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CERTAIN CLASSES OF ANALYTIC FUNCTIONS OF COMPLEX ORDER INVOLVING A FAMILY OF GENERALIZED DIFFERENTIAL OPERATORS

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Abstract. A new class of analytic functions of complex order is defined using a generalized differential operator. Coefficient inequalities, sufficient condition and an interesting subordination result are obtained.

1. Introduction, definitions and preliminaries

Let $\mathscr{A}_1 = \mathscr{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad a_k \geqslant 0, \tag{1.1}$$

which are analytic in the open disc $\mathscr{U} = \{z \in \mathbb{C} \setminus |z| < 1\}$ and $\mathscr{S}$ be the class of function $f(z) \in \mathscr{A}$ which are univalent in $\mathscr{U}$.

The Hadamard product of two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ is given by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k\, b_k\, z^k. \tag{1.2}$$

For complex parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \in \mathbb{C} \setminus \mathbb{Z}_0^-; \mathbb{Z}_0^- = 0, -1, -2, \ldots; j = 1, \ldots, s)$, we define the generalized hypergeometric function ${}_qF_s(\alpha_1, \ldots, \alpha_q; \beta_1, \ldots, \beta_s; z)$ by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \ \ldots \ (\alpha_q)_k}{(\beta_1)_k \ \ldots \ (\beta_s)_k} \frac{z^k}{k!},$$

$$(q \leqslant s + 1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}; z \in \mathscr{U})$$

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where $\mathbb{N}$ denotes the set of all positive integers and $(x)_k$ is the Pochhammer symbol defined, in terms of the Gamma function Γ, by

$$(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = \begin{cases} 1 & \text{if } k = 0 \\ x(x+1)(x+2)\ \ldots\ (x+k-1) & \text{if } k \in \mathbb{N} = \{1, 2, , \ldots\}. \end{cases}$$

Corresponding to a function $\mathscr{G}_{q,s}(\alpha_1,\ \beta_1;\ z)$ defined by

$$\mathscr{G}_{q,s}(\alpha_1,\ \beta_1;\ z) := z\,{}_qF_s(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_q;\ \beta_1,\ \beta_2,\ \ldots,\ \beta_s; z), \tag{1.3}$$

Recently, the authors [11] defined the linear operator $D_\lambda^m(\alpha_1\, \beta_1)f\ :\ \mathscr{A}_1 \longrightarrow \mathscr{A}_1$ by

$$D_\lambda^0(\alpha_1,\ \beta_1)f(z) = f(z) * \mathscr{G}_{q,s}(\alpha_1,\ \beta_1;\ z),$$

$$D_\lambda^1(\alpha_1,\ \beta_1)f(z) = (1-\lambda)(f(z)*\mathscr{G}_{q,s}(\alpha_1,\ \beta_1;\ z)) + \lambda\, z(f(z)*\mathscr{G}_{q,s}(\alpha_1,\ \beta_1;\ z))', \tag{1.4}$$

$$D_\lambda^m(\alpha_1,\ \beta_1)f(z) = D_\lambda^1(D_\lambda^{m-1}(\alpha_1,\ \beta_1)f(z)). \tag{1.5}$$

If $f \in \mathscr{A}_1$, then from (1.4) and (1.5) we may easily deduce that

$$D_\lambda^m(\alpha_1,\ \beta_1)f(z) = z + \sum_{k=2}^{\infty} \left[1 + (k-1)\lambda\right]^m \frac{(\alpha_1)_{k-1}\ \ldots\ (\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ \ldots\ (\beta_s)_{k-1}} \frac{a_k z^k}{(k-1)!}, \tag{1.6}$$

where $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\lambda \geqslant 0$. We remark that, for a choice of the parameter $m = 0$, the operator $D_\lambda^0(\alpha_1,\ \beta_1)f(z)$ reduces to the well-known Dziok- Srivastava operator [5], for $q = 2,\ s = 1;\ \alpha_1 = \beta_1,\ \alpha_2 = 1$, we get the operator introduced by F.M.Al-oboudi [1] and for $q = 2,\ s = 1;\ \alpha_1 = \beta_1,\ \alpha_2 = 1$ and $\lambda = 1$, we get the operator introduced by G. Ş. Sălăgean [9]. Also many (well known and new) integral and differential operators can be obtained by specializing the parameters.

