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EXTENSIONS OF THE GENERALIZED WILKER INEQUALITY TO BESSEL FUNCTIONS

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Abstract. In this note our aim is to extend the weighted and exponential generalization of Wilker's inequality and the Sándor-Bencze conjectured inequality to Bessel functions of the first kind. Our main motivation to write this note is a recent publication of Wu and Srivastava, which we wish to complement.

1. Introduction and Preliminaries

The inequality

$$\left(\frac{\sin x}{x}\right)^2 + \frac{\tan x}{x} \geqslant 2, \tag{1}$$

which holds for all $x \in (-\pi/2, \pi/2)$, is known in literature as Wilker's inequality [15]. Here and throughout this paper, it should be understood that functions such as $(\sin x)/x$, which have removable singularities at $x = 0$, have had these singularities removed in statements like (1). Wilker's inequality has generated considerable interest and has been investigated by many mathematicians in the last few decades. For more details the interested reader is referred to the papers [1, 8, 12, 14, 17] and to the references therein. Another inequality which is of interest in this paper is the so-called Huygens inequality [9], i.e.

$$2\left(\frac{\sin x}{x}\right) + \frac{\tan x}{x} \geqslant 3, \tag{2}$$

which holds for all $x \in (-\pi/2, \pi/2)$. We note that in fact the inequalities (1) and (2) are simple consequences of the well-known Lazarević-type inequality [11, p. 238]

$$\left(\frac{\sin x}{x}\right)^3 \geqslant \cos x, \tag{3}$$

which holds for all $x \in (-\pi/2, \pi/2)$, and of the arithmetic-geometric mean inequality. Using this idea, recently, Wu and Srivastava [16, Theorem 1], in order to unify and extend the inequalities (1) and (2), proved that if $\lambda, \mu, q > 0$, $x \in (0, \pi/2)$ and

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$p \leqslant 2q\mu/\lambda$, then the following weighted and exponential generalization of Wilker's inequality holds true:

$$\frac{\lambda}{\lambda+\mu}\left(\frac{\sin x}{x}\right)^p+\frac{\mu}{\lambda+\mu}\left(\frac{\tan x}{x}\right)^q>1. \tag{4}$$

Since the left hand side of (4) is even in x, and if x tends to zero, then the left hand side of (4) tends to 1, we can conclude that (4) in fact holds true for all $x \in (-\pi/2,\pi/2)$, provided that when $x \to 0$, then we have equality in (4). Moreover, the authors in [16, Theorem 1] proved that the weighted and exponential generalization of Wilker's inequality (1), i.e. the inequality (4) holds true if we replace the condition $q>0$ with $q \leqslant \min\{-\lambda/\mu,-1\}$. The key tool in the proof of this it was the following Wilker-type inequality [16, Lemma 3]

$$\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}>2, \tag{5}$$

which holds for each $x \in (0,\pi/2)$. We note that inequality (5) holds true for each $x \in (-\pi/2,\pi/2)$. Moreover, (5) is in fact an immediate consequence of Wilker's inequality (1). Namely, if we consider the function $f:(-\pi/2,\pi/2)\to\mathbf{R}$, defined by

$$f(x)=\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}-2,$$

then we easily have

$$f'(x)=\frac{x^2\cos x}{\sin^3 x}\left[\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}-2\right].$$

This in turn together with (1) implies that f is decreasing on $(-\pi/2,0]$ and increasing on $[0,\pi/2)$, and consequently one has $f(x)\geqslant f(0)=0$ for all $x\in(-\pi/2,\pi/2)$, as we requested. With other words, the Lazarević-type inequality (3) implies the Wilker inequality (1), which implies the Wilker-type inequality (5). Moreover, the inequalities (1), (3) and (5) can be rewritten in terms of the arithmetic, geometric and harmonic means of the values $[x/(\sin x)]^2$ and $x/\tan x$, i.e.

$$\frac{1}{2}\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]\geqslant 1\geqslant\sqrt{\left(\frac{x}{\sin x}\right)^2\left(\frac{x}{\tan x}\right)}\geqslant 2\left[\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\right]^{-1}$$

or in terms of the arithmetic, geometric and harmonic means of the values $[(\sin x)/x]^2$ and $(\tan x)/x$, i.e.

$$\frac{1}{2}\left[\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}\right]\geqslant\sqrt{\left(\frac{\sin x}{x}\right)^2\left(\frac{\tan x}{x}\right)}\geqslant 1\geqslant 2\left[\left(\frac{x}{\sin x}\right)^2+\frac{x}{\tan x}\right]^{-1},$$

where in both of inequalities $x\in(-\pi/2,\pi/2)$.

