geometric, logarithmic, identric, arithmetic and centroidal means of order one. They will be denoted by H, G, L, I, A and C, respectively. They are special cases of the Stolarsky mean $D_{p,q}$. We have

$$\begin{gathered} H = D_{-2,-1}, \quad G = D_{0,0}, \quad L = D_{0,1}, \quad \mathscr{H}_1 = D_{1/2,3/2} \\ I = D_{1,1}, \quad A = D_{1,2}, \quad C = D_{2,3}\,. \end{gathered} \tag{2.7}$$

The Comparison Theorem for the Stolarsky means (see, eg., [15]) implies the chain of inequalities

$$H < G < L < \mathscr{H}_1 < I < A < C \tag{2.8}$$

provided $x \neq y$.

Another mean used in this paper is commonly referred to as the Schwab-Borchardt mean. Now let $x \geqslant 0$ and $y > 0$. The latter mean, denoted by $SB(x,y) \equiv SB$, is defined as the common limit of two sequences $\{x_n\}_0^\infty$ and $\{y_n\}_0^\infty$, i.e.,

$$SB = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n+y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1}y_n}\,, \tag{2.9}$$

$n \geqslant 0$ (see [2]). It is known that the mean under discussion can be expressed in terms of the elementary transcendental functions

$$SB(x,y) = \begin{cases} \frac{\sqrt{y^2-x^2}}{\arccos(x/y)}, & 0 \leqslant x < y \\ \frac{\sqrt{x^2-y^2}}{\operatorname{arc\,cosh}(x/y)}, & y < x \\ x, & x = y \end{cases}$$

(see [1, Theorem 8.4], [2, (2.3)]). The Schwab-Borchardt mean has been studied extensively in recent papers [12] and [14].

The lemniscatic mean of $x > 0$ and $y \geqslant 0$, denoted by $LM(x,y) \equiv LM$, is also the iterative mean, i.e.,

$$LM = \lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n,$$

where

$$x_0 = x, \quad y_0 = y, \quad x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \sqrt{x_{n+1}x_n} \quad n \geqslant 0.$$

The explicit formula

$$\left[LM(x,y)\right]^{-1/2} = \begin{cases} (x^2-y^2)^{-1/4} \operatorname{arc\,sl}\left(1 - \frac{y^2}{x^2}\right)^{1/4}, & y < x \\ (y^2-x^2)^{-1/4} \operatorname{arc\,slh}\left(\frac{y^2}{x^2} - 1\right)^{1/4}, & x < y \\ x^{-1/2}, & x = y \end{cases}$$

involves two incomplete symmetric integrals of the first kind

$$\operatorname{arc\,sl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}}, \quad |x| \leqslant 1$$

and

$$\operatorname{arc\,slh} x = \int_0^x \frac{dt}{\sqrt{1+t^4}},$$

which are also called the Gauss lemniscate functions, (see [2, (2.5)–(2.6)], [1, p. 259]). It is known [2, (4.1)] that

$$\operatorname{arc\,sl} x = x R_B(1, 1-x^4) \tag{2.10}$$

and

$$\operatorname{arc\,slh} x = x R_B(1, 1+x^4), \tag{2.11}$$

where

$$R_B(x,y) = \frac{1}{4}\int_0^\infty (t+x)^{-3/4}(t+y)^{-1/2} dt \tag{2.12}$$

(see [2, (3.14)]). The lemniscatic mean has been studied extensively in [9].

For later use let us record the fact that both SM and LM are homogeneous of degree 1, however, they are not symmetric in their variables. We shall make use of the inequality which has been established in [9, Theorem 5.2]:

$$SB(x,y) \leqslant LM(y,x) \leqslant A \leqslant LM(x,y) \leqslant SB(y,x) \tag{2.13}$$

provided $0 < y \leqslant x$. Inequalities (2.13) are reversed if $y \geqslant x > 0$.

3. Main Results

Before we state and prove one of the main results of this section (Theorem 3.3) we shall investigate a function $u(t)$ which is defined as follows

$$u(t) \equiv u(t;x) = \frac{d}{dx} I_t(x,1)$$

($0 < x < 1$), where I_t is the identric mean defined in (2.3). It follows from (2.1) that

$$u(t) = \begin{cases} \dfrac{x^{2t-1} - x^{t-1} - tx^{t-1}\ln x}{(x^t-1)^2}, & t \neq 0 \\ \dfrac{1}{2x}, & t = 0. \end{cases} \tag{3.1}$$

We need the following.