Henceforth, we let $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ $(\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, s)$ to be real.

Using the operator $D_\lambda^m(\alpha_1,\ \beta_1)f(z)$, we define $\mathscr{H}_\lambda^m(b;\ \alpha_1, \beta_1; A, B)$ to be the class of functions $f \in \mathscr{A}_1$ satisfying the inequality

$$1 + \frac{1}{b}\left(\frac{D_\lambda^{m+1}(\alpha_1,\ \beta_1)f(z)}{D_\lambda^m(\alpha_1,\ \beta_1)f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \quad z \in \mathscr{U}, \tag{1.7}$$

where $b \in \mathbb{C}\setminus\{0\}$, A and B are arbitrary fixed numbers, $-1 \leqslant B < A \leqslant 1$, $m \in \mathbb{N}_0$.

For a choice of the parameter $\lambda = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$ and $\alpha_2 = 1$, the class $\mathscr{H}_\lambda^m(b;\ \alpha_1, \beta_1; A, B)$ reduces to a subclass of analytic function recently introduced and studied by Attiya [4] and for $m = 0$, $\lambda = 1$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, the class $\mathscr{H}_\lambda^m(b;\ \alpha_1, \beta_1; A, B)$ reduces to class of Janowski's starlike functions of complex order, a well known subclass of univalent function. Further, we note that by specializing $b,\ m,\ \lambda,\ q, s,\ \alpha_1, \beta_1,\ A, B$, we obtain several subclasses of analytic and univalent functions studied by various authors in earlier papers (see e.g. [2, 3, 7, 8, 10]).

We use Ω to denote the class of bounded analytic functions $w(z)$ in $\mathscr{U}$ which satisfy the conditions $w(0) = 1$ and $\mid w(z) \mid < 1$ for $z \in \mathscr{U}$.

2. Coefficient estimates

THEOREM 2.1. *Let the function $f(z)$ defined by (1.1) be in the class $\mathscr{H}_{\lambda}^{m}(b;\ \alpha_1,\beta_1;A,B)$.*

(a) If $(A-B)^2 \mid b \mid^2 > (k-1)\left\{2B(A-B)\lambda\, Re\{b\} + (1-B^2)\lambda^2(k-1)\right\}$, let

$$G = \frac{(A-B)^2 \mid b \mid^2}{(k-1)\left\{2B(A-B)\lambda\, Re\{b\} + (1-B^2)\lambda^2(k-1)\right\}}, \quad k = 2,3,\ldots,m-1,$$

$M = [G]$ (Gauss symbol) and $[G]$ is the greatest integer not greater than G and

$$\Gamma_k = \frac{(\alpha_1)_{k-1}\ \ldots\ (\alpha_q)_{k-1}}{(\beta_1)_{k-1}\ \ldots\ (\beta_s)_{k-1}\,(k-1)!}.$$

Then

$$\mid a_j \mid \leqslant \frac{1}{[1+(j-1)\lambda]^m\,\lambda^{j-1}\,(j-1)!\Gamma_j}\prod_{k=2}^{j} \mid (A-B)b-(k-2)B \mid \qquad (2.1)$$
$$j = 2,3,\ldots,M+2;$$

and

$$\mid a_j \mid \leqslant \frac{1}{[1+(j-1)\lambda]^m\,\lambda^{j-1}(j-1)\,(M+1)!\,\Gamma_j}\prod_{k=2}^{M+3} \mid (A-B)b-(k-2)B \mid \qquad (2.2)$$
$$j > M+2.$$

(b) If $(A-B)^2 \mid b \mid^2 > (k-1)\left\{2B(A-B)\lambda\, Re\{b\} + (1-B^2)\lambda^2(k-1)\right\}$, then

$$\mid a_j \mid \leqslant \frac{(A-B)\mid b \mid}{\lambda\,(j-1)\,[1+(j-1)\lambda]^m\,\Gamma_j} \quad j \geqslant 2. \qquad (2.3)$$

The bounds in (2.1) and (2.3) are sharp for all admissible A, B, $b \in \mathbb{C}\setminus\{0\}$ and for each j.