2. Extensions of the generalized Wilker inequality to Bessel functions

In this section our aim is to extend the inequality (4) to Bessel functions of the first kind. This is motivated by the simple fact that the above inequalities can be rewritten in terms of Bessel functions. For this suppose that $\nu > -1$ and consider the function $\mathscr{J}_\nu : \mathbf{R} \to (-\infty, 1]$, defined by

$$\mathscr{J}_\nu(x) = 2^\nu \Gamma(\nu+1) x^{-\nu} J_\nu(x) = \sum_{n \geqslant 0} \frac{(-1/4)^n}{(\nu+1)_n n!} x^{2n},$$

where Γ is the Euler gamma function, $(\nu+1)_n = \Gamma(\nu+n+1)/\Gamma(\nu+1)$ for each $n \geqslant 0$ is the well-known Pochhammer (or Appell) symbol, and J_ν, defined by

$$J_\nu(x) = \sum_{n \geqslant 0} \frac{(-1)^n (x/2)^{\nu+2n}}{n! \Gamma(\nu+n+1)},$$

stands for the Bessel function of the first kind of order ν. It is worth mentioning that in particular the function $\mathscr{J}_\nu$ reduces to some elementary functions, like sine and cosine. More precisely, in particular we have

$$\mathscr{J}_{-1/2}(x) = \sqrt{\pi/2} \cdot x^{1/2} J_{-1/2}(x) = \cos x, \tag{6}$$

$$\mathscr{J}_{1/2}(x) = \sqrt{\pi/2} \cdot x^{-1/2} J_{1/2}(x) = \frac{\sin x}{x}, \tag{7}$$

$$\mathscr{J}_{3/2}(x) = 3\sqrt{\pi/2} \cdot x^{-3/2} J_{3/2}(x) = 3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right), \tag{8}$$

respectively, which can verified easily by using the series representation of the function $\mathscr{J}_\nu$ and of the cosine and sine functions, respectively. Taking into account the relations (6) and (7), as we mentioned above, the inequalities (1), (2), (3) and (5) can be rewritten in terms of $\mathscr{J}_{-1/2}$ and $\mathscr{J}_{1/2}$. For example, using (6) and (7) the inequality (4) can be rewritten as

$$\frac{\lambda}{\lambda+\mu} \left[\mathscr{J}_{-1/2+1}(x)\right]^p + \frac{\mu}{\lambda+\mu} \left[\frac{\mathscr{J}_{-1/2+1}(x)}{\mathscr{J}_{-1/2}(x)}\right]^q \geqslant 1$$

and thus it is natural to ask what is the general form of the inequalities (4) and (5) for arbitrary ν. We note that, recently the inequalities (1) and (3) has been extended to Bessel functions [5, Theorem 3] (see inequalities (17) and (14) below).

Our first main result provides an affirmative answer to the above problem and it is an extension of inequalities (4) and (5) to Bessel functions of the first kind. We note this result improves the earlier result of the first author [5, Eq. 2.11].

THEOREM 1. *Let $\nu > -1$ and let $j_{\nu,1}$ the first positive zero of the Bessel function J_ν of the first kind. Then the following assertions are true:*

1. *If $\nu \geqslant -1/2$ and $x \in (-j_{\nu+1,1}, j_{\nu+1,1})$, then the following inequality holds*

$$\frac{1}{\left[\mathscr{J}_{\nu+1}(x)\right]^2} + \frac{\mathscr{J}_\nu(x)}{\mathscr{J}_{\nu+1}(x)} \geqslant 2. \tag{9}$$

2. *If* $\nu \in (-1,\nu_0]$, *where* $\nu_0 = (-3+\sqrt{5})/2 \simeq -0.381966011$ *and* $x \in (-j_{\nu+1,1}, j_{\nu+1,1})$, *then*

$$\frac{1}{[\mathscr{J}_{\nu+1}(x)]^{1/(\nu+1)}} + \frac{\mathscr{J}_\nu(x)}{\mathscr{J}_{\nu+1}(x)} \geqslant 2. \tag{10}$$

