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A NONLINEAR INEQUALITY

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Abstract. A quadratic inequality is formulated in the paper. An estimate of the rate of decay of solutions to this inequality is obtained. This inequality is of interest in a study of dynamical systems and nonlinear evolution equations. It can be applied to the study of global existence of solutions to nonlinear PDE.

1. Introduction

In [2] the following differential inequality

$$\dot{g}(t) \leqslant -\gamma(t)g(t) + \alpha(t)g^2(t) + \beta(t), \qquad t \geqslant t_0, \tag{1}$$

was studied and applied to various evolution problems. In (1) $\alpha(t), \beta(t), \gamma(t)$ and $g(t)$ are continuous nonnegative functions on $[t_0, \infty)$ where $t_0 \geqslant 0$ is a fixed number. In [2], an upper bound for $g(t)$ is obtained under some conditions on α, β, γ:

THEOREM 1. ([2] p. 97) *If there exists a monotonically growing function* $\mu(t)$,

$$\mu \in C^1[t_0, \infty), \quad \mu > 0, \quad \lim_{t\to\infty} \mu(t) = \infty,$$

such that

$$0 \leqslant \alpha(t) \leqslant \frac{\mu}{2}\left[\gamma - \frac{\dot{\mu}(t)}{\mu(t)}\right], \qquad \dot{u} := \frac{du}{dt}, \tag{2}$$

$$\beta(t) \leqslant \frac{1}{2\mu}\left[\gamma - \frac{\dot{\mu}(t)}{\mu(t)}\right], \tag{3}$$

$$\mu(t_0)g(t_0) < 1, \tag{4}$$

where $\alpha(t), \beta(t), \gamma(t)$ *and* $g(t)$ *are continuous nonnegative functions on* $[t_0, \infty)$, $t_0 \geqslant 0$, *and* $g(t)$ *satisfies* (1), *then:*

$$0 \leqslant g(t) < \frac{1}{\mu(t)}, \qquad \forall t \geqslant t_0. \tag{5}$$

If inequalities (2)–(4) *hold on an interval* $[t_0, T)$, *then* $g(t)$ *exists on this interval and inequality* (5) *holds on* $[t_0, T)$.

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This result allows one to establish global existence of a solution to a nonlinear inequality (1) and to estimate the rate of its decay when $t \to \infty$. These results are very useful in a study of nonlinear equations. For example, in [1] an operator equation $F(u) = f$ was studied by the Dynamical Systems Method (DSM), which consists of a study of an evolution equation whose solution converges as $t \to \infty$ to a solution of equation $F(u) = f$. Theorem 1 has been used in the above study.

In this paper we consider a *discrete analog* of Theorem 1. We study the following inequality:

$$\frac{g_{n+1} - g_n}{h_n} \leqslant -\gamma_n g_n + \alpha_n g_n^2 + \beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1,$$

or an inequality:

$$g_{n+1} \leqslant (1 - \gamma_n)g_n + \alpha_n g_n^2 + \beta_n, \quad n \geqslant 0, \qquad 0 < \gamma_n < 1,$$

where g_n, β_n, γ_n and α_n are positive sequences of real numbers. Under suitable conditions on α_n, β_n and γ_n, we obtain an upper bound for g_n as $n \to \infty$. This result can be used in a study of evolution problems, in a study of iterative processes, and in a study of nonlinear PDE.

In Section 2, the main result, namely, Theorem 2, is formulated and proved. In Section 3, an application of Theorem 2 is presented.

2. Results

THEOREM 2. *Let $\alpha_n, \beta_n, \gamma_n$ and g_n be nonnegative sequences satisfying the inequality:*

$$\frac{g_{n+1} - g_n}{h_n} \leqslant -\gamma_n g_n + \alpha_n g_n^2 + \beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1, \tag{6}$$

or equivalently

$$g_{n+1} \leqslant (1 - h_n\gamma_n)g_n + \alpha_n h_n g_n^2 + h_n\beta_n, \qquad h_n > 0, \quad 0 < h_n\gamma_n < 1. \tag{7}$$

If there is a monotonically growing sequence $(\mu_n)_{n=1}^{\infty} > 0$ such that the following conditions hold:

$$g_0 \leqslant \frac{1}{\mu_0}, \tag{8}$$

$$\alpha_n \leqslant \frac{\mu_n}{2}\left(\gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n h_n}\right), \tag{9}$$

$$\beta_n \leqslant \frac{1}{2\mu_n}\left(\gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n h_n}\right), \tag{10}$$

then

$$g_n \leqslant \frac{1}{\mu_n} \quad \forall n \geqslant 0. \tag{11}$$

Therefore, if $\lim_{n\to\infty} \mu_n = \infty$ *then* $\lim_{n\to\infty} g_n = 0$.

