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A CLASS OF NEW TRIGONOMETRIC INEQUALITIES AND THEIR SHARPENINGS

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Abstract. Some sufficient or necessary conditions for Schur-convexity of a function of two variables $F(x,y) = (f(y)-f(x))/(g(y)-g(x))$ were considered. These results are applied to establish a class new inequalities in a triangle. In the fourth section we prove two theorems for a kind of symmetric function. These theorems are used to sharpen some of the inequalities and yield two inequalities in the last section.

1. Introduction

In the paper [5], the author consider necessary and sufficient conditions for convexity of a function $x \mapsto f(x)$ in terms of some properties of the associated function of two variables $F(x,y) = (f(y)-f(x))/(y-x)$. These results are applied to the theory of the Gamma function. This paper follows the ideas in [5] and discusses the conditions for Schur-convexity and some statements of the function $F(x,y) = (f(y)-f(x))/(g(y)-g(x))$, and use them to yield some new inequalities in a triangle. For example it is shown that in an acute $\triangle ABC$,

$$1 < \frac{\sin A - \sin B}{A-B} + \frac{\sin B - \sin C}{B-C} + \frac{\sin C - \sin A}{C-A} \leqslant \frac{3}{2}.$$

Actually the best lower bound of the inequality above is not 1 but $\frac{4}{\pi}$. The similar case is happened to some other inequalities. In the fourth section, we prove two theorems to find the optimal lower or upper bound of each inequality.

Firstly, we recall the concepts of majorization and Schur-convexity. For any $x = (x_1, x_2, \cdots, x_n) \in \mathbf{R}^n$, let $x_{[1]} \geqslant x_{[2]} \geqslant \cdots \geqslant x_{[n]}$, denote the components of x in decreasing order.

DEFINITION. The vector x is said to be majorized by the vector y (denoted $x \prec y$) if

$$\sum_{i=1}^{k} x_{[i]} \leqslant \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1, \tag{1.1}$$

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and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_i. \tag{1.2}$$

DEFINITION. If $x \prec y$ but x is not a permutation of y, then x is said to be strictly majorized by y (denoted $x \prec\prec y$).

DEFINITION. A function $\varphi : \mathbf{R}^n \to \mathbf{R}$ is called Schur-convex if $x \prec\prec y \Rightarrow \varphi(x) \leqslant \varphi(y)$.

If, in addition, $\varphi(x) < \varphi(x)$ whenever $x \prec\prec y$, then $\varphi(x)$ is said to be strictly Schur-convex. Of course $\varphi(x)$ is said to be (strictly) Schur-concave if and only if $-\varphi(x)$ is (strictly) Schur-convex.

2. Conditions for Schur-convexity of the function

Let f, g be functions defined on an interval I and their derivative f', g' exist, and $g' \neq 0$. Define the function F of two variables by

$$F(x,y) = \frac{f(y) - f(x)}{g(y) - g(x)}, \quad (x \neq y), \qquad F(x,x) = \frac{f'(x)}{g'(x)}, \tag{2.1}$$

where $(x,y) \in I^2$. Let us now consider the following statements:

(A) f' is convex, g' is concave, $f' \geqslant 0$ and $g' > 0$ on I; or f' is concave, g' is convex, $f' \leqslant 0$ and $g' < 0$ on I,

(B) $F(x,y) \geqslant \dfrac{f'(\frac{x+y}{2})}{g'(\frac{x+y}{2})}$ for all $x, y \in I$,

(C) $F(x,y) \leqslant \dfrac{f'(x) + f'(y)}{g'(x) + g'(y)}$ for all $x, y \in I$,

(D) F is Schur-convex on I^2,

and

(A') f' is concave, g' is convex, $f' \geqslant 0$ and $g' > 0$ on I; or f' is convex, g' is concave, $f' \leqslant 0$ and $g' < 0$ on I,

(B') $F(x,y) \leqslant \dfrac{f'(\frac{x+y}{2})}{g'(\frac{x+y}{2})}$ for all $x, y \in I$,

(C') $F(x,y) \geqslant \dfrac{f'(x) + f'(y)}{g'(x) + g'(y)}$ for all $x, y \in I$,

(D') F is Schur-concave on I^2.