LEMMA 3.1. *Function $u(t)$ has the following properties*

$$u(t) \geqslant 0, \quad t \in \mathbf{R}, \tag{3.2}$$

$$u(-t) + u(t) = 2u(0), \tag{3.3}$$

$$u(t) \text{ is strictly decreasing for every } t \neq 0, \tag{3.4}$$

$$u(t) \text{ is strictly convex for } t > 0 \text{ and strictly concave for } t < 0. \tag{3.5}$$

Proof. In order to establish the inequality (3.2) it suffices to apply the inequality $\ln x^t \leqslant x^t - 1$ to the right side of (3.1). Formula (3.3) follows easily from (3.1). For the proof of monotonicity property (3.4) we differentiate (3.1) to obtain

$$\frac{(x^t-1)^3}{x^{t-1}\ln x} u'(t) = y\ln y + \ln y - 2y + 2, \tag{3.6}$$

where $y = x^t$. Letting $z = x^{-t}$ we can rewrite the right side of (3.6) as

$$\frac{(x^t-1)^3}{x^{t-1}\ln x} u'(t) = \frac{(z-1)(z+1)}{z} \left[\frac{1}{A(z,1)} - \frac{1}{L(z,1)} \right]. \tag{3.7}$$

Let $t > 0$. Then $0 < x^t < 1$. This in turn implies that $z > 1$. Application of the well-known inequality $L(z,1) < A(z,1)$ shows that the right side of (3.7) is negative. Hence $u'(t) < 0$ for $t > 0$. The same argument can be used that $u'(t) < 0$ for positive t. This completes the proof of (3.4). For the proof of (3.5) we differentiate (3.6) to obtain

$$\frac{(x^t-1)^4}{x^{t-1}(\ln x)^2} u''(t) = 3(y^2-1) - (\ln y)(y^2+4y+1). \tag{3.8}$$

The right side of (3.8) can also be written as

$$6(\ln y) \left[L(y^2,1) - \frac{A(y^2,1) + 2G(y^2,1)}{3} \right] =: R.$$

Let $t > 0$. Then $y < 1$. This in turn implies that $R > 0$ because

$$L < \frac{A+2G}{3} \tag{3.9}$$

(see [3], [12]). This in conjunction with (3.8) shows that $u''(t) > 0$ for $t > 0$. Since the proof of strict concavity of $u(t)$ when $t < 0$ goes along the lines introduced above, it is omitted. □

For later use let us record a generalization of the classical Hermite-Hadamard inequalities.

PROPOSITION 3.2. ([4]) *Let $f(t)$ be a real-valued function which is concave for $t < 0$, convex for $t > 0$, and satisfies the symmetry condition $f(-t) + f(t) = 2f(0)$. Then for any r and s ($r \neq s$) in the domain of $f(t)$ the following inequalities*

$$f\left(\frac{r+s}{2}\right) \leqslant \frac{1}{s-r}\int_r^s f(t)\,dt \leqslant \frac{1}{2}\big[f(r)+f(s)\big] \tag{3.10}$$

hold true provided $r + s \geqslant 0$. Inequalities (3.10) are reversed if $r + s \leqslant 0$.

We are in a position to prove the following.

THEOREM 3.3. *Let the vectors $a = (a_1, a_2)$ and $b = (b_1, b_2)$ of positive numbers be such that the inequalities (1.1) are satisfied. Further, let the numbers p, q, r and s satisfy the conditions $p \leqslant q$ and $r \leqslant s$. Then the following inequality*

$$\frac{D_{r,s}(a)}{D_{r,s}(b)} \leqslant \frac{D_{p,q}(a)}{D_{p,q}(b)} \tag{3.11}$$

is satisfied if either

(i) $r + s \geqslant 0$ *and* $p \geqslant \frac{r+s}{2}$

or

(ii) $r + s \leqslant 0$ *and* $p \geqslant r$

or

(iii) $p + q \geqslant 0$ *and* $s \leqslant p$

or

(iv) $p + q \leqslant 0$ *and* $s \leqslant \frac{p+q}{2}$.