Proof. Since $f(z) \in \mathscr{H}_{\lambda}^{m}(b;\ \alpha_1,\beta_1;A,B)$, the inequality (1.7) gives

$$\begin{aligned} &\mid D_{\lambda}^{m+1}(\alpha_1,\ \beta_1)f(z) - D_{\lambda}^{m}(\alpha_1,\ \beta_1)f(z) \mid \qquad (2.4)\\ &\qquad = \left\{[(A-B)b+B]D_{\lambda}^{m}(\alpha_1,\ \beta_1)f(z) - BD_{\lambda}^{m+1}(\alpha_1,\ \beta_1)f(z)\right\}w(z). \end{aligned}$$

Equation (2.4) may be written as

$$\begin{aligned} &\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)\,\Gamma_k a_k z^k \qquad (2.5)\\ &\qquad = \left\{(A-B)bz + \sum_{k=2}^{\infty}[(A-B)b-B(k-1)\lambda][1+(k-1)\lambda]^m\Gamma_k\, a_k z^k\right\}w(z). \end{aligned}$$

Or equivalently

$$\sum_{k=2}^{j}[1+(k-1)\lambda]^m\lambda(k-1)\,\Gamma_k a_k z^k+\sum_{k=j+1}^{\infty}c_k z^k$$
$$=\left\{(A-B)bz+\sum_{k=2}^{j-1}[(A-B)b-B(k-1)\lambda][1+(k-1)\lambda]^m\Gamma_k\,a_k z^k\right\}w(z), \quad (2.6)$$

for certain coefficients c_k. Explicitly $c_k=[1+(k-1)\lambda]^m\lambda(k-1)\Gamma_k a_k-[(A-B)b-B(k-2)\lambda][1+(k-2)\lambda]^m\Gamma_{k-1}a_{k-1}z^{-1}$.

Since $\mid w(z)\mid<1$, we have

$$\left|\sum_{k=2}^{j}[1+(k-1)\lambda]^m\lambda(k-1)\,\Gamma_k a_k z^k+\sum_{k=j+1}^{\infty}c_k z^k\right| \quad (2.7)$$
$$\leqslant\left|(A-B)bz+\sum_{k=2}^{j-1}[(A-B)b-B(k-1)\lambda][1+(k-1)\lambda]^m\Gamma_k\,a_k z^k\right|.$$

Let $z=re^{i\theta}$, $r<1$, applying the Parseval's formula (see [6] p.138) on both sides of the above inequality and after simple computation , we get

$$\sum_{k=2}^{j}[1+(k-1)\lambda]^{2m}\lambda^2(k-1)^2\,\Gamma_k^2\mid a_k\mid^2 r^{2k}+\sum_{k=j+1}^{\infty}\mid c_k\mid^2 r^{2k}$$
$$\leqslant(A-B)^2\mid b\mid^2 r^2+\sum_{k=2}^{j-1}\mid(A-B)b-B(k-1)\lambda\mid^2$$
$$\times[1+(k-1)\lambda]^{2m}\Gamma_k^2\mid a_k\mid^2 r^{2k}.$$

Let $r\longrightarrow 1^-$, then on some simplification we obtain

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\,\Gamma_j^2\mid a_j\mid^2 \quad (2.8)$$
$$\leqslant(A-b)^2\mid b\mid^2+\sum_{k=2}^{j-1}\left\{\mid(A-B)b-B(k-1)\lambda\mid^2-(k-1)^2\lambda^2\right\}$$
$$\times[1+(k-1)\lambda]^{2m}\Gamma_k^2\mid a_k\mid^2\quad j\geqslant 2.$$