THEOREM 2.1. *if $f'''(t), g'''(t)$ is continuous on I,*

1. *if (A) holds then the conditions $(B)-(D)$ hold;*

2. *if (A') holds then the conditions $(B')-(D')$ hold.*

Proof. $(A) \Rightarrow (C)$: Let f' is convex, g' is concave and $x, y \in I, x < y$. Then for each $t \in [x,y]$ we have $t = \lambda(t)x + (1 - \lambda(t))y$, where $\lambda(t) = (y-t)/(y-x)$.

By Jensen's inequality, we have $f'(t) \leqslant f'(x)\lambda(t) + f'(y(1-\lambda(t)))$. The integration over $t \in [x,y]$ yields

$$\int_x^y f'(t)dt \leqslant f'(x)\int_x^y \lambda(t)dt + f'(y)\int_x^y [1-\lambda(t)]dt, \tag{2.2}$$

which is equivalent to

$$f(y) - f(x) \leqslant \frac{f'(x)+f'(y)}{2}(y-x). \tag{2.3}$$

Similarly,

$$g(y) - g(x) \geqslant \frac{g'(x)+g'(y)}{2}(y-x). \tag{2.4}$$

And if $f' \geqslant 0, g' > 0$ on I, then $f(y)-f(x) \geqslant 0$ and $g(y)-g(x) > 0$.We get

$$\frac{f(y)-f(x)}{g(y)-g(x)} \leqslant \frac{f'(x)+f'(y)}{g'(x)+g'(y)}, \tag{2.5}$$

which is the statement (C).

The second part follows upon replacing f by $-f$ and g by $-g$.

$(C) \Leftrightarrow (D)$: It's evident that $F(x,y)$ is symmetric on I^2. Since

$$\frac{\partial F(x,y)}{\partial x} = \frac{g'(x)(f(y)-f(x)) - f'(x)(g(y)-g(x))}{(g(y)-g(x))^2}, \tag{2.6}$$

$$\frac{\partial F(x,y)}{\partial y} = \frac{f'(y)(g(y)-g(x)) - g'(y)(f(y)-f(x))}{(g(y)-g(x))^2}, \tag{2.7}$$

we see that the partial derivatives are continuous in each point $(x,y) \in I^2, x \neq y$. Let $x < y$, then $F(x,y)$ is Schur-convex on I^2 if and only if ([4,3.A.4]) $\frac{\partial F(x,y)}{\partial y} - \frac{\partial F(x,y)}{\partial x} \geqslant 0$, which is equivalent to (C).

$(D) \Rightarrow (B)$: Suppose that F is Schur-convex, for sufficiently small $\varepsilon > 0$ it follows that $(\frac{x+y}{2} - \varepsilon, \frac{x+y}{2} + \varepsilon) \prec (x,y)$, so

$$F(\frac{x+y}{2} - \varepsilon, \frac{x+y}{2} + \varepsilon) \leqslant F(x,y). \tag{2.8}$$

Letting $\varepsilon \to 0$, we get $\frac{f'(\frac{x+y}{2})}{g'(\frac{x+y}{2})} \leqslant F(x,y)$, which is the statement (B).

This ends the proof of (1). The proof of (2) is similar to (1). □

Let us now consider the other four statements:

(A_1) g' is concave on I,
(A_1') g' is convex on I,
(A_2) f' is convex on I,
(A_2') f' is concave on I.