3. *Suppose that* $\lambda,\mu > 0$ *and* $x \in (-j_{\nu,1}, j_{\nu,1})$. *Then the generalized and extended Wilker's inequality*

$$\frac{\lambda}{\lambda+\mu}[\mathscr{J}_{\nu+1}(x)]^p + \frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_\nu(x)}\right]^q \geqslant 1 \tag{11}$$

holds in the following cases:

a) *if* $q > 0,\ \nu > -1$ *and* $p(\nu+1) \leqslant q\mu/\lambda$;
b) *if* $q \leqslant \min\{-\lambda/\mu, -1\},\ \nu \geqslant -1/2$ *and* $p \leqslant 2q\mu/\lambda$;
c) *if* $q \leqslant \min\{-\lambda/\mu, -1\},\ \nu \in (-1,\nu_0]$ *and* $p(\nu+1) \leqslant q\mu/\lambda$.

Proof. **1.** Consider the function $\varphi_\nu : (-j_{\nu+1,1}, j_{\nu+1,1}) \to \mathbf{R}$, defined by

$$\varphi_\nu(x) = \frac{1}{[\mathscr{J}_{\nu+1}(x)]^2} + \frac{\mathscr{J}_\nu(x)}{\mathscr{J}_{\nu+1}(x)} - 2.$$

Clearly φ_ν is an even function and $\varphi_\nu(0) = 0$. Moreover, a simple computation yields

$$\begin{aligned} 2(\nu+1)(\nu+2)\varphi_\nu'(x) &= \frac{x}{[\mathscr{J}_{\nu+1}(x)]^2}\left[2(\nu+1)\frac{\mathscr{J}_{\nu+2}(x)}{\mathscr{J}_{\nu+1}(x)}\right. \\ &\quad \left. +(\nu+1)\mathscr{J}_\nu(x)\mathscr{J}_{\nu+2}(x) - (\nu+2)[\mathscr{J}_{\nu+1}(x)]^2\right] \\ &\geqslant \frac{x}{[\mathscr{J}_{\nu+1}(x)]^2}\left[2(\nu+1)\frac{\mathscr{J}_{\nu+2}(x)}{\mathscr{J}_{\nu+1}(x)} - 1\right] \\ &\geqslant \frac{(2\nu+1)x}{[\mathscr{J}_{\nu+1}(x)]^2} \geqslant 0, \end{aligned}$$

where $\nu \geqslant -1/2$ and $x \in [0, j_{\nu+1,1})$. Here we have used that for each $\nu > -1$ and $x \in \mathbf{R}$ the differentiation formula

$$\mathscr{J}_\nu'(x) = -\frac{x}{2(\nu+1)}\mathscr{J}_{\nu+1}(x) \tag{12}$$

holds, which can be verified easily by using the series representation of the function $\mathscr{J}_\nu$. Moreover, we used on the one hand the Turán-type inequality [10, Eq. 2.9]

$$(\nu+1)\mathscr{J}_\nu(x)\mathscr{J}_{\nu+2}(x) - (\nu+2)[\mathscr{J}_{\nu+1}(x)]^2 \geqslant -1, \tag{13}$$

which holds for all $x \in \mathbf{R}$ and $\nu > -1$, and on the other hand the well-known fact [5, Theorem 3] that the function $\nu \mapsto \mathscr{J}_\nu(x)$ is increasing on $(-1,\infty)$ for all fixed $x \in (-j_{\nu,1}, j_{\nu,1})$, i.e. for each $\nu > -1$ and $x \in (-j_{\nu+1,1}, j_{\nu+1,1})$ we have $\mathscr{J}_{\nu+2}(x) \geqslant \mathscr{J}_{\nu+1}(x)$. Thus we have proved that the function φ_ν is increasing on $[0, j_{\nu+1,1})$ and since is even it is decreasing on $(-j_{\nu+1,1}, 0]$. Consequently for each

$\nu \geqslant -1/2$ and $x \in (-j_{\nu+1,1}, j_{\nu+1,1})$ one has $\varphi_\nu(x) \geqslant \varphi_\nu(0) = 0$, and with this the proof of the inequality (9) is done.

