Journal of
Mathematical
Inequalities
Volume 2, Number 3 (2008), 407–427

IMPLICIT DIFFERENCE FUNCTIONAL INEQUALITIES AND APPLICATIONS

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(*communicated by B. Pachpatte*)

Abstract. We give a theorem on implicit difference functional inequalities of the Volterra type for functions of several variables. We apply this general result in the investigation of the stability of implicit difference functional equations with initial boundary conditions.

Classical solutions of parabolic functional differential equations are approximated in the paper by solutions of suitable implicit difference schemes. The proofs of the convergence of difference methods are based on comparison technique and results on difference functional inequalities are used. Numerical examples are presented.

1. Introduction

Differential inequalities found applications in several topics concerning differential or functional differential equations. Such problems as: estimates of solutions of ordinary or partial differential or functional differential equations, estimates of the domain of the existence of classical or generalized solutions, criteria of uniqueness and continuous dependence, are classical examples, however not the only ones. Moreover discrete versions of differential inequalities, the so called difference inequalities, are frequently used to prove the convergence of numerical methods.

Explicit difference schemes for evolution functional differential equations consist in replacing partial derivatives with difference operators. Moreover, because differential equations contain functional variables, some interpolating operators are needed. This leads to difference functional problems which satisfy consistency conditions on sufficiently regular solutions of original equations. The main task in these investigations is to find functional difference problems which are stable. Methods of difference inequalities are used in the investigation of the stability of nonlinear difference functional equations generated by initial or initial boundary value problems for functional differential equations see [4] Chapter 3 and [5], [9], [10] [18]. Explicit difference inequalities and explicit difference schemes are investigated in these papers.

The aim of the paper is to show a theorem on implicit difference inequalities corresponding to nonlinear parabolic functional differential problems. We give also

Mathematics subject classification (2000): 35R10, 65M12.

Key words and phrases: Difference inequalities, initial boundary value problems, stability and convergence, implicit difference methods.

Paper JMI-02-36

applications of a result on implicit difference inequalities. More precisely, we propose implicit difference schemes for the numerical solving of functional differential equations. We give a complete convergence analysis for the methods and we show by examples that new difference schemes are considerable better than classical methods.

Results presented in the paper are new also in the case of differential equations without the functional dependence.

We formulate our functional differential problems. For any metric spaces X and Y we denote by $C(X,Y)$ the class of all continuous functions from X into Y. We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Write

$$E = [0,a] \times (-b,\, b), \quad D = [-d_0, 0] \times [-d, d],$$

where $a > 0$, $b = (b_1, \ldots, b_n) \in \mathbf{R}^n$, $b_i > 0$ for $1 \leqslant i \leqslant n$ and $d_0 \in \mathbf{R}_+$, $R_+ = [0, +\infty)$. Let $c = b + d$ and

$$E_0 = [-d_0, 0] \times [-c,\, c], \quad \partial_0 E = [0,a] \times \big([-c,\, c] \setminus (-b,\, b) \big), \quad \Omega = E \cup E_0 \cup \partial_0 E.$$

For a function $z : \Omega \to \mathbf{R}^k$, $z = (z_1, \ldots, z_k)$, and for a point $(t,x) \in \bar{E}$ where $\bar{E}$ is the closure of E, we define a function $z_{(t,x)} : D \to \mathbf{R}^k$ by $z_{(t,x)}(\tau, y) = z(t+\tau, x+y)$, $(\tau, y) \in D$. Then $z_{(t,x)}$ is the restriction of z to the set $[t-d_0, t] \times [x-d, x+d]$ and this restriction is shifted to the set D.

Let us denote by $M_{n\times n}$ the class of all $n \times n$ matrices with real elements. Write $\Xi = E \times C(D, \mathbf{R}^k) \times \mathbf{R}^n \times M_{n\times n}$ and suppose that $F = (F_1, \ldots, F_k) : \Xi \to \mathbf{R}^k$ and $\varphi : E_0 \cup \partial_0 E \to \mathbf{R}^k$ are given functions. We consider the system of functional differential equations

$$\partial_t z_i(t,x) = F_i\big(t, x, z_{(t,x)}, \partial_x z_i(t,x), \partial_{xx} z_i(t,x)\big), \; i = 1, \ldots, k, \tag{1}$$

with the initial boundary condition

$$z(t,x) = \varphi(t,x) \;\text{ on }\; E_0 \cup \partial_0 E, \tag{2}$$

where

$$\partial_x z_i = \big(\partial_{x_1} z_i, \ldots, \partial_{x_n} z_i\big), \quad \partial_{xx} z_i = \big[\, \partial_{x_\mu x_\nu} z_i \,\big]_{\mu,\nu=1,\ldots,n}, \; i = 1, \ldots, k.$$

We consider classical solutions of (1), (2). We give examples of equations which can be obtained from (1) by specializing the operator F.

EXAMPLE 1.1. *Suppose that the function $\alpha : E \to \mathbf{R}^{1+n}$ satisfies the condition: $\alpha(t,x) - (t,x) \in D$ for $(t,x) \in E$. For a given $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_k) : E \times \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^n \times M_{n\times n} \to \mathbf{R}^k$, we put*

$$F(t,x,w,q,s) = \tilde{F}(t,x,w(0,\theta), w(\alpha(t,x) - (t,x)), q, s) \;\text{ on }\; \Sigma,$$

where $\theta = (0, \ldots, 0) \in \mathbf{R}^n$, $w \in C(D, \mathbf{R}^k)$, $q \in \mathbf{R}^n$, $s \in M_{n\times n}$. Then (1) reduces to the system of differential equations with deviated variables

$$\partial_t z_i(t,x) = \tilde{F}_i(\, t, x, z(t,x), z(\alpha(t,x)), \partial_x z_i(t,x), \partial_{xx} z_i(t,x)\,), \quad i = 1, \ldots, k.$$

EXAMPLE 1.2. *For the above* $\tilde{F}$ *we define*

$$F(t,x,w,q,s) = \tilde{F}(t,x,w(0,\theta), \int_D w(\tau,y)\,dy\,d\tau, q, s) \quad on \quad \Sigma.$$

Then (1) is equivalent to the system of differential integral equations

$$\partial_t z_i(t,x) = \tilde{F}_i(t,x,z(t,x), \int_D z(t+\tau, x+y)\,dy\,d\tau, \partial_x z_i(t,x), \partial_{xx} z_i(t,x)\,),\ i=1,\dots,k.$$

It is clear that more complicated differential systems with deviated variables and differential integral systems can be obtained from (1) by a suitable definition of F. Sufficient conditions for the existence and uniqueness of classical or generalized solutions of parabolic functional differential problems can be found in [1], [2], [3], [6], [13], [14]. Functional differential inequalities and applications were studied in [11], [12], [15] - [17].

Our motivations for investigations of implicit difference functional inequalities and for the construction of implicit difference schemes are the following. Two types of assumptions are needed in theorems on the stability of difference functional equations generated by (1), (2). The first type conditions concern regularity of F. It is assumed that

(i) the function F of the variables (t,x,w,q,s), $q=(q_1,\dots,q_n)$, $s=[s_{\mu\nu}]_{\mu,\nu=1,\dots,n}$, is of class C^1 with respect to (q,s) and the functions

$$\partial_q F_i = (\partial_{q_1} F_i, \dots, \partial_{q_n} F_i),\ \partial_s F_i = \left[\,\partial_{s_{\mu\nu}} F\,\right]_{\mu,\nu=1,\dots,n},\ 1 \leqslant i \leqslant k,$$

are bounded,

(ii) F satisfies the Perron type estimates with respect to the functional variable w.

The second type conditions concern the mesh. The following condition is needed in the analysis of the convergence of explicit difference schemes for (1), (2):

$$1 - 2h_0 \sum_{\mu=1}^{n} \frac{1}{h_\mu^2}\,\partial_{s_{\mu\mu}} F_i(P) + h_0 \sum_{\mu,\nu=1\,\mu\neq\nu}^{n} \frac{1}{h_\mu h_\nu}\left|\partial_{s_{\mu\nu}} F_i(P)\right| \geqslant 0,\ P \in \Xi,\ \ i=1,\dots,k, \tag{3}$$

see [5] and [9], [10]. It is clear that strong assumptions on relations between h_0 and $h' = (h_1,\dots,h_n)$ are required in (3). It is important in our considerations that assumption (3) is omitted in a theorem on difference functional inequalities and in a theorem on the convergence of implicit difference methods for (1), (2).

It important in the paper that we use nonlinear estimates for F with respect to the functional variable and ordinary differential functional equations are used as comparison problems.

The paper is organized as follows. In Section 2 we prove a general theorem on implicit difference functional inequalities with unknown function of several variables. In the next section we prove a theorem on the existence and uniqueness of a solution of implicit difference equation with an initial boundary condition. We establish also some estimates for the difference between exact and approximate solutions to difference functional problems. They are used in the investigation of the stability of difference schemes generated by (1), (2).

The second part of the paper deals with applications of the above general results. We propose implicit difference schemes for the numerical solving of evolution functional differential equations. Convergence results and error estimates are presented in Sections 3 and 4. Theorems on difference inequalities are used in the investigation of the stability of implicit difference methods. Numerical examples are given in the last part of the paper.