THEOREM 2.2. *if $g'''(t)$ is continuous on I and $f(t) = t$, then the conditions $(A_1), (B), (C), (D)$ are equivalent, and the conditions $(A_1'), (B'), (C'), (D')$ are equivalent.*

Proof. If g' is concave on I, let $x, y (x < y)$ be arbitrary points in I. By Taylor's expansion around $c = (x+y)/2$, we have

$$g(x) = g(c) + g'(c)(x-c) + \frac{g''(c)}{2}(x-c)^2 + g'''(\xi_1)(x-c)^3, \tag{2.9}$$

$$g(y) = g(c) + g'(c)(y-c) + \frac{g''(c)}{2}(y-c)^2 + g'''(\xi_2)(y-c)^3, \tag{2.10}$$

where $\xi_1 \in (x,c), \xi_2 \in (c,y)$ and $g'''(\xi_1)(x-c)^3 \geqslant 0, g'''(\xi_2)(y-c)^3 \leqslant 0$. Then from (2.9) and (2.10) we get

$$g(y) - g(x) = g'(c)(y-x) + g'''(\xi_2)(y-c)^3 - g'''(\xi_1)(x-c)^3 \leqslant g'(c)(y-x). \tag{2.11}$$

Therefore $\frac{y-x}{g(y)-g(x)} \geqslant \frac{1}{g'(c)}$ wherefrom (B) follows.

$(B) \Rightarrow (A_1)$: Suppose that (B) holds and that (A_1) does not hold. Therefore, there exists $t \in I$ with $g'''(t) < 0$. By continuity of g''', there is an interval $I^* \subset I$ so $g'''(t) < 0$ for $t \in I^*$. Then the function $-g'$ is convex on I^* and by the above proof of $(A_1) \Rightarrow (B)$, we conclude that (B) holds for $-g$ on I^*, hence (B) does not hold for g, which is a contradiction.

$(A_1) \Rightarrow (C)$: Similar to prove $(A) \Rightarrow (C)$, if g' is concave on I, then (2.4) holds. (2.4) is equivalent to (C).

The proofs of $(C) \Rightarrow (D)$ and $(D) \Rightarrow (B)$ are the same with theorem 2.1.

This is the end of the proof of the first part. The second part follows upon replacing g by $-g$. □

Similarly we have the theorem 2.3, which has been proved by Milan Merkle[5].

THEOREM 2.3. *if $f'''(t)$ is continuous on I and $g(t) = t$, then the conditions $(A_2), (B), (C), (D)$ are equivalent, and the conditions $(A_2'), (B'), (C'), (D')$ are equivalent.*

3. Some new trigonometric inequalities

A, B, C are the angles of an acute triangle ABC.

PROPOSITION 3.1.1.

$$1 < \frac{\sin A - \sin B}{A - B} + \frac{\sin B - \sin C}{B - C} + \frac{\sin C - \sin A}{C - A} \leqslant \frac{3}{2} \tag{3.1}$$

Proof. Let $f(t) = \sin t$, $g(t) = t$ and $I = (0, \frac{\pi}{2})$, then $f' = \cos t$ is convex on I. From theorem 2.3, we get

$$\frac{\cos A + \cos B}{2} \leqslant \frac{\sin A - \sin B}{A - B} \leqslant \cos\frac{A+B}{2} = \sin\frac{C}{2}.$$

Similarly we have

$$\frac{\cos B + \cos C}{2} \leqslant \frac{\sin B - \sin C}{B - C} \leqslant \sin\frac{A}{2},$$

and

$$\frac{\cos C + \cos A}{2} \leqslant \frac{\sin C - \sin A}{C - A} \leqslant \sin \frac{B}{2}.$$

Adding the three inequalities upon, it yields

$$\sum_{\text{cyclic}} \cos A \leqslant \sum_{\text{cyclic}} \frac{\sin A - \sin B}{A - B} \leqslant \sum_{\textit{cyclic}} \sin \frac{A}{2}.$$

$\sum\limits_{\text{cyclic}} \cos A > 1$ had been shown in [1] and $\sum\limits_{\text{cyclic}} \sin \frac{A}{2} \leqslant \frac{3}{2}$ in [3]. □

Similarly we can get

PROPOSITION 3.1.2.

$$-\frac{3\sqrt{3}}{2} \leqslant -\sum_{\text{cyclic}} \cos A \leqslant \sum_{\text{cyclic}} \frac{\cos A - \cos B}{A - B} \leqslant -\sum_{\text{cyclic}} \sin A < 0, \tag{3.2}$$

$$\sum_{\text{cyclic}} \csc^2 \frac{A}{2} \leqslant \sum_{\text{cyclic}} \frac{\tan A - \tan B}{A - B} \leqslant \sum_{\text{cyclic}} \sec^2 A, \tag{3.3}$$

$$-\sum_{\text{cyclic}} \csc^2 A \leqslant \sum_{\text{cyclic}} \frac{\cot A - \cot B}{A - B} \leqslant -\sum_{\text{cyclic}} \sec^2 \frac{A}{2} \leqslant -4. \tag{3.4}$$