Proof. The following function

$$\phi(x) = \frac{D_{p,q}(x,1)}{D_{r,s}(x,1)}, \quad 0 < x < 1, \tag{3.12}$$

plays a crucial role in the proof of the inequality (3.11). Logarithmic differentiation together with the use of (P5) yields

$$\frac{\phi'(x)}{\phi(x)} = \begin{cases} \frac{1}{q-p}\int_p^q u(t)\,dt - \frac{1}{s-r}\int_r^s u(t)\,dt, & p \neq q \text{ and } r \neq s \\ u(p) - \frac{1}{s-r}\int_r^s u(t)\,dt, & p = q \text{ and } r \neq s \\ \dfrac{1}{q-p}\int_p^q u(t)\,dt - u(r), & p \neq q \text{ and } r = s \\ u(p) - u(r), & p = q \text{ and } r = s, \end{cases} \tag{3.13}$$

where $u(t) = \frac{d}{dx}I_t(x,1)$. We shall prove that $\phi(x)$ is a decreasing function on its domain. Consider the case when $r+s \geqslant 0$ and $p \geqslant (r+s)/2$. Taking into account that the function $u(t)$ is strictly decreasing for $t \neq 0$ (see (3.4)) we have

$$\frac{1}{q-p}\int_p^q u(t)\,dt \leqslant u(p). \tag{3.14}$$

This in conjuction with the first inequality in (3.10) and the first line of (3.13) gives

$$\frac{\phi'(x)}{\phi(x)} \leqslant u(p) - u\left(\frac{r+s}{2}\right) \leqslant 0,$$

where the last inequality holds true because $p \geqslant (r+s)/2$. Hence $\phi'(x) \leqslant 0$ for $0 < x < 1$. Assume now that $r+s \leqslant 0$. Making use of (3.14) and the second inequality in (3.10) applied to the expression on the right side in the second line of (3.13) we obtain

$$\frac{\phi'(x)}{\phi(x)} \leqslant u(p) - \frac{1}{2}\big[u(r)+u(s)\big] = \frac{1}{2}\big[u(p)-u(r)\big] + \frac{1}{2}\big[u(p)-u(s)\big] \leqslant 0,$$

where the last inequality holds true provided $p \geqslant r$ and $p \geqslant s$. Since $r \leqslant p$, $\phi'(x) \leqslant 0$ provided $p \geqslant r$. Assume now that $p+q \geqslant 0$. Utilizing monotonicity of the function $u(t)$ together with the use of $r \leqslant s$ gives

$$\frac{1}{s-r}\int_r^s u(t)\,dt \geqslant u(s). \tag{3.15}$$

This in conjunction with the third member of (3.13) and the second inequality in (3.10) gives

$$\frac{\phi'(x)}{\phi(x)} \leqslant \frac{1}{2}\big[u(p)+u(q)\big] - u(s) = \frac{1}{2}\big[u(p)-u(s)\big] + \frac{1}{2}\big[u(q)-u(s)\big] \leqslant 0,$$

where the last inequality is valid provided $p \geqslant s$ and $q \geqslant s$. Thus $\phi'(x) \leqslant 0$ if $s \leqslant p$. Finally, let $p+q \leqslant 0$. Then

$$\frac{\phi'(x)}{\phi(x)} \leqslant u\left(\frac{p+q}{2}\right) - u(s), \tag{3.16}$$

where the last inequality follows from the first inequality in (3.10) and from (3.15). Since $u(t)$ is strictly decreasing, the right side of (3.16) is nonpositive if $s \leqslant (p+q)/2$. The desired property of the function $\phi(x)$ now follows. In order to establish the inequality (3.11) we employ the inequality $\phi(x) \geqslant \phi(y)$ with

$$x = \frac{a_2}{a_1} \leqslant \frac{b_2}{b_1} = y < 1.$$

Making use of (3.12) and properties (P2) and (P3) we obtain the assertion. The proof is complete. □

We shall establish now an inequality between the ratios of the Stolarsky and Gini means.

THEOREM 3.4. *Let the vectors* a *and* b *satisfy assumptions of Theorem 3.3. If* $p+q \geqslant 0$, *then*

$$\frac{D_{p,q}(a)}{D_{p,q}(b)} \leqslant \frac{S_{p,q}(a)}{S_{p,q}(b)}. \tag{3.17}$$

Inequality (3.17) *is reversed if* $p+q \leqslant 0$.