2. Functional difference inequalities

For any two sets U and W we denote by $\mathbf{F}(U,W)$ the class of all functions defined on U and taking values in W. If $A \subset U$ and $\alpha \in \mathbf{F}(U,W)$ then $\alpha\,|_A$ is the restriction of α to the set A. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers respectively. For $x = (x_1,\ldots,x_n) \in \mathbf{R}^n$, $p = (p_1,\ldots,p_k) \in \mathbf{R}^k$ we put $\|x\| = |x_1| + \ldots + |x_n|$ and $\|p\|_\infty = \max\{\,|p_i| : 1 \leqslant i \leqslant k\,\}$. We define a mesh on Ω in the following way. Suppose that (h_0,h'), $h' = (h_1,\ldots,h_n)$, stand for steps of the mesh. For $(r,m) \in \mathbf{Z}^{1+n}$ where $m = (m_1,\ldots m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \quad x^{(m)} = (x_1^{(m_1)},\ldots,x_n^{(m_n)}) = (m_1h_1,\ldots,m_nh_n).$$

Let us denote by H the set of all $h = (h_0,h')$ such that there are $K_0 \in \mathbf{Z}$ and $K = (K_1,\ldots,K_n) \in \mathbf{Z}^n$ satisfying the conditions: $K_0h_0 = d_0$ and $(K_1h_1,\ldots,K_nh_n) = d$. Set

$$\mathbf{R}_h^{1+n} = \{\,(t^{(r)},x^{(m)}) : \quad (r,m) \in \mathbf{Z}^{1+n}\,\}$$

and

$$D_h = D \cap \mathbf{R}_h^{1+n}, \quad E_h = E \cap \mathbf{R}_h^{1+n}, \quad E_{0.h} = E_0 \cap \mathbf{R}_h^{1+n},$$

$$\partial_0 E_h = \partial_0 E \cap \mathbf{R}_h^{1+n}, \quad \Omega_h = E_h \cup E_{0.h} \cup \partial_0 E_h.$$

Let $N_0 \in \mathbf{N}$ be defined by the relations: $N_0h_0 \leqslant a < (N_0+1)h_0$ and

$$E_h' = \{\,(t^{(r)},x^{(m)}) \in E_h : \quad 0 \leqslant r \leqslant N_0 - 1\,\}.$$

For functions $w : D_h \to \mathbf{R}^k$ and $z : \Omega_h \to \mathbf{R}^k$ we write $w^{(r,m)} = w(t^{(r)},x^{(m)})$ on D_h and $z^{(r,m)} = z(t^{(r)},x^{(m)})$ on Ω_h. We need a discrete version of the operator $(t,x) \to z_{(t,x)}$. For a function $z : \Omega_h \to \mathbf{R}^k$ and for a point $(t^{(r)},x^{(m)}) \in E_h$ we define a function $z_{[r,m]} : D_h \to \mathbf{R}^k$ by $z_{[r,m]}(\tau,y) = z(t^{(r)}+\tau, x^{(m)}+y)$, $(\tau,y) \in D_h$. Write $\chi = 1 + 2n^2$ and

$$\Lambda = \{\,\lambda = (\lambda_1,\ldots,\lambda_n) : \lambda_i \in \{-1,0,1\} \text{ for } 1 \leqslant i \leqslant n \text{ and } \|\lambda\| \leqslant 2\,\},$$

$$\Lambda' = \Lambda \setminus \{\,\theta\,\}$$

Note that χ is the number of elements of Λ. Let $\psi : \Lambda \to \{\,1,2,\ldots,\chi\,\}$ be a function such that $\psi(\lambda) \neq \psi(\tilde\lambda)$ for $\lambda \neq \tilde\lambda$. We assume that $\prec$ is an order in Λ defined in the following way: $\lambda \prec \tilde\lambda$ if $\psi(\lambda) < \psi(\tilde\lambda)$.

Elements of the space $\mathbf{R}^{\chi}$ will be denoted by $\xi = \{\,\xi^{(\lambda)}\,\}_{\lambda\in\Lambda}$. Write $Y_h = E'_h \times \mathbf{F}(D_h,\mathbf{R}^k)\times\mathbf{R}^{\chi}$ and suppose that the function $G_h = (G_{h.1},\dots,G_{h.k}) : Y_h \to \mathbf{R}^k$ of the variables (t,x,w,ξ) is given. For $(t^{(r)},x^{(m)},w,\xi)\in Y_h$ we write $G_{h.i}[w,\xi]^{(r,m)} = G_{h.i}(t^{(r)},x^{(m)},w,\xi)$, $i = 1,\dots,k$. For a function $\omega : \Omega_h \to \mathbf{R}$ and for a point $(t^{(r)},x^{(m)})\in E_h$ we put $\omega_{\langle\, r,m\rangle} = \{\,\omega^{(r,m+\lambda)}\,\}_{\lambda\in\Lambda}$ and

$$\delta_0\omega^{(r,m)} = \frac{1}{h_0}\left[\,\omega^{(r+1,m)} - \omega^{(r,m)}\,\right]. \tag{4}$$

Given $\varphi_h \in \mathbf{F}(E_{0.h}\cup\partial_0E_h,\mathbf{R}^k)$, we consider the system of functional difference equations

$$\delta_0 z_i^{(r,m)} = G_{h.i}[z_{[r,m]},(z_i)_{\langle\, r+1,m\rangle}\,]^{(r,m)},\ i = 1,\dots,k, \tag{5}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h}\cup\partial_0E_h. \tag{6}$$

Note that the numbers $z_i^{(r+1,m+\lambda)}$ where $\lambda\in\Lambda$ appear in $(z_i)_{\langle\, r+1,m\rangle}$. Then (5), (6) is an implicit difference functional problem.

REMARK 2.1. Let us denote by $\delta = (\delta_1,\dots,\delta_n)$ and $\delta^{(2)} = \left[\,\delta_{\mu\nu}\,\right]_{\mu,\nu=1,\dots,n}$ difference operators corresponding to the derivatives $\partial_x = (\partial_{x_1},\dots,\partial_{x_n})$ and $\partial_{xx} = \left[\,\partial_{x_\mu x_\nu}\,\right]_{\mu,\nu=1,\dots,n}$. Then implicit difference scheme corresponding to (1) has the form

$$\delta_0 z_i^{(r,m)} = F_i\big(t^{(r)},x^{(m)},T_hz_{[r,m]},\delta z_i^{(r+1,m)},\delta^{(2)}z_i^{(r+1,m)}\big),\ i = 1,\dots,k, \tag{7}$$

where $T_h : \mathbf{F}(D_h,\mathbf{R}^k)\to C(D,\mathbf{R}^k)$ is an interpolating operator. It is clear that system (7) is a particular case of (5).

We prove a theorem on difference inequalities generated by (5), (6). For $w,\bar{w}\in \mathbf{F}(D_h,\mathbf{R}^k)$ we write $w\leqslant\bar{w}$ if $w^{(r,m)}\leqslant\bar{w}^{(r,m)}$ where $(t^{(r)},x^{(m)})\in D_h$ and $\|w\|_{D_h} = \max\{\|w(t^{(r)},x^{(m)})\|_\infty :\ (t^{(r)},x^{(m)})\in D_h\}$.

ASSUMPTION $H[G_h]$. The function $G_h : Y_h\to\mathbf{R}^k$ satisfies the conditions:

1) there exist the partial derivatives

$$\left\{\,\partial_{\xi^{(\lambda)}}G_{h.i}[w,\xi]^{(r,m)}\,\right\}_{\lambda\in\Lambda},\ i = 1,\dots,k,$$

and $\partial_{\xi^{(\lambda)}}G_{h.i}[w,\,\cdot\,]^{(r,m)}\in C(\mathbf{R}^{\chi},R)$ for $\lambda\in\Lambda$, $(t^{(r)},x^{(m)},w)\in E'_h\times\mathbf{F}(D_h,\mathbf{R}^k)$, $i = 1,\dots,k$,

2) for each $(t^{(r)},x^{(m)},w)\in E'_h\times\in\mathbf{F}(D_h,\mathbf{R}^k)$ and $\lambda\in\Lambda$, $1\leqslant i\leqslant k$ the function $\partial_{\xi^{(\lambda)}}G_{h.i}[w,\,\cdot\,]^{(r,m)}$ is bounded on $\mathbf{R}^{\chi}$,

3) the conditions

$$\partial_{\xi^{(\lambda)}}G_{h.i}[w,\xi]^{(r,m)}\geqslant 0 \ \text{ for } \ \lambda\in\Lambda', \tag{8}$$

$$\sum_{\lambda\in\Lambda}\partial_{\xi^{(\lambda)}}G_{h.i}[w,\xi]^{(r,m)} = 0 \tag{9}$$

are satisfied on Y_h for $i = 1,\dots,k$,

4) the following monotonicity condition holds: if $w,\ \bar{w} \in \mathbf{F}(D_h, \mathbf{R}^k)$, $w = (w_1, \ldots, w_k)$, $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_k)$, and $w \leqslant \bar{w}$ then

$$w_i^{(0,\theta)} + h_0 G_{h.i}[w, \xi]^{(r,m)} \leqslant \bar{w}_i^{(0,\theta)} + h_0 G_{h.i}[\bar{w}, \xi]^{(r,m)} \text{ on } E'_h \times \mathbf{R}^k \text{ for } i = 1, \ldots, k.$$

REMARK 2.2. Suppose that $\tilde{G}_h : E'_h \times \mathbf{R} \times \mathbf{F}(D_h, \mathbf{R}^k) \times \mathbf{R}^\chi \to \mathbf{R}^k$, $\tilde{G}_h = (\tilde{G}_{h.1}, \ldots, \tilde{G}_{h.k})$, is a given function of the variables (t, x, p, w, ξ) and G_h is defined by

$$G_{h.i}(t, x, w, \xi) = \tilde{G}_{h.i}(t, x, w_i(0, \theta), w, \xi) \text{ on } Y_h, \quad i = 1, \ldots, k.$$

Then system (5) has the form

$$\delta_0 z_i^{(r,m)} = \tilde{G}_{h.i}(t^{(r)}, x^{(m)}, z_i^{(r,m)}, z_{[r,m]}, (z_i)_{\langle r,m \rangle}), \ i = 1, \ldots, k.$$

The dependence of the right hand of (5) on the classical variable $z^{(r,m)}$ is distinguished in the above system. Suppose that

1) $\tilde{G}_h$ is nondecreasing with respect to the functional variable,
2) there exist the derivatives $\left(\partial_p \tilde{G}_{h.1}, \ldots, \partial_p \tilde{G}_{h.k}\right) = \partial_p \tilde{G}_h$ and there is $\tilde{L} \in \mathbf{R}_+$ such that

$$\partial_p \tilde{G}_{h.i}(t, x, p, w, \xi) \geqslant \tilde{L}, \quad 1 \leqslant i \leqslant k,$$

and $1 + \tilde{L} h_0 \geqslant 0$.