And using theorem 2.2, we get

PROPOSITION 3.2.

$$6 \leqslant \sum_{\text{cyclic}} \csc \frac{A}{2} \leqslant \sum_{\text{cyclic}} \frac{A - B}{\sin A - \sin B} \leqslant \sum_{\text{cyclic}} \frac{2}{\cos A + \cos B} \tag{3.5}$$

$$-\sum_{\text{cyclic}} \frac{2}{\sin A + \sin B} \leqslant \sum_{\text{cyclic}} \frac{A - B}{\cos A - \cos B} \leqslant -\sum_{\text{cyclic}} \sec \frac{A}{2} \leqslant -2\sqrt{3}, \tag{3.6}$$

$$\sum_{\text{cyclic}} \frac{2}{\sec^2 A + \sec^2 B} \leqslant \sum_{\text{cyclic}} \frac{A - B}{\tan A - \tan B} \leqslant \sum_{\text{cyclic}} \sin^2 \frac{A}{2} < 1, \tag{3.7}$$

$$-\frac{9}{4} \leqslant -\sum_{\text{cyclic}} \cos^2 \frac{A}{2} \leqslant \sum_{\text{cyclic}} \frac{A - B}{\cot A - \cot B} \leqslant -\sum_{\text{cyclic}} \frac{2}{\csc^2 A + \csc^2 B}. \tag{3.8}$$

Using theorem 2.1, we have

PROPOSITION 3.3

$$\sum_{\text{cyclic}} \frac{\cos A + \cos B}{\sec^2 A + \sec^2 B} \leqslant \sum_{\text{cyclic}} \frac{\sin A - \sin B}{\tan A - \tan B} \leqslant \sum_{\text{cyclic}} \sin^3 \frac{A}{2} \leqslant \frac{\sqrt{2}}{2}, \tag{3.9}$$

$$24 \leqslant \sum_{\text{cyclic}} \frac{\sec^2 A + \sec^2 B}{\cos A + \cos B} \leqslant \sum_{\text{cyclic}} \frac{\tan A - \tan B}{\sin A - \sin B} \leqslant \sum_{\text{cyclic}} \csc^3 \frac{A}{2}, \tag{3.10}$$

$$\sum_{\text{cyclic}} \cos^3 \frac{A}{2} \leqslant \sum_{\text{cyclic}} \frac{\cos A - \cos B}{\cot A - \cot B} \leqslant \sum_{\text{cyclic}} \frac{\sin A + \sin B}{\csc^2 A + \csc^2 B}, \tag{3.11}$$

$$\frac{8\sqrt{3}}{3} \leqslant \sum_{\text{cyclic}} \sec^3 \frac{A}{2} \leqslant \sum_{\text{cyclic}} \frac{\cot A - \cot B}{\cos A - \cos B} \leqslant \sum_{\text{cyclic}} \frac{\csc^2 A + \csc^2 B}{\sin A + \sin B}. \tag{3.12}$$

Till this section, we have found 12 new inequalities. But some of these inequalities don't have the best lower or upper bounds. This problem will be solved in the next two sections.

4. Theorems for Symmetric Function

In this section, we prove the theorem of a kind of symmetric function. Moreover, improvement of certain known results is also presented.

LEMMA 4.1. (Schur, 1923) *If $I \subset \mathbf{R}$ is an interval and $g : I \to \mathbf{R}$ is convex, then*

$$\varphi(x) = \sum_{i=1}^{n} g(x_i)$$

is Schur-convex on I^n.

Define the symmetric function G of three variables by

$$G(x,y,z) = \sum_{\text{cyclic}} F(x,y) = F(x,y) + F(y,z) + F(z,x)$$

where $(x,y,z) \in I^3$.