2. Consider the function $\phi_\nu : (-j_{\nu+1,1}, j_{\nu+1,1}) \to \mathbf{R}$, defined by

$$\phi_\nu(x) = \frac{1}{[\mathscr{J}_{\nu+1}(x)]^{1/(\nu+1)}} + \frac{\mathscr{J}_\nu(x)}{\mathscr{J}_{\nu+1}(x)} - 2.$$

Clearly ϕ_ν is an even function and $\phi_\nu(0) = 0$. Moreover, using (12) and (13) we have

$$\begin{aligned}
2(\nu+1)(\nu+2)\phi_\nu'(x) &= \frac{x}{[\mathscr{J}_{\nu+1}(x)]^2}\Big[[\mathscr{J}_{\nu+1}(x)]^{\frac{\nu}{\nu+1}}\mathscr{J}_{\nu+2}(x) \\
&\quad +(\nu+1)\mathscr{J}_\nu(x)\mathscr{J}_{\nu+2}(x) - (\nu+2)[\mathscr{J}_{\nu+1}(x)]^2\Big] \\
&\geqslant \frac{x}{[\mathscr{J}_{\nu+1}(x)]^2}\Big[[\mathscr{J}_{\nu+1}(x)]^{\frac{\nu}{\nu+1}}\mathscr{J}_{\nu+2}(x) - 1\Big] \\
&\geqslant \frac{x}{[\mathscr{J}_{\nu+1}(x)]^2}\Big[[\mathscr{J}_{\nu+1}(x)]^{\frac{2(\nu^2+3\nu+1)}{\nu^2+4\nu+3}} - 1\Big] \geqslant 0,
\end{aligned}$$

where $\nu \in (-1, \nu_0]$ and $x \in [0, j_{\nu+1,1})$. Here we have used the fact that for all $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$ the generalized Lazarević-type inequality [5, Theorem 3]

$$[\mathscr{J}_{\nu+1}(x)]^{\nu+2} \geqslant [\mathscr{J}_\nu(x)]^{\nu+1} \tag{14}$$

holds and that for each $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$ we have $\mathscr{J}_\nu(x) \in (0,1]$, since the function $x \mapsto \mathscr{J}_\nu(x)$ is increasing [5, Theorem 3] on $(-j_{\nu,1}, 0]$ and is decreasing on $[0, j_{\nu,1})$. Thus we have proved that the function ϕ_ν is increasing on $[0, j_{\nu+1,1})$ and since is even it is decreasing on $(-j_{\nu+1,1}, 0]$. Consequently for each $\nu \in (-1, \nu_0]$ and $x \in (-j_{\nu+1,1}, j_{\nu+1,1})$ one has $\phi_\nu(x) \geqslant \phi_\nu(0) = 0$, and with this the proof of inequality (10) is complete.

3.a) First consider the case when $\lambda, \mu, q > 0$, $\nu > -1$ and $p(\nu+1) \leqslant q\mu/\lambda$. Using the well-known weighted arithmetic-geometric inequality and the generalized Lazarević-type inequality (14) we easily obtain that

$$\begin{aligned}
\frac{\lambda}{\lambda+\mu}[\mathscr{J}_{\nu+1}(x)]^p + \frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_\nu(x)}\right]^q &\geqslant [\mathscr{J}_{\nu+1}(x)]^{\frac{p\lambda}{\lambda+\mu}}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_\nu(x)}\right]^{\frac{q\mu}{\lambda+\mu}} \\
&= [\mathscr{J}_{\nu+1}(x)]^{\frac{p\lambda+q\mu}{\lambda+\mu}}[\mathscr{J}_\nu(x)]^{-\frac{q\mu}{\lambda+\mu}} \\
&\geqslant [\mathscr{J}_\nu(x)]^{\frac{p\lambda+q\mu}{\lambda+\mu}\cdot\frac{\nu+1}{\nu+2}}[\mathscr{J}_\nu(x)]^{-\frac{q\mu}{\lambda+\mu}} \\
&= [\mathscr{J}_\nu(x)]^{\frac{(\nu+1)p\lambda-q\mu}{(\nu+2)(\lambda+\mu)}} \geqslant 1.
\end{aligned}$$

Here we have used that from hypothesis $(\nu+1)p\lambda - q\mu \leqslant 0$ and for each $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$ we have $\mathscr{J}_\nu(x) \in (0,1]$, since the function $x \mapsto \mathscr{J}_\nu(x)$ is increasing [5, Theorem 3] on $(-j_{\nu,1}, 0]$ and is decreasing on $[0, j_{\nu,1})$.