Proof. Let now

$$\phi(x) = \frac{D_{p,q}(x,1)}{S_{p,q}(x,1)}, \tag{3.18}$$

where $0 < x < 1$. Using (P5) and (P6) we obtain

$$\ln \phi(x) = \begin{cases} \frac{1}{q-p} \int_p^q \big[\ln I_t(x,1) - \ln J_t(x,1)\big]\, dt, & p \neq q \\ \ln I_p(x,1) - \ln J_p(x,1), & p = q. \end{cases}$$

Differentiation with respect to x gives

$$\frac{\phi'(x)}{\phi(x)} = \begin{cases} \frac{1}{q-p} \int_p^q u(t)\, dt, & p \neq q \\ u(p), & p = q, \end{cases} \tag{3.19}$$

where now

$$u(t) = \frac{d}{dx}\big[\ln I_t(x,1) - \ln J_t(x,1)\big].$$

Making use of (2.3), (2.1), (2.4), and (2.2) we obtain

$$\ln I_t(x,1) - \ln J_t(x,1) = -\frac{1}{t} + \frac{2x^t \ln x}{x^{2t}-1}, \quad t \neq 0.$$

Hence

$$u(t) = \frac{2x^{t-1}}{(x^{2t}-1)^2}\big[x^{2t} - 1 - (x^{2t}+1)\ln x^t\big]. \tag{3.20}$$

We shall prove that the function $u(t)$ has the following properties

$$u(t) \begin{cases} > 0 & \text{if } t > 0, \\ < 0 & \text{if } t < 0 \end{cases} \tag{3.21}$$

and

$$u(-t) = -u(t). \tag{3.22}$$

For the proof of (3.21) we substitute $y = x^t$ into (3.20) to obtain

$$u(t) = \frac{4x^{t-1}\ln x}{(x^{2t}-1)^2}\left(\frac{y^2-1}{\ln y^2} - \frac{y^2+1}{2}\right) = \frac{4x^{t-1}\ln x}{(x^{2t}-1)^2}\big[L(y^2,1) - A(y^2,1)\big].$$

Since $0 < x < 1$, $0 < y < 1$ for $t > 0$ and $y > 1$ for $t < 0$, the inequality of the logarithmic and arithmetic means implies (3.21). For the proof of (3.22) we rewrite (3.20) as

$$u(t) = \frac{2}{x}\frac{y}{(y^2-1)^2}\big[y^2 - 1 - (y^2+1)\ln y\big],$$

where $y = x^t$. Easy computations give the assertion. It follows from (3.19), (3.21) and (3.22) that $\phi'(x) \geqslant 0$ if $p+q \geqslant 0$ and $\phi'(x) \leqslant 0$ if $p+q \leqslant 0$ with equalities if $p+q=0$. To complete the proof of (3.17) we let

$$x = \frac{a_2}{a_1} \leqslant \frac{b_2}{b_1} = y < 1$$

in $\phi(x) \leqslant \phi(y)$ when $p+q \geqslant 0$. This in conjunction with (3.18) and properties (P2) and (P3) completes the proof. The case when $p+q \leqslant 0$ can be treated in an analogous manner. This completes the proof. □

Our next result reads as follows.

THEOREM 3.5. *Let the vectors a and b satisfy monotonicity conditions* (1.1). *Then*

$$\begin{aligned} \frac{H(a)}{H(b)} &\leqslant \frac{G(a)}{G(b)} \leqslant \left[\frac{G^2(a)A(a)}{G^2(b)A(b)}\right]^{1/3} \leqslant \frac{L(a)}{L(b)} \leqslant \frac{\mathscr{H}_4(a)}{\mathscr{H}_4(b)} \leqslant \frac{\mathscr{H}_1(a)}{\mathscr{H}_1(b)} \\ &\leqslant \frac{A_{2/3}(a)}{A_{2/3}(b)} \leqslant \frac{I(a)}{I(b)} \leqslant \frac{\mathscr{H}_{e-2}(a)}{\mathscr{H}_{e-2}(b)} \leqslant \frac{\mathscr{H}_\omega(a)}{\mathscr{H}_\omega(b)} \leqslant \frac{A(a)}{A(b)} \leqslant \frac{C(a)}{C(b)} \end{aligned} \tag{3.23}$$

provided $0 \leqslant \omega \leqslant e-2$.