Then the above G_h satisfies the monotonicity condition 4) from Assumption $H[G_h]$.

THEOREM 2.3. *Suppose that Assumption $H[G_h]$ is satisfied and*

1) $h \in H$ and the functions $u,\ v : \Omega_h \to \mathbf{R}^k$, $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_k)$, satisfy on E'_h the difference functional inequalities

$$\delta_0 u_i^{(r,m)} - G_{h.i}[u_{[r,m]}, (u_i)_{\langle r+1,m \rangle}]^{(r,m)} \leqslant \delta_0 v_i^{(r,m)} - G_{h.i}[v_{[r,m]}, (v_i)_{\langle r+1,m \rangle}]^{(r,m)} \quad (10)$$

where $i = 1, \ldots, k$,

2) the initial boundary estimate $u^{(r,m)} \leqslant v^{(r,m)}$ holds on $E_{0.h} \cup \partial_0 E_h$.

Then

$$u^{(r,m)} \leqslant v^{(r,m)} \text{ on } E_h. \quad (11)$$

Proof. We prove (11) by induction on r. It follows from assumption 2) that estimate (11) is satisfied for $r = 0$ and $(t^{(0)}, x^{(m)}) \in E_h$. Assume that $u^{(j,m)} \leqslant v^{(j,m)}$ for $(t^{(j)}, x^{(m)}) \in E_h \cap \left([0, t^{(r)}] \times \mathbf{R}^n\right)$. We prove that $u^{(r+1,m)} \leqslant v^{(r+1,m)}$ for $(t^{(r+1)}, x^{(m)}) \in E_h$. Write

$$U_i^{(r,m)} = u_i^{(r,m)} + h_0 G_{h.i}[u_{[r,m]}, (u_i)_{\langle r+1,m \rangle}]^{(r,m)} - v_i^{(r,m)} - h_0 G_{h.i}[v_{[r,m]}, (u_i)_{\langle r+1,m \rangle}]^{(r,m)}$$

where $i = 1, \ldots, k$. It follows from (10) that

$$(u_i - v_i)^{(r+1,m)} \leqslant U_i^{(r,m)} + h_0 \Big[G_{h.i}[v_{[r,m]}, (u_i)_{\langle r+1,m \rangle}]^{(r,m)} - G_{h.i}[v_{[r,m]}, (v_i)_{\langle r+1,m \rangle}]^{(r,m)}$$

where $i = 1, \ldots, k$. The monotonicity condition 4) of Assumption $H[G_h]$ implies the inequalities $U_i^{(r,m)} \leqslant 0$ for $(t^{(r)}, x^{(m)}) \in E'_h$, $i = 1, \ldots, k$. Then there exist intermediate points $Q_i^{(r+1,m)} \in \mathbf{R}^\chi$ such that

$$(u_i - v_i)^{(r+1,m)} \Big[1 - h_0 \partial_{\xi(\theta)} G_{h.i}[v_{[r,m]}, Q_i^{(r+1,m)}]^{(r,m)} \Big] \leqslant h_0 \sum_{\lambda \in \Lambda'} \partial_{\xi(\lambda)} G_{h.i}\big[v_{[r,m]}, Q_i^{(r+1,m)}\big]^{(r,m)} \big(u_i - v_i\big)^{(r+1,m+\lambda)}, \ i = 1, \ldots, k. \tag{12}$$

We define $\tilde{m} \in \mathbf{Z}^n$ and $j \in \mathbf{N}$, $1 \leqslant j \leqslant k$, as follows

$$(u_j - v_j)^{(r+1,\tilde{m})} = \max_{1 \leqslant i \leqslant k} \max\{ (u_i - v_i)^{(r+1,m)} : (t^{(r+1)}, x^{(m)}) \in \Omega_h \}.$$

If $(t^{(r+1)}, x^{(\tilde{m})}) \in \partial_0 E_h$ then assumption 2) implies that $(u_j - v_j)^{(r+1,\tilde{m})} \leqslant 0$. Let us consider the case when $(t^{(r+1)}, x^{(\tilde{m})}) \in E_h$. Then we have from (12) that

$$(u_j - v_j)^{(r+1,\tilde{m})} \Big[1 - h_0 \partial_{\xi(\theta))} G_{h.j}[v_{[r,\tilde{m}]}, Q_j^{(r+1,\tilde{m})}]^{(r,\tilde{m})} \Big] \leqslant (u_j - v_j)^{(r+1,\tilde{m})} \sum_{\lambda \in \Lambda'} \partial_{\xi(\lambda)} G_{h.j}[v_{[r,\tilde{m}]}, Q_j^{(r+1,\tilde{m})}]^{(r,\tilde{m})}.$$

It follows from (8), (9) that $(u_j - v_j)^{(r+1,\tilde{m})} \leqslant 0$. Then the proof of (11) is completed by induction. □

3. Approximate solutions of difference functional equations

We define $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ by the relations: $(N_1 h_1, \ldots, N_n h_n) < (b_1, \ldots, b_n) \leqslant ((N_1 + 1)h_1, \ldots, (N_n + 1)h_n)$ and we assume that $(N_i + 1)h_i = b_i$ if $d_i = 0$. We first prove a theorem on the existence and uniqueness of solutions to (5), (6).

THEOREM 3.1. *If conditions 1) - 3) of Assumption $G[G_h]$ are satisfied and $\varphi_h \in \mathbf{F}(E_{0.h} \cup \partial_0 E_h, \mathbf{R}^k)$ then there exists exactly one solution $u_h = (u_{h.1}, \ldots, u_{h.k}) : \Omega_h \to \mathbf{R}^k$ of (5), (6).*

Proof. Suppose that $0 \leqslant r \leqslant N_0 - 1$ is fixed and that the solution u_h of problem (5), (6) is given on the set $\Omega_h \cap \big([-d_0, t^{(r)}] \times \mathbf{R}^n \big)$. We prove that the vectors $u_h^{(r+1,m)}$, $-N \leqslant m \leqslant N$, exists and that they are unique. It is sufficient to show that there exists exactly one solution of the system of equations

$$\frac{1}{h_0}\big(z_i^{(r+1,m)} - u_{h.i}^{(r,m)}\big) = G_{h.i}[(u_h)_{[r,m]}, (z_i)_{\langle r+1,m \rangle}]^{(r,m)}, \ \ -N \leqslant m \leqslant N, \ \ i = 1, \ldots, k, \tag{13}$$

with the initial boundary condition (6). There exists $Q_h > 0$ such that

$$Q_h \geqslant -h_0 \partial_{\xi(\theta)} G_{h.i}[(u_h)_{[r,m]}, \xi]^{(r,m)}, \ \xi \in \mathbf{R}^\chi, \ -N \leqslant m \leqslant N, \ \lambda \in \Lambda, i = 1, \ldots, k.$$

It is clear that system (13) is equivalent to the following one

$$z_i^{(r+1,m)} = \frac{1}{Q_h+1}\Big[\, Q_h z_i^{(r+1,m)} + u_{h.i}^{(r,m)} + h_0 G_{h.i}[(u_h)_{[r,m]}, (z_i)_{\langle\, r+1,m\rangle}\,]\Big]^{(r,m)} \tag{14}$$

where $-N \leqslant m \leqslant N,\ i = 1,\dots,k.$ Write $S_h = \{x^{(m)} :\ x^{(m)} \in [-c,c]\,\}$. Elements of the space $\mathbf{F}(S_h,\mathbf{R}^k)$ are denoted by $\zeta, \bar{\zeta}$. For $\zeta : S_h \to \mathbf{R}^k$, $\zeta = (\zeta_1,\dots,\zeta_k)$, we write $\zeta^{(m)} = \zeta(x^{(m)})$ and

$$(\zeta_i)_{\langle\, m\rangle} = \{\ \zeta_i^{(m+\lambda)}\ \}_{\lambda\in\Lambda},\ i = 1,\dots,k.$$

The norm in the space $\mathbf{F}(S_h,\mathbf{R}^k)$ is defined by $\|\zeta\|_\star = \max\{\,\|\zeta^{(m)}\|_\infty :\ \ x^{(m)} \in S_h\,\}$. Let us consider the set

$$X_h = \{\ \zeta \in \mathbf{F}(S_h,\mathbf{R}^k) :\ \zeta^{(m)} = \varphi^{(r+1,m)} \text{ for } x^{(m)} \in [-c,c]\setminus(-b,b)\,\}.$$

We apply the operator $W_h : X_h \to X_h$, $W_h = (W_{h.1},\dots,W_{h.k})$, defined by

$$W_{h.i}[\zeta]^{(m)} = \frac{1}{Q_h+1}\Big[\, Q_h\zeta_i^{(m)} + u_{h.i}^{(r,m)} + h_0 G_{h.i}[(u_h)_{[r,m]}, (\zeta_i)_{\langle\, m\rangle}\,]^{(r,m)}\Big], \tag{15}$$

where $-N \leqslant m \leqslant N,\ i = 1,\dots,k,$ and

$$W_h[\zeta]^{(m)} = \varphi_h^{(r+1,m)} \ \text{ for } \ x^{(m)} \in [-c,c]\setminus(-b,b) \tag{16}$$

where $\zeta = (\zeta_1,\dots,\zeta_k) \in \mathbf{F}(S_h,\mathbf{R}^k)$. We prove that

$$\|W_h[\zeta] - W_h[\bar{\zeta}]\|_\star \leqslant \frac{Q_h}{Q_h+1}\|\zeta - \zeta\|_\star \ \text{ on } \ \mathbf{F}(S_h,\mathbf{R}^k). \tag{17}$$