Now the following two cases arises:

(a) Let $(A-B)^2\mid b\mid^2>(k-1)\lambda\left\{2B(A-B)Re(b)+(1-B^2)\lambda(k-1)\right\}$, suppose that $j\leqslant M+2$, then for $j=2$, (2.8) gives

$$\mid a_2\mid\leqslant\frac{(A-B)\mid b\mid}{(1+\lambda)^m\lambda\,\Gamma_2},$$

which gives (2.1) for $j=2$. We establish (2.1) for $j\leqslant M+2$ from (2.8), by mathematical induction. Suppose (2.1) is valid for $j=2,3,\ldots,(k-1)$. Then it follows from (2.8)

$$\begin{aligned}
&[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\,\Gamma_j^2 \mid a_j \mid^2 \\
&\leqslant (A-B)^2 \mid b \mid^2 + \sum_{k=2}^{j-1}\left\{ \mid (A-B)b - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\} \\
&\quad \times [1+(k-1)\lambda]^{2m}\Gamma_k^2 \mid a_k \mid^2 \\
&\leqslant (A-B)^2 \mid b \mid^2 + \sum_{k=2}^{j-1}\left\{ \mid (A-B)b - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\} \\
&\quad \times [1+(k-1)\lambda]^{2m}\Gamma_k^2 \\
&\quad \times \left\{ \frac{1}{[1+(k-1)\lambda]^{2m}\Gamma_k^2\{\lambda^{k-1}(k-1)!\}^2} \prod_{n=2}^{k} \mid (A-B)b-(n-2)B \mid^2 \right\} \\
&= (A-B)^2 \mid b \mid^2 + \sum_{k=2}^{j-1}\left\{ \mid (A-B)b - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\} \\
&\quad \times \left\{ \frac{1}{\{\lambda^{k-1}(k-1)!\}^2} \prod_{n=2}^{k} \mid (A-B)b-(n-2)B \mid^2 \right\} \\
&= (A-B)^2 \mid b \mid^2 + \left(\mid (A-B)b - B\lambda \mid^2 - \lambda^2 \right) \frac{1}{\lambda^2\,(1!)^2}(A-B)^2 \mid b \mid^2 \\
&\quad + \left(\mid (A-B)b-2B\lambda \mid^2 - 4\lambda^2 \right) \frac{1}{\lambda^4(2!)^2}\left((A-B)^2 \mid b \mid^2 \mid (A-B)b-B\lambda \mid^2 \right) + \ldots \\
&\qquad\qquad \text{up to } (k=j-1) \\
&= \frac{1}{\{\lambda^{j-2}(j-2)!\}^2} \prod_{k=2}^{j} \mid (A-B)b-(k-2)B \mid^2 .
\end{aligned}$$

Thus, we get

$$\mid a_j \mid \leqslant \frac{1}{[1+(j-1)\lambda]^m\,\lambda^{j-1}\,(j-1)!\Gamma_j} \prod_{k=2}^{j} \mid (A-B)b-(k-2)B \mid,$$

which completes the proof of (2.1).

Next, we suppose $j > M+2$. Then (2.8) gives

$$\begin{aligned}
&[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\,\Gamma_j^2 \mid a_j \mid^2 \\
&\leqslant (A-B)^2 \mid b \mid^2 + \sum_{k=2}^{M-2}\left\{ \mid (A-B)b - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\} \\
&\quad \times [1+(k-1)\lambda]^{2m}\Gamma_k^2 \mid a_k \mid^2 \\
&\quad + \sum_{k=M+3}^{j-1}\left\{ \mid (A-B)b - B(k-1)\lambda \mid^2 - (k-1)^2\lambda^2\right\} \\
&\quad \times [1+(k-1)\lambda]^{2m}\Gamma_k^2 \mid a_k \mid^2 .
\end{aligned}$$

On substituting upper estimates for $a_2, a_3, \ldots, a_{M+2}$ obtained above and simplifying, we obtain (2.2).