3.b) Now consider the case when $\lambda,\mu>0,\ \nu\geqslant-1/2,\ q\leqslant\min\{-\lambda/\mu,-1\}$ and $p\leqslant 2q\mu/\lambda.$ Then from (9) we easily obtain that

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\left[\mathscr{J}_{\nu+1}(x)\right]^{p}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_{\nu}(x)}\right]^{q}\\
&\quad\geqslant\frac{\lambda}{\lambda+\mu}\left[\mathscr{J}_{\nu+1}(x)\right]^{\frac{2q\mu}{\lambda}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_{\nu}(x)}\right]^{q}\\
&\quad=\frac{\lambda}{\lambda+\mu}\left[\frac{1}{\mathscr{J}_{\nu+1}(x)}\right]^{-\frac{2q\mu}{\lambda}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-q}\\
&\quad\geqslant\frac{\lambda}{\lambda+\mu}\left[2-\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-\frac{q\mu}{\lambda}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-q}\geqslant 1.
\end{aligned}$$

Here we used that the function $g:[0,1]\to\mathbf{R},$ defined by

$$g(t)=\frac{\lambda}{\lambda+\mu}(2-t)^{-q\mu/\lambda}+\frac{\mu}{\lambda+\mu}t^{-q},\tag{15}$$

is decreasing [16, p. 533] and hence for all $t\in[0,1]$ we have $g(t)\geqslant g(1)=1,$ and the fact [5, Theorem 3] that the function $\nu\mapsto\mathscr{J}_{\nu}(x)$ is increasing on $(-1,\infty)$ for all fixed $x\in(-j_{\nu,1},j_{\nu,1}),$ i.e. for each $\nu>-1$ and $x\in(-j_{\nu,1},j_{\nu,1})$ we have $\mathscr{J}_{\nu+1}(x)\geqslant\mathscr{J}_{\nu}(x).$ All that remains is to observe that for each $\nu>-1$ we have $(-j_{\nu,1},j_{\nu,1})\subset(-j_{\nu+1,1},j_{\nu+1,1}),$ since the function $\nu\mapsto j_{\nu,n},$ where $j_{\nu,n}$ is the $n-$th positive root of $J_{\nu},$ is strictly increasing [7] on $[0,\infty),$ and consequently in particular we have $j_{\nu+1,n}>j_{\nu,n}$ for each $n\geqslant 1$ integer.

3.c) Finally, consider the case when $\lambda,\mu>0,\ \nu\in(-1,\nu_0],\ q\leqslant\min\{-\lambda/\mu,-1\}$ and $p(\nu+1)\leqslant q\mu/\lambda.$ The proof of this part is similar to the proof of part b, so we just sketch the proof. Using (10) we easily obtain that

$$\begin{aligned}
&\frac{\lambda}{\lambda+\mu}\left[\mathscr{J}_{\nu+1}(x)\right]^{p}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_{\nu}(x)}\right]^{q}\\
&\quad\geqslant\frac{\lambda}{\lambda+\mu}\left[\mathscr{J}_{\nu+1}(x)\right]^{\frac{q\mu}{\lambda(\nu+1)}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_{\nu}(x)}\right]^{q}\\
&\quad=\frac{\lambda}{\lambda+\mu}\left[\frac{1}{\mathscr{J}_{\nu+1}(x)}\right]^{-\frac{q\mu}{\lambda(\nu+1)}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-q}\\
&\quad\geqslant\frac{\lambda}{\lambda+\mu}\left[2-\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-\frac{q\mu}{\lambda}}+\frac{\mu}{\lambda+\mu}\left[\frac{\mathscr{J}_{\nu}(x)}{\mathscr{J}_{\nu+1}(x)}\right]^{-q}\geqslant 1,
\end{aligned}$$

where we have used again that the function $g,$ defined by (15), satisfies $g(t)\geqslant g(1)=1$ for all $t\in[0,1].$ $\square$

3. Concluding remarks

1. First note that choosing $\nu = -1/2$ in (9) or in (10) in view of the relations (6) and (7) we reobtain the Wilker-type inequality (5). It is important to note here that the interval of validity for this inequality from (9) or (10) becomes $(-j_{1/2,1}, j_{1/2,1})$, i.e. $(-\pi, \pi)$, and not $(-j_{-1/2,1}, j_{-1/2,1})$, i.e. $(-\pi/2, \pi/2)$, as it was stated in [16, Lemma 3]. Here we used that for each $n \geqslant 1$ integer $j_{-1/2,n} = (2n-1)\pi/2$ and $j_{1/2,n} = n\pi$, which can be deduced easily from (6) and (7) or from the infinite product representation of the Bessel function J_ν and from the relations (6) and (7) keeping in mind the infinite product representations of the cosine and sine functions. Now, choosing $\nu = -1/2$ in (11) we get back the inequality (4). Here we used again the formulae (6) and (7).