It follows from (15) that we have for $-N \leqslant m \leqslant N$:

$$\begin{aligned} W_{h.i}[\zeta]^{(m)} - W_{h.i}[\bar{\zeta}]^{(m)} &\\ = \frac{1}{Q_h+1}\Big\{& Q_h(\zeta_i - \bar{\zeta}_i)^{(m)} \\ &+ h_0\sum_{\lambda\in\Lambda} \partial_{\xi(\lambda)} G_{h.i}\big[(u_h)_{[r,m]}, P_i^{(r,m)}\big]^{(r,m)}\,\big(\zeta_i - \bar{\zeta}_i\big)^{(m+\lambda)}\Big\} \\ = \frac{1}{Q_h+1}\Big\{&\Big[\, Q_h + h_0\partial_{\xi(\theta)} G_{h.i}\big[(u_h)_{[r,m]}, P_i^{(r,m)}\big]^{(r,m)}\Big]\,\big(\zeta_i - \bar{\zeta}_i\big)^{(m)} \\ &+ h_0\sum_{\lambda\in\Lambda'} \partial_{\xi(\lambda)} G_{h.i}\big[(u_h)_{[r,m]}, P_i^{(r,m)}\,\big]^{(r,m)}\,\big(\zeta_i - \bar{\zeta}_i\big)^{(m+\lambda)}\Big\},\ i = 1,\dots,k, \end{aligned}$$

where $P_i^{(r,m)} \in \mathbf{R}^k$ are intermediate points. It follows from the above relations and from (8), (9) that

$$\big|W_{h.i}[\zeta]^{(m)} - W_{h.i}[\bar{\zeta}]^{(m)}\big| \leqslant \frac{Q_h}{Q_h+1}\,\|\zeta - \bar{\zeta}\|_\star \text{ for } -N \leqslant m \leqslant N,\ i = 1,\dots,k.$$

According to (16) we have

$$W_h[\zeta]^{(m)} - W_h[\bar{\zeta}]^{(m)} = 0 \ \text{ for } \ x^{(m)} \in [-c,c]\setminus(-b,b).$$

This completes the proof of (17). □

It follows from the Banach fixed point theorem that there exists exactly one solution $\tilde{\zeta} : S_h \to \mathbf{R}^k$ of the equation $\zeta = W_h[\zeta]$ and consequently, there exists exactly one solution of (6), (14). Then the vectors $u_h^{(r+1,m)}$, $-N \leqslant m \leqslant N$, exist and they are unique. Then the proof is completed by induction with respect to r, $0 \leqslant r \leqslant N_0$.

Suppose that the functions $v_h : \Omega_h \to \mathbf{R}^k$, $v_h = (v_{h.1}, \ldots, v_{h.k})$, and α_0, $\gamma : H \to \mathbf{R}_+$ satisfy the conditions:

$$\left| \, \delta_0 v_{h.i}^{(r,m)} - G_{h.i}[(v_h)_{[r,m]}, (v_{h.i})_{\langle \, r+1,m \rangle} \,]^{(r,m)} \, \right| \leqslant \gamma(h) \text{ on } E'_h, \quad i = 1, \ldots, k, \tag{18}$$

$$\left\| \varphi_h^{(r,m)} - v_h^{(r,m)} \right\|_\infty \leqslant \alpha_0(h) \;\; \text{on} \;\; E_{0.h} \cup \partial_0 E_h \tag{19}$$

and

$$\lim_{h \to 0} \alpha_0(h) = 0, \quad \lim_{h \to 0} \gamma(h) = 0. \tag{20}$$

The function v_h satisfying the above relations is considered as an approximate solution of (5), (6).

We give a theorem on the estimate of the difference between the exact and approximate solutions of (5), (6). Write $I = [-d_0, 0]$, $J = [0, a]$ and

$$I_h = \{ \, t^{(r)} : \; -K_0 \leqslant r \leqslant 0 \, \}, \;\; J_h = \{ \, t^{(r)} : \; 0 \leqslant r \leqslant N_0 \, \}, \; J'_h = J_h \setminus \{ \, t^{(N_0)} \, \}.$$

For $\eta : I_h \cup J_h \to \mathbf{R}$ we write $\eta^{(r)} = \eta(t^{(r)})$. We will need the following operator $V_h : \mathbf{F}(D_h, \mathbf{R}^k) \to \mathbf{F}(I_h, \mathbf{R}_+)$. Put

$$V_h[w]^{(r)} = V_h[w](t^{(r)}) = \max \{ \, \|w^{(r,m)}\|_\infty : \; -K \leqslant m \leqslant K \, \}, \quad -K_0 \leqslant r \leqslant 0,$$

where $w \in \mathbf{F}(D_h, \mathbf{R}^k)$. For a function $\eta : I \cup J \to \mathbf{R}$ and for a point $t \in I$ we denote by $\eta_t : I \to \mathbf{R}$ the function defined by $\eta_t(\tau) = \eta(t + \tau)$, $\tau \in I$. The maximum norm in the space $C(I, \mathbf{R})$ we denote by $\| \cdot \|_I$. We will need a discrete version of the operator $t \to \eta_t$. For a function $\eta : I_h \cup J_h \to \mathbf{R}$ and for a point $t^{(r)} \in J_h$ we define a function $\eta_{[r]} : I_h \to \mathbf{R}$ by $\eta_{[r]}(\tau) = \eta(t^{(r)} + \tau)$, $\tau \in I_h$. Let $T_{h_0} : \mathbf{F}(I_h, \mathbf{R}) \to C(I, \mathbf{R}$ denote the interpolating operator defined by

$$T_{h_0}[\eta](t) = \frac{t - t^{(r)}}{h_0} \eta^{(r+1)} + \big(1 - \frac{t - t^{(r)}}{h_0} \big) \eta^{(r)} \text{ for } t^{(r)} \leqslant t \leqslant t^{(r+1)},$$

where $\eta \in \mathbf{F}(I_h, \mathbf{R})$. We will use the operator $U_h : \mathbf{F}(D_h, \mathbf{R}^k) \to \mathbf{C}(I, \mathbf{R}_+)$ given by

$$U_h[w](t) = T_{h_0} \big[\, V_h[w] \, \big](t), \quad t \in I,$$

where $w \in \mathbf{F}(D_h, \mathbf{R}^k)$. We formulate assumptions on comparison functions corresponding to (5), (6).

ASSUMPTION $H[\sigma]$. The function $\sigma : J \times C(I, \mathbf{R}_+) \to \mathbf{R}_+$ satisfies the conditions:

1) σ is continuous and it is nondecreasing with respect to the both variables,
2) $\sigma(t, \mathbf{0}) = 0$ for $t \in I$ where $\mathbf{0} \in C(I, \mathbf{R}_+)$ is given by $\mathbf{0}(\tau) = 0$ for $\tau \in I$,

3) the maximal solution of the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t), \quad \eta(t) = 0 \text{ for } t \in I,$$

is $\tilde{\eta}(t) = 0$ for $t \in I \cup J$.

THEOREM 3.2. *Supose that Assumption $H[G_h]$ is satisfied and*

1) $h \in H$ and $u_h : \Omega_h \to \mathbf{R}^k$ is a solution of (5), (6) where $\varphi_h \in \mathbf{F}(E_{0.h} \cup \partial_0 E_h, \mathbf{R}^k)$,

2) $v_h : \Omega_h \to \mathbf{R}^k$ and there are $\alpha_0, \gamma : H \to \mathbf{R}_+$ such that conditions (18) - (20) are satisfied,

3) there exists $\sigma : J \times C(I, \mathbf{R}_+) \to \mathbf{R}$ such that Assumption $H[\sigma]$ is satisfied and

$$G_{h.i}[w, \xi]^{(r,m)} - G_{h.i}[\bar{w}, \xi]^{(r,m)} \leqslant \sigma(t^{(r)}, U_h[w - \bar{w}]), \; i = 1, \ldots, k, \tag{21}$$

where $(t^{(r)}, x^{(m)}, \xi) \in E'_h \times \mathbf{R}^\chi$, $w, \bar{w} \in \mathbf{F}(D_h, \mathbf{R}^k)$ and $w \geqslant \bar{w}$.

Then

$$\| (u_h - v_h)^{(r,m)} \|_\infty \leqslant \beta_h^{(r)} \;\; on \;\; E_h \tag{22}$$

where $\beta_h : I_h \cup J_h \to \mathbf{R}_+$ is a solution of the difference problem

$$\beta^{(r+1)} = \beta^{(r)} + h_0 \sigma(t^{(r)}, T_{h_0}[\beta_{[r]}]) + h_0 \gamma(h), \;\; 0 \leqslant r \leqslant N_0 - 1, \tag{23}$$

$$\beta^{(r)} = \alpha_0(h) \;\; for \;\; -K_0 \leqslant r \leqslant 0, \tag{24}$$

and there is $\alpha : H \to \mathbf{R}_+$ such that

$$\|(u_h - v_h)^{(r,m)}\|_\infty \leqslant \alpha(h) \; on \;\; E_h \; and \; \lim_{h \to 0} \alpha(h) = 0. \tag{25}$$

Proof. The proof falls naturally into two parts.