Proof. The first inequality in (3.23) follows from (3.11) and (2.7) with $r = -2$, $s = -1$, $p = q = 0$ while the second one is an immediate consequence of $G(a)/G(b) \leqslant A(a)/A(b)$ which is a part of (3.23). For the proof of the third inequality in (3.23) we define a function

$$\phi(x) = \frac{L^3(x,1)}{G^2(x,1)A(x,1)}, \tag{3.24}$$

$0 < x < 1$. We shall prove that $\phi(x)$ is a decreasing function on the stated domain. Logarithmic differentiation gives

$$\frac{\phi'(x)}{\phi(x)} = 3\left(\frac{1}{x-1} - \frac{1}{x\ln x}\right) - \frac{2x+1}{x(x+1)}.$$

Letting $x = 1/t$ ($t > 1$) we see that the last formula can be written as

$$\frac{\phi'(x)}{\phi(x)} = \frac{3t}{t-1}\left[\frac{t-1}{\ln t} - \frac{t^2+4t+1}{3(t+1)}\right]. \tag{3.25}$$

To complete the proof of monotonicity of $\phi(x)$ we apply Carlson's inequality (3.9) to obtain

$$\frac{t-1}{\ln t} \leqslant \frac{t^2+4t+1}{3(t+1)}.$$

This in conjunction with (3.25) gives the desired result. To complete the proof of the inequality in question we follow the lines introduced at the end of the proofs of Theorems 3.3 and 3.4. The fourth, sixth, and eighth inequalities in (3.23) are established in [6]. (See Theorems 3.2, 3.1, and 3.3, respectively.) The fifth, ninth, and the tenth inequalities in (3.23) are a consequence of the monotonicity in ω of the ratio $\mathscr{H}_\omega(a)/\mathscr{H}_\omega(b)$. We have

$$\frac{\mathscr{H}_\alpha(a)}{\mathscr{H}_\alpha(b)} \leqslant \frac{\mathscr{H}_\beta(a)}{\mathscr{H}_\beta(b)} \tag{3.26}$$

provided $\alpha > \beta \geqslant 0$ and $0 < x \leqslant 1$. For, let

$$\phi(x) = \frac{\mathscr{H}_\alpha(x,1)}{\mathscr{H}_\beta(x,1)}. \tag{3.27}$$

Differentiating we obtain

$$\phi'(x) = \frac{(2+\beta)(\alpha-\beta)}{2+\alpha} \frac{1-x}{2\sqrt{x}(x+1+\beta\sqrt{x})^2}.$$

Thus $\phi(x)$ is increasing for $0 < x \leqslant 1$. Letting in (3.27) $x = a_2/a_1 \leqslant b_2/b_1 = y \leqslant 1$ we obtain the inequality (3.26). The seventh inequality in (3.23) is a consequence of the fact that $A_{2/3}(x,1)/I(x,1)$ is a decreasing function for $0 < x < 1$ (see [13, p. 104]). The remaining part of the proof of the inequality in question goes along the lines introduced in the proofs of Theorems 3.3 and 3.4. The last inequality in (3.23) is a special case of (3.11) when $r = 1$, $s = 2$, $p = 2$ and $q = 3$. The proof is complete. □

We shall now derive inequalities involving ratios of the Schwab-Borchardt means and the lemniscatic means. The following result, sometimes called the L'Hospital-type rule for monotonicity, will be utilized in the sequel.

PROPOSITION 3.6. ([21]) *Let f and g be continuous functions on $[c,d]$. Assume that they are differentiable and $g'(t) \neq 0$ on (c,d). If f'/g' is strictly increasing (decreasing) on (c,d), then so are*

$$\frac{f(t)-f(c)}{g(t)-g(c)} \quad \textit{and} \quad \frac{f(t)-f(d)}{g(t)-g(d)}.$$

(See also [16].)

We are in a position to prove the following.