(b) Let $(A-B)^2 \mid b \mid^2 \leqslant (k-1)\lambda\left\{2B(A-B)Re(b)+(1-B^2)\lambda(k-1)\right\}$, then it follows from (2.8)

$$[1+(j-1)\lambda]^{2m}\lambda^2(j-1)^2\,\Gamma_j^2 \mid a_j \mid^2 \leqslant (A-B)^2 \mid b \mid^2 \qquad (j \geqslant 2),$$

which proves (2.3)

The bounds in (2.1) are sharp for the functions $f(z)$ given by

$$D_\lambda^m(\alpha_1,\, \beta_1)f(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B}} & \text{if } B \neq 0, \\ z\exp(Abz) & \text{if } B = 0. \end{cases}$$

Also, the bounds in (2.3) are sharp for the functions $f_k(z)$ given by

$$D_\lambda^m(\alpha_1,\, \beta_1)f_k(z) = \begin{cases} z(1+Bz)^{\frac{(A-B)b}{B\lambda(k-1)}} & \text{if } B \neq 0, \\ z\exp\left(\frac{Ab}{\lambda(k-1)}\, z^{k-1}\right) & \text{if } B = 0. \end{cases}$$

□

REMARK 2.1. Putting $\lambda = 1, q = 2, s = 1,\ \alpha_1 = \beta_1$ and $\alpha_2 = 1$ in Theorem 2.1, we get the result due to Attiya [4].

3. A sufficient condition for a function to be in $\mathscr{H}_\lambda^m(b;\ \alpha_1, \beta_1; A, B)$

THEOREM 3.1. *Let the function $f(z)$ defined by (1.1) and let*

$$\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\left\{(k-1)+ \mid (A-B)b-B(k-1) \mid \right\}\lambda\,\Gamma_k \mid a_k \mid \leqslant (A-B) \mid b \mid \quad (3.1)$$

holds, then $f(z)$ belongs to $\mathscr{H}_\lambda^m(b;\ \alpha_1, \beta_1; A, B)$.

Proof. Suppose that the inequality holds. Then we have for $z \in \mathscr{U}$

$$\begin{aligned} &\mid D_\lambda^{m+1}(\alpha_1,\, \beta_1)f(z) - D_\lambda^m(\alpha_1,\, \beta_1)f(z) \mid \\ &\quad - \left|(A-B)bD_\lambda^m(\alpha_1,\, \beta_1)f(z) - B\left[D_\lambda^{m+1}(\alpha_1,\, \beta_1)f(z) - D_\lambda^m(\alpha_1,\, \beta_1)f(z)\right]\right| \\ &= \left|\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)\,\Gamma_k\, a_k z^k\right| \\ &\quad - \left|(A-B)b\left[z+\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\Gamma_k a_k z^k\right] - B\sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\lambda(k-1)\Gamma_k a_k z^k\right| \\ &\leqslant \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\left\{(k-1)\lambda+ \mid (A-B)b-B(k-1)\lambda \mid \right\}\Gamma_k \mid a_k \mid r^k-(A-B) \mid b \mid r. \end{aligned}$$

Letting $r \longrightarrow 1^-$, then we have

$$\begin{aligned}&\mid D_\lambda^{m+1}(\alpha_1,\,\beta_1)f(z) - D_\lambda^m(\alpha_1,\,\beta_1)f(z)\mid\\&\quad - \left|(A-B)bD_\lambda^m(\alpha_1,\,\beta_1)f(z) - B\left[D_\lambda^{m+1}(\alpha_1,\,\beta_1)f(z) - D_\lambda^m(\alpha_1,\,\beta_1)f(z)\right]\right|\\&\quad\leqslant \sum_{k=2}^{\infty}[1+(k-1)\lambda]^m\left\{(k-1)\lambda + \mid (A-B)b - B(k-1)\lambda\mid\right\}\Gamma_k \mid a_k\mid - (A-B)\mid b\mid \leqslant 0.\end{aligned}$$