I. The existence of u_h follows from Theorem 3.1. Let $\tilde{v}_h = (\tilde{v}_{h.1}, \ldots, \tilde{v}_{h.k}) : \Omega_h \to \mathbf{R}^k$ be defined by

$$\tilde{v}_{h.i}^{(r,m)} = v_{h.i}^{(r,m)} + \beta_h^{(r)} \;\; \text{on} \;\; \Omega_h \;\; \text{for} \;\; i = 1, \ldots, k.$$

We prove that the difference functional inequalities

$$\delta_0 \tilde{v}_{h.i}^{(r,m)} \geqslant G_{h.i}[(\tilde{v})_{[r,m]}, (\tilde{v}_{h.i})_{\langle r+1,m \rangle}]^{(r,m)}, \;\; i = 1, \ldots, k, \tag{26}$$

are satisfied on E'_h. It follows from Assumption $H[G_h]$ and (18) that

$$\begin{aligned} \delta_0 \tilde{v}_{h.i}^{(r,m)} &= \delta_0 v_{h.i}^{(r,m)} + \frac{1}{h_0}\big(\beta_h^{(r+1)} - \beta_h^{(r)}\big) \\ &\geqslant G_{h.i}[(\tilde{v}_h)_{[r,m]}, (\tilde{v}_{h.i})_{\langle r+1,m \rangle}]^{(r,m)} - \gamma(h) + \frac{1}{h_0}\big[\beta_h^{(r+1)} - \beta_h^{(r)}\big] \\ &\quad + G_{h.i}[(v_h)_{[r,m]}, (v_{h.i})_{\langle r+1,m \rangle}]^{(r,m)} - G_{h.i}[(\tilde{v}_h)_{[r,m]}, (v_{h.i})_{\langle r+1,m \rangle}]^{(r,m)} \\ &\quad + \sum_{\lambda \in \Lambda} \partial_{\xi(\lambda} G_{h.i}[(\tilde{v}_h)_{[r,m]}, Q_i^{(r+1,m)}]^{(r,m)}, \; i = 1, \ldots, k, \end{aligned} \tag{27}$$

where $Q_i^{(r+1,m)}$ are intermediate points. It is easily seen that

$$U_{h.i}[(\tilde{v}_h)_{[r,m]} - (v_h)_{[r,m]}](\tau) \leqslant T_{h_0}[(\beta_h)_{[r]}](\tau), \;\; i = 1, \ldots, k, \;\; \text{for} \;\; \tau \in I_0. \tag{28}$$

We conclude from (21), (27), (28) that

$$\begin{aligned}\delta_0 \tilde{v}_{h.i}^{(r,m)} &\geqslant G_{h.i}[(\tilde{v}_h)_{[r,m]}, (\tilde{v}_{h.i})_{\langle r+1,m\rangle}]^{(r,m)} - \gamma(h) + \frac{1}{h_0}\big(\beta_h^{(r+1)} - \beta_h^{(r)}\big) \\ &\quad - \sigma(t^{(r)}, T_{h_0}[(\beta_h)_{[r]}]) - \beta_h^{(r+1)} \sum_{\lambda\in\Lambda} \partial_{\xi(\lambda} G_{h.i}[(\tilde{v}_{h.i})_{[r,m]}, Q^{(r+1,m)}]^{(r,m)} \\ &= G_h[(\tilde{v}_h)_{[r,m]}, (\tilde{v}_h)_{\langle r+1,m\rangle}]^{(r,m)}, \quad i = 1,\dots,k.\end{aligned}$$

where $(t^{(r)}, x^{(m)}) \in E'_h$. This completes the proof of (26). Since $v_h^{(r,m)} \leqslant \tilde{v}_h^{(r,m)}$ on $E_{0.h} \cup \partial_0 E_h$, it follows from Theorem 2.3 that

$$u_h^{(r,m)} \leqslant v_h^{(r,m)} + \beta_h^{(r)} \text{ on } E_h.$$

In a similar way we prove that

$$v_h^{(r,m)} - \beta_h^{(r)} \leqslant u_h^{(r,m)} \text{ on } E_h.$$

The above estimates imply (22).

II. Now we prove (25). Consider the Cauchy problem

$$\eta'(t) = \sigma(t, \eta_t) + (\kappa(h))_t) + \gamma(h), \quad \eta(t) = \alpha_0(t) \text{ for } t \in I \tag{29}$$

where $\kappa : H \to \mathbf{R}_+$, $\lim_{h\to 0} \kappa(h) = 0$ and the symbol $(\kappa(h))_t$ denotes a constant function: $(\kappa(h))_t(\tau) = \kappa(h)$ for $\tau \in I$. It follows from Assumption $H[\sigma]$ that there is $\tilde{\varepsilon} > 0$ such that the maximal solution $\eta(\cdot, h)$ of (29) is defined on $I \cup J$ for $\|h\| < \tilde{\varepsilon}$ and

$$\lim_{h\to 0} \eta(t,h) = 0 \text{ uniformly on } I \cup J.$$

Suppose that $\tilde{h} \in H$ is fixed and $\|\tilde{h}\| < \tilde{\varepsilon}$. Let us denote by $C[\tilde{h}]$ the set of all $h \in H$ such that $\|h\| < \tilde{\varepsilon}$ and $\gamma(h) \leqslant \gamma(\tilde{h})$, $\kappa(h) \leqslant \kappa(\tilde{h})$. It follows easily from theorems on differential functional inequalities that for $h \in C[\tilde{h}]$ we have $\eta(t,h) \leqslant \eta(t,\tilde{h})$ on $I \cup J$. Let $\eta_{h_0}(\,\cdot\,, h)$ denote the restriction of $\eta(\,\cdot\,, h)$ to the set $I_h \cup J_h$. Since $\eta(\,\cdot\,, h)$ is a convex function, the definition of T_{h_0} shows that for $h \in C[\tilde{h}]$ we have

$$\big(\eta(\,\cdot\,, h)\big)_{t^{(r)}}(\tau) - T_{h_0}[(\eta_{h_0}(\,\cdot\,, h))_{[r]}](\tau) \geqslant -h_0 \eta'(a,h) \geqslant -h_0 \eta'(a,\tilde{h})$$

where $t^{(r)} \in I_h$ and $\tau \in I$. There is $\varepsilon_0 > 0$ such that for $h \in C[\tilde{h}]$ and $\|h\| < \varepsilon_0$ we have

$$\kappa(\tilde{h}) > \kappa(h) \geqslant h_0 \eta'(a,\tilde{h}).$$

It follows from condition 1) of Assumption $H[\sigma]$ and from the above inequalities that for $h \in C[\tilde{h}]$ and $\|h\| < \varepsilon_0$ we have

$$\begin{aligned}\eta'(t^{(r)}, h) &= \sigma\big(t^{(r)}, (\eta(\,\cdot\,, h))_{t^{(r)}} + (\kappa(h))_{t^{(r)}}\big) + \gamma(h) \\ &= \sigma\big(t^{(r)}, T_{h_0}[(\eta_{h_0}(\,\cdot\,, h))_{[r]}] + (\eta(\,\cdot\,, h))_{t^{(r)}} \\ &\quad - T_{h_0}[(\eta_{h_0}(\,\cdot\,, h))_{[r]}] + (\kappa(h))_{t^{(r)}}\big) + \gamma(h) \\ &\geqslant \sigma\big(t^{(r)}, T_{h_0}[(\eta_{h_0}(\,\cdot\,, h))_{[r]}] - (h_0 \eta'(a,\tilde{h}))_{t^{(r)}} + (\kappa(h))_{t^{(r)}}\big) + \gamma(h)\end{aligned}$$

and consequently

$$\eta'(t^{(r)}h) \geqslant \sigma\big(t^{(r)}, T_{h_0}[\,(\eta_{h_0}(\,\cdot\,,h))_{[r]}]\big) + \gamma(h), \;\; 0 \leqslant r \leqslant N_0.$$

Since $\eta(\,\cdot\,,h$ is a convex function then the above relations imply the difference functional inequality

$$\eta_{h_0}(t^{(r+1)},h) \geqslant \eta_{h_0}(t^{(r)},h) + h_0\sigma(t^{(r)}, T_{h_0}[(\eta_{h_0}(\,\cdot\,,h))_{[r]}]) + h_0\gamma(h)$$

where $0 \leqslant r \leqslant N_0 - 1$. Since β_h satisfies (23), (24) then the above relations and (29) show that $\beta_h^{(r)} \leqslant \eta(t^{(r)},h)$ for $0 \leqslant r \leqslant N_0$. It follows from (22) that condition (25) is satisfied with $\alpha(h) = \eta(a,h)$, where $h \in C[\tilde{h}]$ and $\|h\| < \varepsilon_0$.

This completes the proof. □

The following particular case of Theorem 3.2 is important in simple applications.

LEMMA 3.3. *Suppose that Assumption $H[G_h]$ is satisfied and*

1) $u_h : \Omega_h \to \mathbf{R}$ *is a solution of (5), (6) where* $\varphi_h \in \mathbf{F}(E_{0.h} \cup \partial_0 E_h, \mathbf{R}^k)$,
2) $v_h : \Omega_h \to \mathbf{R}$ *and there are* $\alpha_0,\ \alpha : H \to \mathbf{R}_+$ *such that conditions (18) - (20) are satisfied,*
3) *there exists* $L \in \mathbf{R}_+$ *such that the estimates*

$$G_{h.i}[w,\xi]^{(r,m)} - G_{h.i}[\bar{w},\xi]^{(r,m)} \leqslant L\|w-\bar{w}\|_{D_h}, \;\; i = 1,\dots,k,$$

are satisfied for $(t^{(r)},x^{(m)},\xi) \in E'_h$, $w,\ \bar{w} \in \mathbf{F}(D_h,\mathbf{R}^k)$ *and* $w \geqslant \bar{w}$.

Then

$$\big\|(u_h - v_h)^{(r,m)}\big\|_\infty \leqslant \tilde{\alpha}(h) \;\; on \;\; E_h \tag{30}$$

where

$$\tilde{\alpha}(h) = \alpha_0(h)e^{La} + \gamma(h)\frac{e^{La}-1}{L} \;\; if \;\; L > 0, \tag{31}$$

$$\tilde{\alpha}(h) = \alpha_0(h) + a\gamma(h) \;\; if \;\; L = 0. \tag{32}$$

Proof. It follows that the solution $\beta_h : J_h \to \mathbf{R}_+$ of the difference problem

$$\beta^{(r+1)} = (1 + Lh_0)\beta^{(r)} + h_0\gamma(h), \;\; 0 \leqslant r \leqslant N_0 - 1,$$

$$\beta^{(0)} = \alpha_0(h)$$

satisfies the condition: $\beta_h^{(r)} \leqslant \tilde{\alpha}(h)$ for $0 \leqslant r \leqslant N_0$. Then we obtain the assertion (30) from Theorem 3.2. □

REMARK 3.4. It is important in our considerations that differential functional equations appear in comparison problems. Consider the Cauchy problem

$$\eta'(t) = A\sqrt[\alpha]{\eta(t^\beta)} + B\eta(t), \;\; \eta(0) = 0, \tag{33}$$

where $\beta \geqslant \alpha > 1$. It is easy to see that $\bar{\eta}(t) = 0$ for $t \in [0,1]$ is the maximal solution of (33).

If $\alpha > 1$, $\beta = 1$ then problem (33) does not contain a deviated variable and it has a positive maximal solution on $(0,1]$.