2. Taking in (11) the values $\lambda = \mu = 1$ we obtain the following inequality

$$[\mathscr{J}_{\nu+1}(x)]^p + \left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_\nu(x)}\right]^q \geqslant 2, \tag{16}$$

which hold for all $x \in (-j_{\nu,1}, j_{\nu,1})$ in the following cases:

a) if $q > 0$, $\nu > -1$ and $p(\nu+1) \leqslant q$;

b) if $q \leqslant -1$, $\nu \geqslant -1/2$ and $p \leqslant 2q$;

c) if $q \leqslant -1$, $\nu \in (-1, \nu_0]$ and $p(\nu+1) \leqslant q$.

Now, choosing in (16) the values $p = 1/(\nu+1)$ and $q = 1$, then we reobtain the inequality

$$[\mathscr{J}_{\nu+1}(x)]^{1/(\nu+1)} + \frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_\nu(x)} \geqslant 2, \tag{17}$$

which was proved recently by the first author [5, Eq. 2.11]. It is worth mentioning that recently the first author has been extended many elementary inequalities like of Mitrinović, Mahajan, Jordan, Redheffer, Cusa, Wilker, etc. to Bessel functions of the first kind. The interested reader is referred to the papers [2, 3, 4, 5, 6].

3. Using the relations (7) and (8) from the extended Wilker-type inequality (9) for $\nu = 1/2$ we obtain the following inequality

$$\frac{x^6}{(\sin x - x\cos x)^2} + \frac{3x^2}{1 - x\cot x} \geqslant 18,$$

which holds for all $x \in (-j_{3/2,1}, j_{3/2,1})$, where $j_{3/2,1} \simeq 4.493409457$ in view of (8) is in fact the first positive zero of the equation $\tan x = x$.

4. Finally, choosing $\nu = 1/2$ in part 3 of Theorem 1 and using (8) we get the inequality

$$\frac{\lambda}{\lambda+\mu}\left[3\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right)\right]^p + \frac{\mu}{\lambda+\mu}\left[3\left(\frac{1}{x^2} - \frac{\cot x}{x}\right)\right]^q \geqslant 1,$$

which holds in the following cases:

a) if $\lambda, \mu, q > 0$ and $3p \leqslant 2q\mu/\lambda$;

b) if $\lambda, \mu > 0$, $q \leqslant \min\{-\lambda/\mu, -1\}$ and $p \leqslant 2q\mu/\lambda$.

Here the interval of validity for x is $(-j_{1/2,1}, j_{1/2,1})$, i.e. $(-\pi,\pi)$.

Recently, Sándor and Bencze [13] posed the following open problem: prove that for each $x\in(0,\pi/2)$ and $\alpha>0$ the following inequality holds

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{\cos^{\alpha} x}{1+\cos^{\alpha} x}. \tag{18}$$

Motivated, by this problem, Wu and Srivastava [16, Corollary] by using the inequality (4) proved that the following inequalities hold

$$\left(\frac{\sin x}{x}\right)^{\alpha} > \frac{4\cos^{\alpha} x}{1+\sqrt{1+8\cos^{2\alpha} x}} > \frac{2\cos^{\alpha} x}{1+\cos^{\alpha} x} > \frac{\cos^{\alpha} x}{1+\cos^{\alpha} x} \tag{19}$$

for all $x\in(0,\pi/2)$ and $\alpha>0$. Moreover, they [16, Theorem 2] proved that the left hand side of (19) holds true for $\alpha\leqslant-1$ and the following reversed version [16, Corollary 4] of the Sándor-Bencze conjectured inequality holds

$$\left(\frac{\sin x}{x}\right)^{\alpha} < \frac{\cos^{\alpha} x+\sqrt{8+\cos^{2\alpha} x}}{4}, \tag{20}$$

where $x\in(0,\pi/2)$ and $\alpha\geqslant 1$. In what follows our aim is to extend these inequalities to Bessel functions, by using part 3 of Theorem 1. We note that if we choose $\nu=-1/2$ in (21), (22) and (23), respectively, then we reobtain the inequalities (18), (19) and (20), respectively.