Hence it follows that

$$\frac{\left|\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{D_\lambda^m(\alpha_1,\beta_1)f(z)} - 1\right|}{\left|B\left[\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{D_\lambda^m(\alpha_1,\beta_1)f(z)} - 1\right] - (A-B)b\right|} < 1, \quad z \in \mathscr{U}.$$

Letting

$$w(z) = \frac{\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{D_\lambda^m(\alpha_1,\beta_1)f(z)} - 1}{B\left[\frac{D_\lambda^{m+1}(\alpha_1,\beta_1)f(z)}{D_\lambda^m(\alpha_1,\beta_1)f(z)} - 1\right] - (A-B)b},$$

then $w(0) = 0$, $w(z)$ is analytic in $\mid z\mid < 1$ and $\mid w(z)\mid < 1$. Hence we have

$$\frac{D_\lambda^{m+1}(\alpha_1,\,\beta_1)f(z)}{D_\lambda^m(\alpha_1,\,\beta_1)f(z)} = \frac{1 + [B + b(A-B)]w(z)}{1 + Bw(z)}$$

which shows that $f(z)$ belongs to $\mathscr{H}_\lambda^m(b;\,\alpha_1,\beta_1;A,B)$. $\square$

4. Subordination results for the class $\mathscr{H}_\lambda^m(b;\,\alpha_1,\beta_1;A,B)$

DEFINITION 4.1. A sequence $\{b_k\}_{k=1}^{\infty}$ of complex numbers is called a subordinating factor sequence if, whenever $f(z)$ is analytic, univalent and convex in $\mathscr{U}$, we have the subordination given by

$$\sum_{k=1}^{\infty} b_k\, a_k z^k \prec f(z) \qquad (z \in \mathscr{U},\, a_1 = 1). \tag{4.1}$$

LEMMA 4.1. *[12] The sequence $\{b_k\}_{k=1}^{\infty}$ is a subordinating factor sequence if and only if*

$$Re\left\{1 + 2\sum_{k=1}^{\infty} b_k z^k\right\} > 0 \qquad (z \in \mathscr{U}). \tag{4.2}$$

For convenience, we shall henceforth denote

$$\begin{aligned}&\sigma_k(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B) \qquad\qquad (4.3)\\&\quad = [1+(k-1)\lambda]^m\,\lambda\left\{(k-1) + \mid (A-B)b - B(k-1)\mid\right\}\\&\quad\times \frac{(\alpha_1)_{k-1},(\alpha_2)_{k-1},\,\ldots(\alpha_q)_{k-1}}{(\beta_1)_{k-1},(\beta_2)_{k-1},\,\ldots(\beta_s)_{k-1}\,(k-1)!}.\end{aligned}$$

Let $\widetilde{\mathscr{H}}_{\lambda}^{m}(b;\, \alpha_1, \beta_1; A, B)$ denote the class of functions $f(z) \in \mathscr{A}$ whose coefficients satisfy the conditions (3.1).

We note that $\widetilde{\mathscr{H}}_{\lambda}^{m}(b;\, \alpha_1, \beta_1; A, B) \subseteq \mathscr{H}_{\lambda}^{m}(b;\, \alpha_1, \beta_1; A, B)$.