THEOREM 3.7. *Let the vectors satisfy the monotonicity conditions* (1.1). *Then the following inequalities*

$$\frac{SB(a_1,a_2)}{SB(b_1,b_2)} \leqslant \frac{LM(a_2,a_1)}{LM(b_2,b_1)} \leqslant \frac{LM(a_1,a_2)}{LM(b_1,b_2)} \leqslant \frac{SB(a_2,a_1)}{SB(b_2,b_1)} \tag{3.28}$$

hold true.

Proof. In order to establish the first inequality in (3.28) we introduce a function

$$\phi(x) = \frac{SB(x,1)}{LM(1,x)} \tag{3.29}$$

($x \geqslant 1$). Making use of

$$SB(x,1) = \frac{t^2}{\operatorname{arc\,sinh} t^2}$$

(see [12, (1.3)]) and

$$LM(1,x) = \frac{t^2}{(\text{arc slh}\, t)^2}$$

(see [9, (6.2]) we obtain

$$\phi(x) = \frac{(\text{arc slh}\, t)^2}{\text{arc sinh}\, t^2}\,,$$

where $t = \sqrt[4]{x^2-1}$. To prove that $\phi(x)$ is an increasing function on its domain we write $\phi(x) = f(t)/g(t)$, where $f(t) = (\text{arc slh}\, t)^2$ and $g(t) = \text{arc sinh}\, t^2$ $(t \geqslant 0)$. Differentiation gives

$$\frac{f'(t)}{g'(t)} = \frac{\text{arc slh}\, t}{t} = R_B(1, 1+t^4),$$

where in the last step we have used (2.11). Since R_B is a decreasing function in each of its variables (see (2.12)) we conclude, using Proposition 3.6 and the fact that $f(0) = g(0) = 0$, that $\phi(x)$ has the desired property, i.e., $\phi(x) \geqslant \phi(y)$ whenever $x \geqslant y$. Letting

$$x = \frac{a_1}{a_2} \geqslant \frac{b_1}{b_2} = y \geqslant 1$$

and next using (3.29) and the fact that both means SB and LM are homogeneous we obtain the assertion. For the proof of the second inequality in (3.28), we define

$$\phi(x) = \frac{LM(1,x)}{LM(x,1)}$$

($x \geqslant 1$). Using [9, (6.1)–(6.2)] we obtain

$$\phi(x) = \left[\frac{f(t)}{g(t)}\right]^2, \tag{3.30}$$

where

$$f(t) = \text{arc sl}\left(\frac{t}{\sqrt[4]{1+t^4}}\right) = t R_B(1+t^4, 1)$$

and $g(t) = \text{arc slh}\, t = t R_B(1, 1+t^4)$ and $t = \sqrt[4]{x^2-1}$. Taking into account that $f'(t) = (1+t^4)^{-3/4}$ and $g'(t) = (1+t^4)^{-1/2}$ we see that

$$\frac{f'(t)}{g'(t)} = (1+t^4)^{-1/4}$$

is a decreasing function for $t \geqslant 0$. Making use of Proposition 3.6 we conclude that the function $f(t)/g(t)$ decreases with an increase in t. This together with (3.30) implies that $\phi(x) \leqslant \phi(y)$ whenever $x > y$. We now follow the lines introduced in the proof of the first inequality in (3.28) to obtain the desired result. In order to establish the third inequality in (3.28) we define

$$\phi(x) = \frac{LM(x,1)}{SB(1,x)} \tag{3.31}$$

($x \geqslant 1$). In order to prove that $\phi(x)$ is a decreasing function on its domain it suffices to show that the function $\psi(x) = \phi\left(\frac{1}{x}\right)$ is an increasing function on $(0,1]$. Using (3.31) and the fact that LM and SB are homogeneous functions we obtain

$$\psi(x) = \frac{LM(1,x)}{SB(x,1)}$$

($0 < x \leqslant 1$). Making use of [9, (6.1)] and [12, (1.2)] we obtain

$$\psi(x) = \frac{f(t)}{g(t)}, \tag{3.32}$$

where $f(t) = \operatorname{arc\,sin} t^2$, $g(t) = (\operatorname{arc\,sl} t)^2$ and $t = \sqrt[4]{1-x^2}$. Hence

$$\frac{f'(t)}{g'(t)} = \frac{t}{\operatorname{arc\,sl} t} = \frac{1}{R_B(1, 1-t^4)},$$

where the last equality follows from (2.10). We conclude that the ratio $f'(t)/g'(t)$ is a decreasing function of t because R_B is also decreasing in each of its variables. This in conjunction with Proposition 3.6 applied to (3.32) and the fact that t and x satisfy $t = \sqrt[4]{1-x^2}$ leads to the conclusion that $\psi(x)$ is an increasing function on $(0,1]$. This in turn implies that $\phi(x)$ defined in (3.31) is decreasing for every $x \geqslant 1$. We follow the lines introduced earlier in this proof to complete the proof of the last inequality in (3.28). □