4. Parabolic functional difference inequalities

Solutions of difference equations are functions defined on the mesh. On the other hand equations (1) contain the functional variable $z_{(t,x)}$ which is an element of the space $C(D,\mathbf{R}^k)$. Then we need an interpolating operator $T_h : \mathbf{F}(D_h,\mathbf{R}^k) \to C(D,\mathbf{R}^k)$. We define T_h in the following way. Let us denote by $(\vartheta_1,\dots \vartheta_n)$ the family of sets defined by

$$\vartheta_i = \{0,\, 1\} \text{ if } d_i > 0 \text{ and } \vartheta_i = \{0\} \text{ if } d_i = 0, \quad 1 \leqslant i \leqslant n.$$

Set $\upsilon = (\upsilon_1,\dots,\upsilon_n) \in \mathbf{Z}^n$ and $\upsilon_i = 0$ if $d_i = 0$, $\upsilon_i = 1$ if $d_i > 0$ where $1 \leqslant i \leqslant n$. Write

$$\Delta_+ = \{\lambda = (\lambda_1,\dots,\lambda_n) : \lambda_i \in \vartheta_i \text{ for } 1 \leqslant i \leqslant n\}.$$

Let $w \in \mathbf{F}(D_h,\mathbf{R}^k)$ and $(t,x) \in D$. Suppose that $d_0 > 0$. There exists $(t^{(r)},x^{(m)}) \in D_h$ such that $(t^{(r+1)},x^{(m+\upsilon)}) \in D_h$ and $t^{(r)} \leqslant t \leqslant t^{(r+1)}$, $x^{(m)} \leqslant x \leqslant x^{(m+\chi)})$. Write

$$\begin{aligned} T_h[w](t,x) = &\Big(1 - \frac{t-t^{(r)}}{h_0}\Big) \sum_{\lambda\in\Delta_+} w^{(r,m+\lambda)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} \\ &+ \frac{t-t^{(r)}}{h_0} \sum_{\lambda\in\Delta_+} w^{(r+1,m+\lambda)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} \end{aligned}$$

where

$$\Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda} = \prod_{i=1}^{n}\Big(\frac{x_i-x_i^{(m_i)}}{h_i}\Big)^{\lambda_i}, \quad \Big(1\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda} = \prod_{i=1}^{n}\Big(1-\frac{x_i-x_i^{(m_i)}}{h_i}\Big)^{1-\lambda_i}$$

and we take $0^0 = 1$ in the above formulas. If $d_0 = 0$ then we put

$$T_h[w](t,x) = \sum_{\lambda\in\Delta_+} w^{(r,m+\lambda)} \Big(\frac{x-x^{(m)}}{h'}\Big)^{\lambda}\Big(1-\frac{x-x^{(m)}}{h'}\Big)^{1-\lambda}.$$

Then we have defined $T_h[w]$ on D. It is easy to see that $T_h[w] \in C(D,\mathbf{R}^k)$. The above interpolating operator has been first proposed in [4], Chapter 5.

For $z : \Omega_h \to \mathbf{R}^k$ and $(t^{(r)},x^{(m)}) \in E_h$ we write $T_h z_{[r,m]}$ instead of $T_h[z_{[r,m]}]$. For $w \in C(D,\mathbf{R}^k)$ we put $\|w\| = \max\{\|w(t,x)\|_\infty : (t,x) \in D\}$. The following properties of the operator T_h are important in our considerations.

LEMMA 4.1. *Suppose that $w : D \to \mathbf{R}^k$ is of class C^1 and $w_h = w\,|_{D_h}$. Let $\tilde{C}$ be such a constant that $\|\partial_t w\|_D$, $\|\partial_{x_i} w\|_D \leqslant \tilde{C}$ for $1 \leqslant i \leqslant n$. Then $\|T_h[w_h] - w\|_D \leqslant \tilde{C}\|h\|$ where $\|h\| = h_0 + h_1 + \ldots + h_n$.*

LEMMA 4.2. *Suppose that $w : D \to \mathbf{R}^k$ is of class C^2 and $w_h = w\,|_{D_h}$. Let $\tilde{C}$ be such a constant that $\|\partial_{tt} w\|_D$, $\|\partial_{tx_i} w\|_D$, $\|\partial_{x_ix_j} w\|_D \leqslant \tilde{C}$, $i,j = 1,\dots,n$. Then $\|T_h[w_h] - w\|_D \leqslant \tilde{C}\|h\|^2$.*

The above lemmas are consequences of Lemma 3.19 and Theorem 5.27 in [4].

We formulate a difference functional problem corresponding to (1), (2). Write

$$\Gamma = \{ \ (\mu,\nu) \in \mathbf{N}^2 : \ 1 \leqslant \mu,\nu \leqslant n, \ \ \mu \neq \nu \ \}$$

and suppose that we have defined the sets $\Gamma_+, \Gamma_- \subset \Gamma$ such that $\Gamma_+ \cup \Gamma_- = \Gamma$, $\Gamma_+ \cap \Gamma_- = \emptyset$. In particular, it may happens that $\Gamma_+ = \emptyset$ or $\Gamma_- = \emptyset$. Moreover, we assume that $(i,j) \in \Gamma_+$ when $(j,i) \in \Gamma_+$. Let $\omega : \Omega_h \to \mathbf{R}$ and $(t^{(r)}, x^{(m)}) \in E_h$. Let δ_0 be defined by (4) and

$$\delta_i^+ \omega^{(r,m)} = \frac{1}{h_i}\big[\omega^{(r,m+e_i)} - \omega^{(r,m)}\big], \ \delta_i^- \omega^{(r,m)} = \frac{1}{h_i}\big[\omega^{(r,m)} - \omega^{(r,m-e_i)}\big], \ i = 1,\dots,n.$$

We consider the difference operators $(\delta_1,\dots,\delta_n) = \delta$ defined by

$$\delta_i \omega^{(r,m)} = \frac{1}{2h_j}\big[\ \omega^{(r,m+e_j)} - \omega^{(r,m-e_j)}\ \big], \ \ j = 1,\dots,n. \tag{34}$$

We apply the difference operators $\delta^{(2)} = \big[\ \delta_{\mu,\nu}\ \big]_{\mu,\nu=1,\dots,n}$ given by

$$\delta_{\mu\mu}\omega^{(r,m)} = \delta_\mu^+ \delta_\mu^- \omega^{(r,m)} \ \text{ for } \ \mu = 1,\dots,n, \tag{35}$$

and

$$\delta_{\mu\nu}\omega^{(r,m)} = \frac{1}{2}\big[\ \delta_\mu^+ \delta_\nu^- \omega^{(r,m)} + \delta_\mu^- \delta_\nu^+ \omega^{(r,m)}\ \big] \ \text{ for } \ (\mu,\nu) \in \Gamma_-, \tag{36}$$

$$\delta_{\mu\nu}\omega^{(r,m)} = \frac{1}{2}\big[\ \delta_\mu^+ \delta_\nu^+ \omega^{(r,m)} + \delta_\mu^- \delta_\nu^- \omega^{(r,m)}\ \big] \ \text{ for } \ (\mu,\nu) \in \Gamma_+. \tag{37}$$

Given $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbf{R}^k$, we consider the functional difference system

$$\delta_0 z_i^{(r,m)} = F_i\big(\ t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z_i^{(r+1,m)}, \delta^{(2)} z_i^{(r+1,m)}\ \big), \ i = 1,\dots,k, \tag{38}$$

with the initial boundary condition

$$z^{(r,m)} = \varphi_h^{(r,m)} \ \text{ on } \ E_{0.h} \cup \partial_0 E_h. \tag{39}$$

The above problem is considered as an implicit difference scheme for (1), (2). It is important that the difference operators δz_i and $\delta^{(2)} z_i$, $1 \leqslant i \leqslant k$, are calculated in (38) at the point $(t^{(r+1)}, x^{(m)})$ and the functional variable $T_h z_{[r,m]}$ appears in a classical sense.

We begin with a theorem on implicit difference functional inequalities generated by (38), (39). Write

$$F_{h.i}[z]^{(r,m)} = F_i\big(\ t^{(r)}, x^{(m)}, T_h z_{[r,m]}, \delta z_i^{(r+1,m)}, \delta^{(2)} z_i^{(r+1,m)}\ \big), \ i = 1,\dots,k.$$

ASSUMPTION $H[F]$. The function $F = (F_1,\dots,F_k) : \Xi \to \mathbf{R}^k$ of the variables (t,x,w,q,s), $s = [\, s_{\mu,\nu}\,]_{\mu,\nu=1,\dots,n}$, is continuous and

1) the partial derivatives

$$\partial_q F_i = (\partial_{q_1} F_i, \dots, \partial_{q_n} F_i), \ \ \partial_s F_i = \big[\ \partial_{s_{\mu\nu}} F\ \big]_{\mu,\nu=1,\dots,n}, \ i = 1,\dots,k,$$

exist on Ξ and the functions $\partial_q F_i$, $\partial_s F_i$, $1 \leqslant i \leqslant k$, are continuous and bounded on Ξ,

2) for each i, $1 \leqslant i \leqslant k$, the matrix $\partial_s F_i$ is symmetric and

$$\partial_{s_{\mu\nu}} F_i(P) \geqslant 0 \text{ for } (\mu,\nu) \in \Gamma_+, \quad \partial_{s_{\mu\nu}} F_i(P) \leqslant 0 \text{ for } (\mu,\nu) \in \Gamma_- \tag{40}$$

and

$$\frac{1}{h_\mu}\partial_{s_{\mu\mu}} F_i(P) - \frac{1}{2}|\,\partial_{q_\mu} F_i(P)\,| - \sum_{\nu=1,\nu\neq\mu}^{n} \frac{1}{h_\nu}\big|\,\partial_{s_{ij}} F(P)\,\big| \geqslant 0, \quad \mu = 1,\dots,n, \tag{41}$$

where $P = (t,x,w,q,s) \in \Xi$, $i = 1,\dots,k$

3) there is $\varepsilon_0 > 0$ such that for $0 < h_0 < \varepsilon_0$ and $w, \bar{w} \in C(D,\mathbf{R}^k)$, $w \leqslant \bar{w}$, we have

$$w_i(0,\theta) + h_0 F_i(t,x,w,q,s) \leqslant \bar{w}_i(0,\theta) + h_0 F_i(t,x,\bar{w},q,s),$$

where $(t,x,q,s) \in E \times \mathbf{R}^n \times M_{n\times n}$, $i = 1,\dots,k$.