THEOREM 4.2. *Let $x+y+z = s > 0, x,y,z \in I = (0, \frac{s}{2})$,*

1. *if F is Schur-convex on I^2, $\phi_1(t) = F(t,t)$ is convex and $\phi_2(t) = F(\frac{s}{2}, t)$ is strictly convex on I, then $G(\frac{s}{3}, \frac{s}{3}, \frac{s}{3}) \leqslant G(x,y,z) < G(\frac{s}{2}, \frac{s}{2}, 0)$;*
2. *if F is Schur-concave on I^2, $\phi_1(t) = F(t,t)$ is concave and $\phi_2(t) = F(\frac{s}{2}, t)$ is strictly concave on I, then $G(\frac{s}{3}, \frac{s}{3}, \frac{s}{3}) \geqslant G(x,y,z) > G(\frac{s}{2}, \frac{s}{2}, 0)$.*

Proof. (1). It is evident that $(\frac{s-z}{2}, \frac{s-z}{2}) \prec (x,y) \prec (\frac{s}{2}, \frac{s}{2} - z)$ and $F(x,y)$ is Schur-convex on I^2, we get $F(\frac{s-z}{2}, \frac{s-z}{2}) \leqslant F(x,y) \leqslant F(\frac{s}{2}, \frac{s}{2} - z)$. Therefore

$$\sum_{\text{cyclic}} F\left(\frac{s-z}{2}, \frac{s-z}{2}\right) \leqslant G(x,y,z) \leqslant \sum_{\text{cyclic}} F(\frac{s}{2}, \frac{s}{2} - z).$$

Let $X_1 = \frac{s-x}{2}$, $Y_1 = \frac{s-y}{2}$, $Z_1 = \frac{s-z}{2}$, then $(\frac{s}{3}, \frac{s}{3}, \frac{s}{3}) \prec (X_1, Y_1, Z_1)$ and $\sum\limits_{\text{cyclic}} F\left(\frac{s-z}{2}, \frac{s-z}{2}\right) = \sum\limits_{\text{cyclic}} \phi_1(Z_1)$. By the lemma 4.1, we know that $\sum\limits_{\text{cyclic}} \phi_1(Z_1)$ is Schur-convex on I^3. We get

$$\sum_{\text{cyclic}} \phi_1(Z_1) \geqslant 3\phi_1(\frac{s}{3}) = \sum_{\text{cyclic}} F(\frac{s}{3}, \frac{s}{3}) = G(\frac{s}{3}, \frac{s}{3}, \frac{s}{3}),$$

so $G(\frac{s}{3},\frac{s}{3},\frac{s}{3}) \leqslant G(x,y,z)$.
Similarly, let $X_2 = \frac{s}{2}-x, Y_2 = \frac{s}{2}-y, Z_2 = \frac{s}{2}-z$, then $(X_2,Y_2,Z_2) \prec\prec (\frac{s}{2},0,0)$ and $\sum_{\text{cyclic}} F\left(\frac{s}{2},\frac{s}{2}-z\right) = \sum_{\text{cyclic}} \phi_2(Z_2)$, we have

$$\sum_{\text{cyclic}} \phi_2(Z_2) < \phi_2(\frac{s}{2}) + 2\phi_2(0) = 2F(\frac{s}{2},0) + F(\frac{s}{2},\frac{s}{2}) = G(\frac{s}{2},\frac{s}{2},0),$$

so $G(x,y,z) < G(\frac{s}{2},\frac{s}{2},0)$.

This is end of the proof of (1). The (2) follows upon replacing F by $-F$. □

Actually, from above, if F is strictly Schur-convex (concave), and $\phi_2(t)$ is convex (concave), the inequalities are also satisfied. By a similar proof as Theorem 4.2 we get

THEOREM 4.3. *Let $x+y+z=s>0, x,y,z \in I=(0,s)$,*

1. *if F is Schur-convex on I^2, $\phi_1(t)=F(t,t)$ is convex and $\phi_2(t)=F(t,0)$ is strictly convex on I, then $G(\frac{s}{3},\frac{s}{3},\frac{s}{3}) \leqslant G(x,y,z) < G(s,0,0)$;*
2. *if F is Schur-concave on I^2, $\phi_1(t)=F(t,t)$ is concave and $\phi_2(t)=F(t,0)$ is strictly concave on I, then $G(\frac{s}{3},\frac{s}{3},\frac{s}{3}) \geqslant G(x,y,z) > G(s,0,0)$*

This theorem can be formulated for a function in n variables.