Proof. Let us prove (11) by induction. Inequality (11) holds for $n = 0$ by assumption (8). Suppose that (11) holds for $n \leqslant m$. From (6), from the inequalities (9)–(10), and from the induction hypothesis $g_n \leqslant \frac{1}{\mu_n}$, $n \leqslant m$, one gets

$$\begin{aligned}
g_{m+1} &\leqslant g_m(1-h_m\gamma_m) + \alpha_m h_m g_m^2 + h_m\beta_m \\
&\leqslant \frac{1}{\mu_m}(1-h_m\gamma_m) + \frac{h_m\mu_m}{2}\left(\gamma_m - \frac{\mu_{m+1}-\mu_m}{\mu_m h_m}\right)\frac{1}{\mu_m^2} + \frac{h_m}{2\mu_m}\left(\gamma_m - \frac{\mu_{m+1}-\mu_m}{\mu_m h_m}\right) \\
&= \frac{1}{\mu_m} - \frac{\mu_{m+1}-\mu_m}{\mu_m^2} \\
&= \frac{1}{\mu_{m+1}} - (\mu_{m+1}-\mu_m)\big(\frac{1}{\mu_m^2} - \frac{1}{\mu_m\mu_{m+1}}\big) \\
&= \frac{1}{\mu_{m+1}} - \frac{(\mu_{m+1}-\mu_m)^2}{\mu_n^2\mu_{m+1}} \leqslant \frac{1}{\mu_{m+1}}.
\end{aligned}$$

Therefore, inequality (11) holds for $n = m+1$. Thus, inequality (11) holds for all $n \geqslant 0$ by induction. Theorem 2 is proved. □

Setting $h_n = 1$ in Theorem 2, one obtains the following result:

COROLLARY 3. *Let α, β, γ_n and g_n be nonnegative sequences, and*

$$g_{n+1} \leqslant g_n(1-\gamma_n) + \alpha_n g_n^2 + \beta_n, \qquad 0 < \gamma_n < 1. \tag{12}$$

If there is a monotonically growing sequence $(\mu_n)_{n=1}^{\infty} > 0$ such that the following conditions hold

$$g_0 \leqslant \frac{1}{\mu_0}, \tag{13}$$

$$\alpha_n \leqslant \frac{\mu_n}{2}\left(\gamma_n - \frac{\mu_{n+1}-\mu_n}{\mu_n}\right), \tag{14}$$

$$\beta_n \leqslant \frac{1}{2\mu_n}\left(\gamma_n - \frac{\mu_{n+1}-\mu_n}{\mu_n}\right), \tag{15}$$

then

$$g_n \leqslant \frac{1}{\mu_n}, \qquad \forall n \geqslant 0. \tag{16}$$

3. Applications

Let $F : H \to H$ be a twice Fréchet differentiable map in a real Hilbert space H. We also assume that

$$\sup_{u\in B(u_0,R)} \|F^{(j)}(u)\| \leqslant M_j = M_j(R), \quad 0 \leqslant j \leqslant 2, \tag{17}$$

where $B(u_0,R) := \{u : \|u-u_0\| \leqslant R\}$, $u_0 \in H$ is some element, $R > 0$, and there is no restriction on the growth of $M_j(R)$ as $R \to \infty$, i.e., an arbitrarily strong nonlinearities F are admissible.

Consider the equation:

$$F(v) = f, \tag{18}$$

and assume that $F'(\cdot) \geqslant 0$, that is, F is monotone: $\langle F(u) - F(v), u - v\rangle \geqslant 0$, $\forall u, v \in H$, and that (18) has a solution, possibly non-unique. Let y be the unique minimal-norm solution to (18). If F is monotone and continuous, then $\mathcal{N}_f := \{u : F(u) = f\}$ is a closed convex set in H ([2]). Such a set in a Hilbert space has a unique minimal-norm element. So, the solution y is well defined.

Let $a \in C^1[0,\infty)$ be such that

$$a(t) > 0, \qquad a(t) \searrow 0 \qquad \text{as} \qquad t \to \infty. \tag{19}$$

Assume $h_n > 0$. Denote

$$a_n := a(t_n), \quad t_n := \sum_{j=1}^{n-1} h_j, \quad t_0 := 0, \quad \lim_{n\to\infty} t_n = \infty, \quad a_n > a_{n+1}.$$

Let $A_{a_n} := A_n + a_n I$, where $A_n := F'(u_n) \geqslant 0$, and I is the identity operator in H. Consider the following iterative scheme for solving (18):

$$u_{n+1} = u_n - h_n A_{a_n}^{-1}\big[F(u_n) + a_n u_n - f\big], \qquad u_0 = u_0, \tag{20}$$

where $u_0 \in H$ is arbitrary. The operator $A_{a_n}^{-1}$ is well defined and $\|A_{a_n}^{-1}\| \leqslant \frac{1}{a_n}$ if $A_n \geqslant 0$ and $a_n > 0$. Denote $w_n := u_n - V_n$ where V_n solves the equation

$$F(V_n) + a_n V_n - f = 0. \tag{21}$$

One has

$$w_{n+1} = w_n - h_n A_{a_n}^{-1}\big[F(u_n) + a_n u_n - a_n V_n - F(V_n)\big] + V_n - V_{n+1}, \qquad w_0 := u_0 - V_0. \tag{22}$$