REMARK 4.3. It is required in condition 2) of Assumption $H[F]$ that for each $(\mu,\nu) \in \Gamma$ and $1 \leqslant i \leqslant k$ the function $g_{\mu,\nu}(P) = \operatorname{sign} \partial_{s_{\mu\nu}} F_i(P)$, $P \in \Xi$, is constant on Ξ. Relations (40) can be considered as definitions of the sets Γ_+ and Γ_-.

REMARK 4.4. Suppose that for each i, $1 \leqslant i \leqslant k$ the matrix $\partial_s F_i$ satisfies the condition: there is $\tilde{\varepsilon} > 0$ such that

$$\partial_{s_{\mu\mu}} F_i(P) - \sum_{\nu=1\,\nu\neq\mu}^{n} \partial_{s_{\mu\nu}} F_i(P) \geqslant \tilde{\varepsilon}, \quad P \in \Xi.$$

If $h_1 = h_2 = \ldots = h_n$ then there is $\bar{\varepsilon} > 0$ such that condition (41) is satisfied for $\|h'\| < \bar{\varepsilon}$.

REMARK 4.5. Given the function $\tilde{G} = (\tilde{G}_1,\dots,\tilde{G}_k) : E \times \mathbf{R} \times C(D,\mathbf{R}^k) \times \mathbf{R}^n \times M_{n\times n}$ of the wariables (t,x,p,w,q,s) Put

$$F_i(t,x,w,q,s) = \tilde{F}_i(t,x,w_i(0,\theta),w,q,s) \text{ on } \Xi, \quad i = 1,\dots,k..$$

Then system (1) has the form

$$\partial_t z_i(t,x) = \tilde{F}_i\big(\,t,x,z_i(t,x),z_{(t,x)},\partial_x z_i(t,x),\partial_{xx} z_i(t,x)\,\big). \tag{42}$$

It is important that the the dependence of $\tilde{F}$ on the classical variable $z(t,x)$ is distinguished in (42). Suppose that

1) $\tilde{F}$ is nondecreasing with respect to the functional variable,
2) there exists the derivatives $(\partial_p \tilde{F}_1,\dots,\partial_p \tilde{F}_k)$ and $\partial_p \tilde{F}_i(t,x,p,w,q,s) \geqslant L$ for $1 \leqslant i \leqslant k$ and $1 + Lh_0 \geqslant 0$.

Then the monotonicity condition 3) of Assumption $H[F]$ is satisfied.

For $\xi \in \mathbf{R}^\chi$, $\xi = \{\,\xi^{(\lambda)}\,\}_{\lambda\in\Lambda}$, we put

$$\delta_j^+ \xi^{(\theta)} = \frac{1}{h_j}\big[\xi^{(e_j)} - \xi^{(\theta)}\big], \quad \delta_j^- \xi^{(\theta)} = \frac{1}{h_j}\big[\xi^{(\theta)} - \xi^{(-e_j)}\big], \quad j = 1,\dots,n.$$

The expressions

$$\delta\xi^{(\theta)} = (\delta_1\xi^{(\theta)}, \ldots, \delta_n\xi^{(\theta)}), \quad \delta^{(2)}\xi^{(\theta)} = \big[\, \delta_{\mu\nu}\xi^{(\theta)} \,\big]_{\mu,\nu=1,\ldots,n},$$

are defined in the following way:

$$\delta_\mu\xi^{(\theta)} = \frac{1}{2h_\mu}\big[\, \xi^{(e_\mu)} - \xi^{(-e_\mu)}\big] \text{ for } \mu = 1, \ldots, n,$$

$$\delta_{\mu\mu}\xi^{(\theta)} = \delta_\mu^+\delta_\mu^-\xi^{(\theta)} \text{ for } \mu = 1, \ldots, n$$

and

$$\delta_{\mu\nu}\xi^{(\theta)} = \frac{1}{2}\big[\delta_\mu^+\delta_\nu^-\xi^{(\theta)} + \delta_\mu^-\delta_\nu^+\xi^{(\theta)}\big] \text{ for } (\mu,\nu) \in \Gamma_-,$$

$$\delta_{\mu,\nu}\xi^{(\theta)} = \frac{1}{2}\big[\delta_\mu^+\delta_\nu^+\xi^{(\theta)} + \delta_\mu^-\delta_\nu^-\xi^{(\theta)}\big] \text{ for } (\mu,\nu) \in \Gamma_+.$$

We consider the sets

$$\bar\Lambda = \{\lambda \in \Lambda : \text{ there is } i, \ 1 \leqslant i \leqslant n, \text{ such that } \lambda = e_i \text{ or } \lambda = -e_i\,\},$$

$$\Lambda_+ = \{\, \lambda \in \Lambda : \ \text{ there is } (\mu,\nu) \in \Gamma_+ \text{ such that } \lambda = e_\mu + e_\nu \text{ or } \lambda = -e_\mu - e_\nu\,\},$$

$$\Lambda_- = \{\, \lambda \in \Lambda : \ \text{ there is } (\mu,\nu) \in \Gamma_- \text{ such that } \lambda = e_\mu - e_\nu \text{ or } \lambda = -e_\mu + e_\nu\,\},$$

and $\Lambda_\star = \Lambda \setminus \big(\,\{\,\theta\,\} \cup \bar\Lambda \cup \Lambda_+ \cup \Lambda_-\,\big)$. Let the function $G_h : Y_h \to \mathbf{R}_+$ be defined by

$$G_{h.i}[w,\xi]^{(r,m)} = F_i\big(\,t^{(r)}, x^{(m)}, T_h w, \delta\xi^{(\theta)}, \delta^{(2)}\xi^{(\theta)}\,\big), \ i = 1, \ldots, k. \tag{43}$$

LEMMA 4.6. *Let Assumption $H[F]$ holds. Then the function G_h defined by (43) satisfies Assumption $H[G_h]$.*

Proof. It is clear that conditions 1), 2), 4) of Assumption $H[G_h]$ are satisfied. It remains to prove relations (8), (9). Write $Q^{(r,m)}[w,\xi] = \big(\,t^{(r)}, x^{(m)}, T_h w, \delta\xi^{(\theta)}, \delta^{(2)}\xi^{(\theta)}\,\big)$. It follows from (43) that

$$\partial_{\xi^{(\theta)}} G_{h.i}[w,\xi]^{(r,m)} = -2\sum_{\mu=1}^{m} \frac{1}{h_\mu^2}\partial_{s_{\mu\mu}}F_i(Q^{(r,m)}[w,\xi]) + \sum_{(\mu,\nu)\in\Gamma} \frac{1}{h_\mu h_\nu}\big|\,\partial_{s_{\mu\nu}}F_i(Q^{(r,m)}[w,\xi])\,\big|,$$

$$\partial_{\xi^{(e_\mu)}} G_{h.i}[w,\xi]^{(r,m)} = \frac{1}{h_\mu^2}\partial_{s_{\mu\mu}}F_i(Q^{(r,m)}[w,\xi]) - \sum_{nu=1,\nu\neq mu}^{n} \frac{1}{h_\mu h_\nu}\big|\partial_{s_{mu\nu}}F_i(Q^{(r,m)}[w,\xi])\,\big|$$

$$+\frac{1}{2h_\mu}\partial_{q_\mu}F_i(Q^{(r,m)}[w,\xi]) \text{ for } \mu = 1, \ldots, n,$$

$$\partial_{\xi^{(-e_\mu)}} G_{h.i}[w,\xi]^{(r,m)} = \frac{1}{h_\mu^2}\partial_{s_{\mu\mu}}F_i(Q^{(r,m)}[w,\xi]) - \sum_{\nu=1,\nu\neq\mu}^{n} \frac{1}{h_\mu h_\nu}\big|\partial_{s_{\mu\nu}}F_i(Q^{(r,m)}[w,\xi])\,\big|$$

$$-\frac{1}{2h_\mu}\partial_{q_\mu}F_i(Q^{(r,m)}[w,\xi]) \text{ for } \mu = 1, \ldots, n,$$

$$\partial_{\xi(e_\mu+e_\nu)} G_{h.i}[w,\xi]^{(r,m)} = \partial_{\xi(-e_\mu-e_\nu)} G_{h.i}[w,\xi]^{(r,m)}$$
$$= \frac{1}{2h_\mu h_\nu}\partial_{s_{\mu\nu}} F_i(Q^{(r,m)}[w,\xi]) \ \text{ for } \ (\mu,\nu)\in\Gamma_+,$$

$$\partial_{\xi(e_\mu-e_\nu)} G_{h.i}[w,\xi]^{(r,m)} = \partial_{\xi(-e_\mu+e_\nu)} G_{h.i}[w,\xi]^{(r,m)}$$
$$= -\frac{1}{2h_\mu h_\nu}\partial_{s_{\mu\nu}} F_i(Q^{(r,m)}[w,\xi]) \ \text{ for } \ (\mu,\nu)\in\Gamma_-.$$

we put $i=1,\dots,k$ in the above formulas. Moreover, we have

$$\partial_{\xi(\lambda)} G_{h.i}[w,\xi]^{(r,m)} = 0 \ \text{ for } \ \lambda\in\Lambda_\star, \ \ i=1,\dots,k.$$

The above relations and condition 2) of Assumption $H[F]$ show that conditions (8), (9) are satisfied on Y_h. This completes the proof. □

THEOREM 4.7. *Suppose the Assumption $H[F]$ is satisfied and*

1) $h\in H$, $h_0<\varepsilon_0$ and the functions $u,v:\Omega_h\to\mathbf{R}^k$ satisfy the difference functional inequalities

$$\delta_0 u_i^{(r,m)} - F_{h.i}[u]^{(r,m)} \leqslant \delta_0 v_i^{(r,m)} - F_{h.i}[v]^{(r,m)} \ \text{ on } \ E'_h \ \text{ for } \ i=1,\dots,k,$$

2) the initial boundary estimate $u^{(r,m)}\leqslant v^{(r,m)}$ holds on $E_{0.h}\cup\partial_0 E_h$.