Before we state and prove a corollary of Theorem 3.7, let us introduce some special means derived from SB and LM. To this end let $x > 0$, $y > 0$ and let G, A and

$$Q \equiv Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}$$

stand for the geometric mean, arithmetic mean and the root-mean-square mean of x and y. Following [12, (2.8)] let

$$L = SB(A,G),\ P = SB(G,A),\ M = SB(Q,A),\ T = SB(A,Q), \tag{3.33}$$

where L stands for the logarithmic mean and P and T are the Seiffert means (see [19], [20]). Clearly all four means defined above are symmetric and homogeneous of degree 1. The lemniscate counterparts of these means have been introduced in [9, (6.4)]:

$$U = LM(G,A),\ V = LM(A,G),\ R = LM(A,Q),\ S = LM(Q,A). \tag{3.34}$$

It is easy to see that these means are symmetric and homogeneous of degree 1. The following inequalities

$$L \leqslant U \leqslant V \leqslant P \leqslant A \leqslant M \leqslant R \leqslant S \leqslant T \tag{3.35}$$

have been established in [9, (6.10)].

We are in a position to establish the following.

COROLLARY 3.8. *The means defined in* (3.33) *and* (3.34) *satisfy the following inequalities*

$$\frac{L}{M} \leqslant \frac{U}{R} \leqslant \frac{V}{S} \leqslant \frac{P}{T}. \tag{3.36}$$

Proof. Let $a_1 = A$, $a_2 = G$, $b_1 = Q$ and $b_2 = A$. Since $A^2 \geqslant GQ$, the numbers a_i and b_i satisfy the inequalities (1.1). Utilizing (3.28) and (3.33) and (3.34) one obtains the assertion (3.36). □

Let a and b satisfy (1.1). Then the inequalities (3.35) can be obtained immediately from

$$\frac{L(a)}{L(b)} \leqslant \frac{U(a)}{U(b)} \leqslant \frac{V(a)}{V(b)} \leqslant \frac{P(a)}{P(b)} \leqslant \frac{A(a)}{A(b)} \leqslant \frac{M(a)}{M(b)} \leqslant \frac{R(a)}{R(b)} \leqslant \frac{S(a)}{S(b)} \leqslant \frac{T(a)}{T(b)} \tag{3.37}$$

by letting $b_1 = b_2$. Since the proof of (3.37) goes along the lines introduced in [9, Theorem 6.2], it is omitted.

We close this section with a result which shows that the inequality (1.2) implies the Ky Fan inequality for the means Φ and Ψ:

$$\frac{\Phi(a)}{\Phi(a')} \leqslant \frac{\Psi(a)}{\Psi(a')}, \tag{3.38}$$

where $a = (a_1, a_2)$ with $0 < a_1, a_2 \leqslant \frac{1}{2}$ and

$$a' = 1 - a = (1 - a_1, 1 - a_2). \tag{3.39}$$

PROPOSITION 3.9. *Let Φ and Ψ be symmetric homogeneous means of two positive variables and assume that the inequality* (1.2) *holds true for the vectors a and b which satisfy monotonicity conditions* (1.1). *Then the means Φ and Ψ also satisfy the Ky Fan inequalities* (3.38).

Proof. Without a loss of generality let us assume that $a = (a_1, a_2)$ is such that $0 < a_2 < a_1 \leqslant \frac{1}{2}$ and $b = (b_1, b_2) = (1 - a_2, 1 - a_1)$. It is easy to verify that a and b satisfy (1.1). Since ϕ and ψ are symmetric means, inequality (1.2) holds true with b replaced by a' (see (3.39)). □

Application of Proposition 3.9 to Theorems 3.1–3.3 in [6] gives immediately Theorems 4.1, 4.2 and 4.4 in [6].

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