THEOREM 4.2. *Let the function $f(z)$ defined by (1.1) be in the class $\widetilde{\mathscr{H}}_{\lambda}^{m}(b;\, \alpha_1, \beta_1; A, B)$ where $-1 \leqslant B < A \leqslant 1$. Also let $\mathscr{C}$ denote the familiar class of functions $f(z) \in \mathscr{A}$ which are also univalent and convex in $\mathscr{U}$. Then*

$$\frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]}(f * g)(z) \prec g(z) \qquad (z \in \mathscr{U};\, g \in \mathscr{C}), \tag{4.4}$$

and

$$\mathfrak{Re}(f(z)) > -\frac{(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)} \qquad (z \in \mathscr{U}). \tag{4.5}$$

The constant $\dfrac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]}$ *is the best estimate.*

Proof. Let $f(z) \in \widetilde{\mathscr{H}}_{\lambda}^{m}(b;\, \alpha_1, \beta_1; A, B)$ and let $g(z) = z + \sum_{k=2}^{\infty} b_k z^k \in \mathscr{C}$. Then

$$\frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]}(f * g)(z)$$
$$= \frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]}\left(z + \sum_{k=2}^{\infty} a_k b_k z^k\right).$$

Thus, by Definition (4.1), the assertion of the theorem will hold if the sequence

$$\left\{\frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]} a_k\right\}_{k=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma (4.1), this will be true if and only if

$$\mathfrak{Re}\left\{1 + 2\sum_{k=1}^{\infty} \frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{2\big[(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)\big]} a_k z^k\right\} > 0 \qquad (z \in \mathscr{U}). \tag{4.6}$$

Now

$$\mathfrak{Re}\left\{1 + \frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)} \sum_{k=1}^{\infty} a_k z^k\right\}$$
$$= \mathfrak{Re}\left\{1 + \frac{\sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)}{(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)} a_1 z\right.$$
$$\left. + \frac{1}{(A-B)|b| + \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B)} \sum_{k=2}^{\infty} \sigma_2(b,\, \lambda,\, m,\, \alpha_1,\, \beta_1,\, A, B) a_k z^k\right\}$$

$$\geqslant 1-\left\{\left|\frac{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}{(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}\right|r \right.$$
$$\left. +\frac{1}{|(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)|}\sum_{k=2}^{\infty}\sigma_k(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)|a_k|r^k\right\}.$$

Since $\sigma_k(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)$ is a real increasing function of k $(k \geqslant 2)$

$$1-\left\{\left|\frac{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}{(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}\right|r \right.$$
$$\left. +\frac{1}{|(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)|}\sum_{k=2}^{\infty}\sigma_k(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)|a_k|r^k\right\}$$
$$>1-\left\{\frac{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}{(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}r \right.$$
$$\left. +\frac{(A-B)|b|}{(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}r\right\}$$
$$=1-r>0.$$

Thus (4.6) holds true in $\mathscr{U}$. This proves the inequality (4.4). The inequality (4.5) follows by taking the convex function $g(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$ in (4.4). To prove the sharpness of the constant $\frac{\sigma_2(b,\lambda,m,\alpha_1,\beta_1,A,B)}{2\left[(A-B)|b|+\sigma_2(b,\lambda,m,\alpha_1,\beta_1,A,B)\right]}$, we consider $f_0(z)\in \widetilde{\mathscr{H}}_{\lambda}^{m}(b;\,\alpha_1,\beta_1;A,B)$ given by

$$f_0(z)=z-\frac{(A-B)|b|}{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}z^2 \qquad (-1\leqslant B<A\leqslant 1).$$

Thus from (4.4), we have

$$\frac{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}{2\left[(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)\right]}f_0(z)\prec\frac{z}{1-z}. \tag{4.7}$$

It can be easily verified that

$$\min\left\{Re\left(\frac{\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)}{2\left[(A-B)|b|+\sigma_2(b,\,\lambda,\,m,\,\alpha_1,\,\beta_1,\,A,B)\right]}f_0(z)\right)\right\}=-\frac{1}{2} \qquad (z\in\mathscr{U}),$$

This shows that the constant $\frac{\sigma_2(b,\lambda,m,\alpha_1,\beta_1,A,B)}{2\left[(A-B)|b|+\sigma_2(b,\lambda,m,\alpha_1,\beta_1,A,B)\right]}$ is best possible. $\square$

REMARK 4.1. By specializing the parameters, the above result reduces to various other results obtained by several authors.

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