The Taylor's formula yields:

$$F(u_n) - F(V_n) = A_n w_n + K_n, \qquad \|K_n\| \leqslant \frac{M_2\|w_n\|^2}{2}. \tag{23}$$

Thus, (22) can be written as

$$w_{n+1} = w_n(1 - h_n) - h_n A_{a_n}^{-1} K_n + V_n - V_{n+1}. \tag{24}$$

From (21) one derives:

$$F(V_{n+1}) - F(V_n) + a_{n+1}(V_{n+1} - V_n) + (a_{n+1} - a_n)V_n = 0. \tag{25}$$

Multiply (25) by $V_{n+1} - V_n$ and use the monotonicity of F, to get:

$$a_{n+1}\|V_n - V_{n+1}\|^2 \leqslant (a_n - a_{n+1})\|V_n\|\|V_n - V_{n+1}\|. \tag{26}$$

This implies

$$\|V_n - V_{n+1}\| \leqslant \frac{a_n - a_{n+1}}{a_{n+1}}\|V_n\| \leqslant \frac{a_n - a_{n+1}}{a_{n+1}}\|y\|. \tag{27}$$

Here we have used the fact that $\|V_n\| \leqslant \|y\|$ (see [2, Lemma 6.1.7]).

Let $g_n := \|w_n\|$. Set $h_n = \frac{1}{2}$. Then (24) and inequality (27) imply

$$g_{n+1} \leqslant \frac{1}{2} g_n + \frac{c_1}{a_n} g_n^2 + \frac{a_n - a_{n+1}}{a_{n+1}} \|y\|, \qquad c_1 = \frac{M_2}{4}, \quad g_0 = \|u_0 - V_0\|. \tag{28}$$

Let

$$\mu_n = \frac{\lambda}{a_n}, \qquad \lambda = \text{const} > 0. \tag{29}$$

Let us apply Corollary 3 to inequality (28). We have:

$$\gamma_n = \frac{1}{2}, \quad \beta_n = \frac{a_n - a_{n+1}}{a_{n+1}} \|y\|, \quad \alpha_n = \frac{c_1}{a_n}, \tag{30}$$

and

$$\frac{\mu_{n+1} - \mu_n}{\mu_n} = \Big(\frac{\lambda}{a_{n+1}} - \frac{\lambda}{a_n}\Big) \frac{a_n}{\lambda} = \frac{a_n}{a_{n+1}} - 1.$$

Let us choose a_n such that $\frac{a_{n+1}}{a_n} \geqslant \frac{4}{5}$. Then, with $\gamma_n = \frac{1}{2}$, one gets:

$$\frac{1}{4} \leqslant \gamma_n - \frac{\mu_{n+1} - \mu_n}{\mu_n}.$$

Condition (14) is satisfied if

$$\frac{c_1}{a_n} \leqslant \frac{\lambda}{2a_n} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{\lambda}{8a_n}.$$

This inequality holds if $\lambda \geqslant 8c_1$. One may take $\lambda = 8c_1$.

If the following inequality $\frac{1}{a_{n+1}} - \frac{1}{a_n} \leqslant \frac{1}{a_0}$ holds, or equivalently,

$$\frac{a_n - a_{n+1}}{a_{n+1}} \leqslant \frac{a_n}{a_0}, \tag{31}$$

then condition (15) holds, provided that $64c_1\|y\| \leqslant a_0$. This conclusion holds regardless of the previous assumptions about a_n. If, for example, $a_n = \frac{4a_0}{4+n}$, then the earlier assumption $\frac{a_{n+1}}{a_n} \geqslant \frac{4}{5}$ is satisfied, inequality (31) holds, and condition (15) holds provided that

$$\|y\| \leqslant \frac{a_0}{16c_1}. \tag{32}$$

Inequality (32) holds if $16c_1\|y\| \leqslant a_0$, that is, if a_0 is sufficiently large.

Condition (13) holds if $g_0 \leqslant \frac{a_0}{\lambda}$. Thus, if

$$\frac{a_{n+1}}{a_n} \geqslant \frac{4}{5}, \quad \lambda = 8c_1, \quad g_0 \leqslant \frac{a_0}{8c_1}, \quad a_0 \geqslant 16c_1\|y\|, \tag{33}$$

then

$$g_n < \frac{a_n}{\lambda}, \quad \forall n \geqslant 0,$$

so

$$\lim_{n\to\infty} g_n = 0. \tag{34}$$

By the triangle inequality, one has

$$\|u_n - y\| \leqslant \|u_n - V_n\| + \|V_n - y\|. \tag{35}$$

By Lemma 6.1.7 in [2], one has

$$\lim_{n\to 0} \|V_n - y\| = 0. \tag{36}$$

From (34)–(36), one obtains

$$\lim_{n\to\infty} \|u_n - y\| = 0.$$

Thus, equation (18) can be solved by the iterative process (20) with a_n satisfying (33).

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