THEOREM 4.4. *Define the function G_1 of n variables by*

$$G_1(x) = F_1(x_1,\cdots,x_{n-1}) + F_1(x_2,\cdots,x_n) + \cdots +_1 F(x_n,x_1,\cdots,x_{n-2})$$

Let $x \in I^n = [0,s]^n$, $s = \sum_{i=1}^{n} x_i > 0$,

1. *if F_1 is Schur-convex on I^{n-1}, $\phi_1(t) = F_1(t,t,\cdots,t)$ is convex and $\phi_2(t) = F_1(t,0,\cdots,0)$ is convex on I, then $G_1(\frac{s}{n},\frac{s}{n},\cdots,\frac{s}{n}) \leqslant G_1(x) < G_1(s,0,\cdots,0)$;*
2. *if F_1 is Schur-concave on I^{n-1}, $\phi_1(t) = F_1(t,t,\cdots,t)$ is concave and $\phi_2(t) = F_1(t,0,\cdots,0)$ is concave on I, then $G_1(\frac{s}{n},\frac{s}{n},\cdots,\frac{s}{n}) \geqslant G_1(x) \geqslant G_1(s,0,\cdots,0)$.*

5. Sharpening Some Inequalities

In this section we will use the theorem 4.2 and 4.3 to find some inequalities' optimal bounds. From [5] we know that

$$F(x,y) = \begin{cases} \dfrac{\sin x - \sin y}{x-y}, & x \neq y \\ \cos x, & x = y \end{cases}$$

is Schur-concave on $(0,\frac{\pi}{2})^2$. It is easy to prove that $F(\frac{\pi}{2},t)$ is strictly concave on $(0,\frac{\pi}{2})$. Hence by theorem 4.2 we get $G(\frac{\pi}{2},\frac{\pi}{2},0) < G(x,y,z) \leqslant G(\frac{\pi}{3},\frac{\pi}{3},\frac{\pi}{3})$, which is equivalent to the inequality (5.1). Similarly we can prove the inequalities (5.2)-(5.6).

PROPOSITION 5.1. *In an acute $\triangle ABC$*

$$\frac{4}{\pi} < \sum_{\text{cyclic}} \frac{\sin A - \sin B}{A-B} \leqslant \frac{3}{2}. \tag{5.1}$$

$$-\frac{3\sqrt{3}}{2} \leqslant \sum_{\text{cyclic}} \frac{\cos A - \cos B}{A - B} < -1 - \frac{4}{\pi}, \tag{5.2}$$

$$0 < \sum_{\text{cyclic}} \frac{A - B}{\tan A - \tan B} \leqslant \frac{3}{4}, \tag{5.3}$$

$$-\frac{9}{4} \leqslant \sum_{\text{cyclic}} \frac{A - B}{\cot A - \cot B} < -1, \tag{5.4}$$

$$0 < \sum_{\text{cyclic}} \frac{\sin A - \sin B}{\tan A - \tan B} \leqslant \frac{3}{8}, \tag{5.5}$$

$$1 < \sum_{\text{cyclic}} \frac{\cos A - \cos B}{\cot A - \cot B} \leqslant \frac{9\sqrt{3}}{8}. \tag{5.6}$$

And using theorem 4.3 we get

PROPOSITION 5.2. In an arbitrary $\triangle ABC$,

$$-\frac{3\sqrt{3}}{2} \leqslant \sum_{\text{cyclic}} \frac{\cos A - \cos B}{A - B} < -\frac{4}{\pi}, \tag{5.7}$$

$$\sum_{\text{cyclic}} \frac{A - B}{\cos A - \cos B} \leqslant -2\sqrt{3}. \tag{5.8}$$

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