COROLLARY 1. *Suppose that $x\in(-j_{\nu,1}, j_{\nu,1})$. Then the following assertions are true:*

1. *If $\nu\in(-1,-1/2]$ and $\alpha>0$ or $\nu\geqslant-1/2$ and $\alpha\leqslant-1$, then*

$$[\mathscr{J}_{\nu+1}(x)]^{\alpha} \geqslant \frac{4\,[\mathscr{J}_{\nu}(x)]^{\alpha}}{1+\sqrt{1+8\,[\mathscr{J}_{\nu}(x)]^{2\alpha}}}. \tag{21}$$

2. *If $\nu\in(-1,-1/2]$ and $\alpha>0$, then*

$$[\mathscr{J}_{\nu+1}(x)]^{\alpha} \geqslant \frac{4\,[\mathscr{J}_{\nu}(x)]^{\alpha}}{1+\sqrt{1+8\,[\mathscr{J}_{\nu}(x)]^{2\alpha}}} \geqslant \frac{2\,[\mathscr{J}_{\nu}(x)]^{\alpha}}{1+[\mathscr{J}_{\nu}(x)]^{\alpha}} > \frac{[\mathscr{J}_{\nu}(x)]^{\alpha}}{1+[\mathscr{J}_{\nu}(x)]^{\alpha}}. \tag{22}$$

3. *If $\nu\geqslant-1/2$ and $\alpha\geqslant 1$, then*

$$[\mathscr{J}_{\nu+1}(x)]^{\alpha} \leqslant \frac{[\mathscr{J}_{\nu}(x)]^{\alpha}+\sqrt{[\mathscr{J}_{\nu}(x)]^{2\alpha}+8}}{4}. \tag{23}$$

Proof. **1.** Choosing $\lambda=\mu=1$, $p=2\alpha$ and $q=\alpha$ in part 3 of Theorem 1, we obtain the following generalized Wilker-type inequality

$$[\mathscr{J}_{\nu+1}(x)]^{2\alpha}+\left[\frac{\mathscr{J}_{\nu+1}(x)}{\mathscr{J}_{\nu}(x)}\right]^{\alpha}\geqslant 2,$$

which holds for all $x \in (-j_{\nu,1}, j_{\nu,1})$, $\nu \in (-1,-1/2]$ and $\alpha > 0$ or for all $x \in (-j_{\nu,1}, j_{\nu,1})$, $\nu \geqslant -1/2$ and $\alpha \leqslant -1$. Straightforward computations show that the above inequality is equivalent to the following inequality

$$\left[[\mathscr{J}_{\nu+1}(x)]^{\alpha} + \frac{[\mathscr{J}_{\nu}(x)]^{-\alpha} + \sqrt{[\mathscr{J}_{\nu}(x)]^{-2\alpha} + 8}}{2}\right] \cdot \left[[\mathscr{J}_{\nu+1}(x)]^{\alpha} + \frac{[\mathscr{J}_{\nu}(x)]^{-\alpha} - \sqrt{[\mathscr{J}_{\nu}(x)]^{-2\alpha} + 8}}{2}\right] \geqslant 0,$$

which implies that the inequality

$$[\mathscr{J}_{\nu+1}(x)]^{\alpha} + \frac{[\mathscr{J}_{\nu}(x)]^{-\alpha} - \sqrt{[\mathscr{J}_{\nu}(x)]^{-2\alpha} + 8}}{2} \geqslant 0$$

holds, since the for each $\nu > -1$ and $x \in (-j_{\nu,1}, j_{\nu,1})$ we have $\mathscr{J}_{\nu}(x) > 0$, as we mentioned above in the proof of Theorem 1. Now, from the above inequality we deduce the inequality (21), i.e.

$$[\mathscr{J}_{\nu+1}(x)]^{\alpha} \geqslant \frac{\sqrt{[\mathscr{J}_{\nu}(x)]^{-2\alpha} + 8} - [\mathscr{J}_{\nu}(x)]^{-\alpha}}{2} = \frac{4\,[\mathscr{J}_{\nu}(x)]^{\alpha}}{1 + \sqrt{1 + 8\,[\mathscr{J}_{\nu}(x)]^{2\alpha}}}.$$

2.&3. Finally, the inequality (22) follows from (21) by using the fact that $\mathscr{J}_{\nu}$ maps $(-j_{\nu,1}, j_{\nu,1})$ into $(0,1]$, while (23) follows also from (21) by replacing α with $-\alpha$. With this the proof is complete. □

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