Then

$$u^{(r,m)} \leqslant v^{(r,m)} \ \text{ on } \ E_h. \tag{44}$$

Proof. We apply Theorem 2.3 to prove (44). Let $G_h : Y_h\to\mathbf{R}^k$ be defined by (43). Then the difference functional inequality (10) is satisfied. We conclude from Lemma 4.6 that all the assumptions of Theorem 2.3 are satisfied and the assertion (44) follows. □

5. Implicit difference methods

We need the following operator $V : C(D,\mathbf{R}^k)\to C(I,\mathbf{R}_+)$. For $w\in C(D,\mathbf{R}^k)$ we put

$$V[w](t) = \max \|w(t,x)\|_\infty : \ x\in[-d,d]\,\}, \ \ t\in I.$$

For $w,\,\bar w\in C(D\,\mathbf{R}^k)$ we write $w\leqslant\bar w$ if $w(t,x)\leqslant\bar w(t,x)$ for $(t,x)\in D$. We consider now the problem of the convergence of the implicit difference schemes for (1), (2).

ASUMPTION $H[F,\sigma]$. There is $\sigma : J\times C(I,\mathbf{R}_+)\to\mathbf{R}_+$ such that Assumption $H[\sigma]$ is satisfied and for $w,\,\bar w\in C(D,\mathbf{R})$, $w\geqslant\bar w$ we have

$$F_i(t,x,w,q,s) - F_i(t,x,\bar w,q,s) \leqslant \sigma(t,V[w-\bar w]), \ i=1,\dots,k,$$

where $(t,x,q,s)\in E\times\mathbf{R}^n\times M_{n\times n}$, $\bar w\in C(D,\mathbf{R}^k)$ and $w\geqslant\bar w$.

THEOREM 5.1. *Suppose that Assumption $H[F]$ and $H[F,\sigma]$ are satisfied and*

1) $v:\Omega\to\mathbf{R}^k$ is a solution of (1), (2) and v is of class C^2 on Ω,

2) $h \in H$, $h_0 < \varepsilon$ *and* $\varphi_h : E_{0.h} \cup \partial_0 E_h \to \mathbf{R}^k$ *and there is* $\alpha_0 : H \to \mathbf{R}_+$ *such that*

$$\parallel \varphi^{(r,m)} - \varphi_h^{(r,m)} \parallel_\infty \leqslant \alpha_0(h) \text{ on } E_{0.h} \cup \partial_0 E_h \text{ and } \lim_{h\to 0} \alpha_0(h) = 0.$$

Under these assumptions we have

1) *there is a solution* $u_h : \Omega_h \to \mathbf{R}^k$ *of (38), (39),*

2) *there is* $\alpha : H \to \mathbf{R}_+$ *such that*

$$\left\|(u_h - v_h)^{(r,m)}\right\|_\infty \leqslant \alpha(h) \text{ on } E_h \text{ and } \lim_{h\to 0} \alpha(h) = 0, \tag{45}$$

where $v_h = v\,|_{\Omega_h}$.

Proof. Let $G_h : Y_h \to \mathbf{R}$ be defined by (43). The existence of u_h follows from Theorem 3.1 and Lemma 4.6. The assertion (45) is a consequence of Theorem 3.2 and Lemma 4.6. This is our claim. □

REMARK 5.2. Suppose that all the assumption of Theorem 5.1 are satisfied and $\sigma(t,\eta) = L\|\eta\|_0$ on $J \times C(I, \mathbf{R}_+)$. Then $\left|(u_h - v_h)^{(r,m)}\right| \leqslant \tilde{\alpha}(h)$ on E_h. where $\tilde{\alpha}$ is given by (31), (32).

The above observation is a consequence of Lemma 3.3.

REMARK 5.3. Suppose that $a \leqslant 1$ and there are $A, B \in \mathbf{R}_+$ and $\beta \geqslant \alpha > 1$ such that

$$F_i(t,x,q,q,s) - F_i(t,x,\bar{w},q,s) \leqslant A\sqrt[\alpha]{V[w-\bar{w}](t^\beta - t)} + B\|w-\bar{w}\|_D,\ i = 1,\dots,k,$$

where $(t,x,q,s) \in E \times \mathbf{R}^n \times M_{n\times n}$, $w, \bar{w} \in C(D, \mathbf{R}^k)$ and $w \geqslant \bar{w}$. Then comparison problem has the form

$$\eta'(t) = A\sqrt[\alpha]{\eta(t^\beta)} + B\eta(t), \quad \eta(0) = 0,$$

and Assumption $H[F,\sigma]$ is satisfied.

The above observation is a consequence of Remark 33. Then Theorem 5.1 is a generalization of results presented in [7], [8].

Now we give numerical examples.

EXAMPLE 5.4. *Put* $n = 2$ *and* $E = [0,\ 0.5]\times[-0.5,\ 0.5]\times[-0.5,\ 0.5]$. *Consider the differential integral equation*

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) + \cos\left[\partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) + 2\pi^2 z(t,x,y)\right]$$
$$-\frac{1}{\pi^2}\partial_{xy} z(t,x,y) + \pi^2 \int_{-0.5}^{x}\int_{-0.5}^{y} z(t,\xi,\zeta)d\zeta\, d\xi - \int_0^t z(\tau,x,y)d\tau$$
$$+2\pi^2 z(t,x,y) - 1 + \cos(\pi x)\cos(\pi y)$$

with the initial boundary condition

$$z(0,x,y) = 0, \quad (x,y) \in [-0.5,\ 0.5] \times [-0.5,\ 0.5],$$
$$z(t,-0.5,y) = z(t,0,5,y) = 0, \quad (t,y) \in [0,\ 0.5] \times [-0.5,\ a] \times [-0.5,\ 0.5],$$
$$z(t,x,-0.5,) = z(t,x,0.5) = 0, \quad (t,x) \in [0,\ 0.5] \times [-0.5,\ 0.5]$$

The function $v(t,x,y) = \sin t \cos(\pi x)\cos(\pi y)$ is the solution of the above problem. Let us denote by z_h an approximate solution which is obtained by using the implicit difference scheme. The Newton method is used for solving nonlinear systems generated by the implicit scheme. Write $m = (m_1, m_2)$ and

$$\varepsilon_h^{(r)} = \frac{1}{(2N_1-1)(2N_2-1)} \sum_{m\in\Pi} \left| z_h^{(r,m)} - v^{(r,m)} \right|, \quad 0 \leqslant r \leqslant N_0, \tag{46}$$

where

$$\Pi = \{ m = (m_1, m_2) \in \mathbf{Z}^2 : \; -N_1+1 \leqslant m_1 \leqslant N_1-1, \; -N_1+1 \leqslant m_2 \leqslant N_2-1 \}$$

and $N_1h_1 = 0.5$, $N_2h_2 = 0.5$, $N_0h_0 = 0.5$. The numbers $\varepsilon_h^{(r)}$ can be called average errors of the difference method for fixed $t^{(r)}$. We put $h_0 = h_1 = h_2 = 0.005$ and we have the following values of the above defined errors.

Table of errors

$t^{(r)}$	0.25	0.30	0.35	0.40	0.45	0.50
$\varepsilon_h^{(r)}$	0.0047	0.0056	0.0065	0.0074	0.0083	0.0091

Note that our equation and the steps of the mesh do not satisfy condition (3) which is necessary for the explicit difference method to be convergent. In our numerical example the average errors for the explicit difference method exceeded 10^6.

EXAMPLE 5.5. *For $n = 2$ we put $E = [0,\ 0.5] \times [-1,\ 1] \times [-1,\ 1]$. Consider the differential equations with deviated variables*

$$\partial_t z(t,x,y) = \partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) + x^2y^2\partial_{xy} z(t,x,y)$$

$$+ \arctan\left[\partial_{xx} z(t,x,y) + \partial_{yy} z(t,x,y) - 4t^2(x^2+y^2) z(t,x,y)\right]$$

$$+ z(t, 0.5(x+y), 0.5(x-y)) + f(t,x,y) z(t,x,y),$$

$$f(t,x,y) = x^2 - y^2 - 4t^2(x^2+y^2-x^3y^3) - \exp\left[t(xy - x^2 + y^2)\right],$$

with the initial boundary conditions

$$z(0,x,y) = 1, \quad (x,y) \in [-1,\ 1] \times [-1,\ 1],$$

$$z(t,-1,y) = z(t,1,y) = e^{t(1-y^2)}, \quad (t,y) \in [0,\ 0.5] \times [-1,\ 1],$$

$$z(t,x,-1) = z(t,x,1) = e^{t(x^2-1)}, \quad (t,x) \in [0,\ 0.5] \times [-1,\ 1].$$

The function $v(t,x,y) = \exp\left[t(x^2-y^2)\right]$ is the solution of the above problem.

Let us denote by z_h an approximate solution which is obtained by using the implicit difference scheme. The Newton method is used for solving nonlinear systems generated by the implicit scheme. Let ε_h be the average error defined by (46) with $N_0h_0 = 0.5$,

$N_1h_1 = 1$, $N_2h_2 = 1$. We put $h_0 = h_1 = h_2 = 0.005$ and we have the following values of the above defined errors.

Table of errors

$t^{(r)}$:	0.25	0.30	0,35	0.40	0.45	0.50
$\varepsilon_h^{(r)}$:	0.0045	0.0055	0.0064	0.0072	0.0078	0.0083

Note that our equation and the steps of the mesh do not satisfy condition (3) which is necessary for the explicit difference method to be convergent. In our numerical example the average errors for the explicit difference method exceeded 10^6.

The above examples show that there are implicit difference schemes which are convergent and the corresponding classical method are not convergent. This is due to the fact that we need assumption (3) for explicit difference methods. We do not need this this condition in our implicit methods.

Our results show that implicit difference schemes are convergent on all meshes.

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(Received February 27, 2008)

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